(II) Lagrangian formulation of classical Electrodynamics

Classical Electrodynamics is governed by

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^{\mu} A_{\mu}$$

with the field strength tensor

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

and source term J^{μ}

(a) We consider a gauge transformation :

$$A^{\mu}(x) \longrightarrow A'^{\mu}(x) = A^{\mu}(x) + \partial^{\mu} f(x)$$

We're required to show that

$$\mathcal{L} \rightarrow \mathcal{L}' = \frac{1}{4} F'^{\mu\nu} F'_{\mu\nu} - J^{\mu} A'_{\mu}$$

Leads to the same E.o.M for $A^{\mu}(x)$.

First we examine $F'^{\mu\nu} = (\partial^{\mu} A'^{\nu} - \partial^{\nu} A'^{\mu})$

$$\Rightarrow \quad F'^{\mu\nu} = \partial^{\mu}(A^{\nu} + \partial^{\nu} f) - \partial^{\nu}(A^{\mu} + \partial^{\mu} f)$$

$$= \partial^{\mu} A^{\nu} + \cancel{\partial^{\mu}\partial^{\nu}} f - \partial^{\nu} A^{\mu} - \cancel{\partial^{\nu}\partial^{\mu}} f$$

$$= \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = F^{\mu\nu}$$

$$\therefore \quad \frac{1}{4} F'^{\mu\nu} F'_{\mu\nu} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

1st term of the Lagrangian is unchanged

Let's examine $J^\mu A_\mu' = J^\mu A_\mu + \underbrace{J^\mu \partial_\mu f}$

$$= J^\mu A^\mu + \boxed{\partial_\mu (J^\mu \cdot f)} \underbrace{- f \partial_\mu J^\mu}$$

This is a total derivative and the E.L. equations do not get affected by it.

We need to calculate $\partial_\mu J^\mu$:

Consider the E.L equations for $\mathcal{L}$

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right)}_{*} = \frac{\partial \mathcal{L}}{\partial A^\nu}$$

$$* \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right]$$

$$= \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[\frac{1}{4} F^{\kappa\lambda} F_{\kappa\lambda} \right]$$

$$= \frac{1}{4} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[(\partial^\kappa A^\lambda - \partial^\lambda A^\kappa)(\partial_\kappa A_\lambda - \partial_\lambda A_\kappa) \right]$$

$$= \frac{1}{4} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[\underline{\partial^\kappa A^\lambda \partial_\kappa A_\lambda} - \underline{\partial^\kappa A^\lambda \partial_\lambda A_\kappa} - \underline{\partial^\lambda A^\kappa \partial_\kappa A_\lambda} + \underline{\partial^\lambda A^\kappa \partial_\lambda A_\kappa} \right]$$

$$= \frac{2}{4} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[\partial^\kappa A^\lambda \partial_\kappa A_\lambda - \partial^\kappa A^\lambda \partial_\lambda A_\kappa \right]$$

$$= \frac{1}{2} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(\partial^\kappa A^\lambda \partial_\kappa A_\lambda - \partial^\kappa A^\lambda \partial_\lambda A_\kappa \right)$$

But $\dfrac{\partial}{\partial(\partial_\mu A_\nu)} (\partial^\kappa A^\lambda \, \partial_\kappa A_\lambda) = \partial^\kappa A^\lambda \dfrac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\kappa A_\lambda)$

$$+ \partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial^\kappa A^\lambda)$$

$$= \partial^\kappa A^\lambda \delta^\mu_\kappa \delta^\nu_\lambda + g^{\kappa\alpha} g^{\lambda\beta} \partial_\kappa A_\lambda \frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)}$$

$$= \partial^\kappa A^\lambda \delta^\mu_\kappa \delta^\nu_\lambda + g^{\kappa\alpha} g^{\lambda\beta} \partial_\kappa A_\lambda \, \delta^\alpha_\mu \delta^\beta_\nu$$

$$= 2 \partial^\mu A^\nu$$

Similarly $\dfrac{\partial}{\partial(\partial_\mu A_\nu)} (\partial^\kappa A^\lambda \, \partial_\lambda A_\kappa) = 2 \partial^\nu A^\mu$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

Then we have $\dfrac{\partial \mathcal{L}}{\partial A^\nu} = -J^\nu$

$\therefore$ The E.L. equations give:

$$\partial_\mu F^{\mu\nu} = -J^{\cancel{\mu}\nu}$$

$$\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu J^\nu$$

$$0 = \partial_\nu J^\nu$$

$$J^\mu A_\mu' = J^\mu A_\mu + \boxed{\partial_\mu (J^\mu \cdot f)}$$

$\therefore$ $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial_\mu (J^\mu \cdot f)$ will give the same equations of motions for $A_\mu(x)$.

(b) We have

$$\partial_\mu F^{\mu\nu} = J^\nu \qquad \text{and} \qquad \partial_\mu \tilde{F}^{\mu\nu} \equiv \partial_\mu \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$$

with

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

and we need to prove that $(c = \epsilon_0 = \mu_0 = 1)$

Heavyside-Lorentz units

(i) $\vec{\nabla} \cdot \vec{E} = \rho$

(ii) $\vec{\nabla} \times \vec{B} = \vec{J} + \dfrac{\partial \vec{E}}{\partial t}$

(iii) $\vec{\nabla} \cdot \vec{B} = 0$

(iv) $\vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$

Start with $\partial_\mu F^{\mu\nu} = J^\nu$

$$\Rightarrow \partial_0 F^{0\nu} + \partial_i F^{i\nu} = J^\nu$$

$$\Rightarrow \frac{\partial}{\partial t} F^{0\nu} + \frac{\partial}{\partial x_i} F^{i\nu} = J^\nu$$

- For $\nu = 0$ we obtain

$$\cancel{\frac{\partial}{\partial t} F^{00}}_{\,0} + \frac{\partial}{\partial x} F^{10} + \frac{\partial}{\partial y} F^{20} + \frac{\partial}{\partial z} F^{30} = J^0$$

$$\Rightarrow \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_x = J^0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \rho}$$

- For $\nu = 1, 2, 3$ we have

$$\frac{\partial}{\partial t}(-\vec{E}) + \frac{\partial}{\partial x}\begin{pmatrix} 0 \\ -B_z \\ B_y \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} B_z \\ 0 \\ -B_x \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} -B_y \\ B_x \\ 0 \end{pmatrix} = \vec{j}$$

$$\Rightarrow -\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = \vec{j} \Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}}$$

Next, we use $\partial_\mu \tilde{F}^{\mu\nu} = 0 = \frac{1}{2} \partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$$= -\frac{1}{2} \partial_\mu \epsilon^{\nu\mu\rho\sigma} F_{\rho\sigma}$$

- For $\nu = 0$ we have

$$\partial_\mu \tilde{F}^{\mu 0} = -\frac{1}{2} \epsilon^{0\mu\rho\sigma} \partial_\mu F_{\rho\sigma} = 0$$

$$= -\frac{1}{2}\Big(\epsilon^{0123} \partial_1 F_{23} + \epsilon^{0231} \partial_2 F_{31} + \epsilon^{0312} \partial_3 F_{12} + \epsilon^{0132} \partial_1 F_{32} + \epsilon^{0213} \partial_2 F_{13} + \epsilon^{0321} \partial_3 F_{21} \Big)$$

$$= -\frac{1}{\cancel{2}}\Big(\underbrace{\epsilon^{0123}}_{=1} \partial_1 F_{23} + \underbrace{\epsilon^{0231}}_{= -\epsilon^{0213} = \epsilon^{0123} = 1} \partial_2 F_{31} + \underbrace{\epsilon^{0312}}_{= -\epsilon^{0132} = \epsilon^{0123} = 1} \partial_3 F_{12} \Big) \cdot \cancel{2}$$

$$\Rightarrow -\partial_x(-B_x) - \partial_y(-B_y) - \partial_z(-B_z) = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

• For $\nu = 1$:

$$0 = -\frac{1}{\cancel{2}}\left(\underbrace{\epsilon^{1023}}_{=-1} \partial_0 F_{23} + \underbrace{\epsilon^{1230}}_{-1} \partial_2 F_{30} + \underbrace{\epsilon^{1302}}_{-1} \partial_3 F_{02} \right) \cdot \cancel{2}$$

$$\Rightarrow 0 : \partial_t(-B_x) + \partial_y(-E_z) + \partial_z(E_y)$$

$$\Rightarrow 0 = -\frac{\partial B_x}{\partial t} - (\partial_y E_z - \partial_z E_y) \quad \text{——} \quad (a)$$

• For $\nu = 2$:

$$0 = -\frac{1}{\cancel{2}}\left(\underbrace{\epsilon^{2013}}_{=1} \partial_0 F_{13} + \underbrace{\epsilon^{2130}}_{\cdot 1} \partial_1 F_{30} + \underbrace{\epsilon^{2301}}_{-1} \partial_3 F_{01} \right) \cdot \cancel{2}$$

$$0 = -\left(\partial_t B_y \pm \partial_x E_z + \partial_z(+E_x) \right)$$

$$0 = -\frac{\partial B_y}{\partial t} + (\partial_x E_z - \partial_z E_x) \quad \text{——} \quad (b)$$

• For $\nu = 3$:

$$0 = -\frac{1}{\cancel{2}}\left(\epsilon^{3012} \partial_0 F_{12} + \epsilon^{3120} \partial_1 F_{20} + \epsilon^{3201} \partial_2 F_{01} \right) \cdot \cancel{2}$$

$$0 = -\underbrace{\epsilon^{3012}}_{-1}\left(\partial_t(-B_z) + \partial_x(-E_y) + \partial_y E_x \right)$$

$$0 = -\frac{\partial B_z}{\partial t} - (\partial_x E_y - \partial_y E_x) \quad \text{——} \quad (c)$$

$$(a) + (b) + (c) = 0$$

$$\Rightarrow -\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{E} = 0$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0}$$

Next we are given the Bianchi Identity

$$\partial^{[\mu} F^{\nu\rho]} \equiv \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0$$

Permute $\mu \leftrightarrow \nu$: $\partial^\nu F^{\mu\rho} + \partial^\mu F^{\rho\nu} + \partial^\rho F^{\nu\mu}$

$$= - \left(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} \right) = 0$$

$\Rightarrow$ Antisymmetric

Now consider the following permutations of $(\mu\nu\rho)$

$$(\mu\nu\rho) = (012), (013), (023), (123)$$

$\mu\nu\rho$
(123) : $\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0$

$$\Rightarrow \partial^x (-B_x) + \partial^y (-B_y) + \partial^z (-B_z) = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

(012) : $\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0$

$$\Rightarrow \partial^t (-B_z) + \partial^x (E_y) + \partial^y (-E_x) = 0$$

$$\Rightarrow + \left[-\frac{\partial}{\partial t} B_z + \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \right] = 0$$

(013) : $\partial^0 F^{13} + \partial^1 F^{30} + \partial^3 F^{01} = 0$

$$\Rightarrow \partial^t B_y + \partial^x E_z + \partial^z (-E_x) = 0$$

$$\Rightarrow \frac{\partial B_y}{\partial t} + \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0$$

(023) : $\partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} = 0$

$$\Rightarrow \partial^t (-B_x) + \partial^y E_z + \partial^z (-E_y) = 0$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0}$$

(c) An alternative Lagrangian to eq^n (1) is

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu) - J^\mu A_\mu$$

E.o.M from the Euler-Lagrange equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\Rightarrow \quad \partial_\mu \left(-\frac{1}{\cancel{2}} \cdot \cancel{2}\, \partial^\mu A^\nu \right) + J^\nu = 0$$

$$\Rightarrow \quad \boxed{-\partial_\mu \partial^\mu A^\nu + J^\nu = 0} \quad — \quad *$$

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[\frac{1}{2}\partial^\lambda A^\sigma \partial_\lambda A_\sigma\right]$$

$$= \frac{\partial}{\partial(\partial_\mu A_\nu)}\left[\frac{1}{2} g^{\kappa\lambda} g^{\rho\sigma} (\partial_\kappa A_\rho)(\partial_\lambda A_\sigma)\right]$$

$$= \frac{1}{2}\left[g^{\kappa\lambda} g^{\rho\sigma} \left\{ (\delta_\kappa^\mu A_\nu^\rho)\partial_\lambda A_\sigma + \delta_\mu^\lambda \delta_\nu^\sigma (\partial_\kappa A_\rho) \right\}\right]$$

$$= \frac{1}{2}\left[\partial^{\mu} A^\nu + \partial^\mu A^\nu\right]$$

To produce the standard form of Maxwell's equations we need

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\Rightarrow \quad \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\nu$$

$$\Rightarrow \quad \underbrace{\partial_\mu \partial^\mu A^\nu}_{J^\nu \text{ (from } *)} - \partial_\mu \partial^\nu A^\mu = J^\nu$$

$$\Rightarrow \quad J^\nu - \partial_\mu \partial^\nu A^\mu = J^\nu$$

$$\Rightarrow \quad \partial^\nu \partial_\mu A^\mu = 0 \quad \Rightarrow \quad \boxed{\partial_\mu A^\mu = 0} \quad \text{Lorentz Gauge.}$$

(d) Let us speculate an extension of Classical Electrodynamics by adding a new term to the field-strength tensor

$$F'^{\mu\nu} = F^{\mu\nu} - \partial_\rho \epsilon^{\mu\nu\rho\sigma} A'_\sigma \quad \text{———} \; (\star)$$

with $A'^\mu = (\varphi', \vec{A}')^T$

Consider that $\partial_\mu F'^{\mu\nu} = \partial_\mu F^{\mu\nu} - \underbrace{\partial_\mu \partial_\rho \epsilon^{\mu\nu\rho\sigma} A'_\sigma}_{=0}$

Now $\partial_\mu \partial_\rho \epsilon^{\mu\nu\rho\sigma} A'_\sigma$

$= \partial_\rho \partial_\mu \epsilon^{\mu\nu\rho\sigma} A'_\sigma$

$= -\partial_\rho \partial_\mu \epsilon^{\mu\rho\nu\sigma} A'_\sigma$

$= \partial_\rho \partial_\mu \epsilon^{\rho\mu\nu\sigma} A'_\sigma$

$= -\partial_\rho \partial_\mu \epsilon^{\rho\nu\mu\sigma} A'_\sigma$

$\Rightarrow 2.\partial_\mu \partial_\rho \epsilon^{\mu\nu\rho\sigma} A'_\sigma = 0$

$\Rightarrow \partial_\mu F'^{\mu\nu} = \partial_\mu F^{\mu\nu} \Rightarrow$ Inhomogeneous maxwell's eq^{n}s are equivalent.

(e) We now add an <u>additional</u> source term:

$$\partial_\mu \tilde{F}'^{\mu\nu} = J'^\nu$$

$$\Rightarrow \partial_\mu \tilde{F}'^{\mu\nu} = \partial_\mu \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \underbrace{F'_{\rho\sigma}}_{\text{from } (\star)}$$

$$= \partial_\mu \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \left[F_{\rho\sigma} - \partial^\lambda \epsilon_{\rho\sigma\lambda\tau} A'^\tau \right]$$

$$= \underbrace{\partial_\mu \tilde{F}^{\mu\nu}}_{=0} - \frac{1}{2} \underbrace{\partial_\mu \epsilon^{\mu\nu\rho\sigma} \partial^\lambda \epsilon_{\rho\sigma\lambda\tau} A'^\tau}_{*}$$

We use

$$\epsilon^{\mu\nu\rho\sigma}\,\epsilon_{\mu\nu\lambda\tau} = -2\left(g^{\rho}{}_{\lambda}\, g^{\sigma}{}_{\tau} - g^{\rho}{}_{\tau}\, g^{\sigma}{}_{\lambda}\right)$$

$$* \quad = \frac{1}{2}\,\partial_\mu \partial^\lambda (-2)\,\cancel{(g^{\rho}{}_{\lambda} g^{\sigma}{}_{\tau})}\left[g^{\mu}{}_{\lambda}\, g^{\nu}{}_{\tau} - g^{\mu}{}_{\tau}\, g^{\nu}{}_{\lambda}\right] A'^{\tau}$$

$$= \partial_\mu \partial^\mu A'^{\nu} - \partial_\mu \partial^\nu A'^{\mu}$$

$$\Rightarrow \boxed{\partial_\mu \tilde{F}'^{\mu\nu} = \partial_\mu \partial^\mu A'^{\nu} - \partial_\mu \partial^\nu A'^{\mu}}$$

$\nu = 0$: $\left(\partial_t \partial^t + \partial_x \partial^x + \partial_y \partial^y + \partial_z \partial^z\right) A'^{0} - \partial^t\left(\partial_t A'^{0} + \partial_x A'^{1} + \partial_y A'^{2} + \partial_z A'^{3}\right)$

$$= \cancel{\frac{\partial^2 \varphi'}{\partial t^2}} - \vec{\nabla}\cdot(\vec{\nabla}\varphi') - \cancel{\frac{\partial^2 \varphi'}{\partial t^2}} - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}')$$

$$= \vec{\nabla}\cdot\left(-\vec{\nabla}\varphi' - \frac{\partial \vec{A}'}{\partial t}\right)$$

$\nu = 1,2,3 = i$: $\dfrac{\partial^2}{\partial t^2} A'^{i} - \vec{\nabla}(\vec{\nabla} A'^{i}) - \partial^i\left(\partial_t A'^{0} + \partial_j A'^{j}\right)$

$(\partial^t = +\partial/\partial t\ ,\ \partial_t = -\partial/\partial t)$

$$= \frac{\partial^2}{\partial t^2}(\vec{A}') + (\vec{\nabla}\cdot\vec{\nabla})\vec{A}' - \vec{\nabla}\left(-\frac{\partial \varphi'}{\partial t} + \vec{\nabla}\cdot\vec{A}'\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial \vec{A}'}{\partial t} + \vec{\nabla}\varphi'\right) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A}')$$

(Since $\vec{\nabla}\times(\vec{\nabla}\times\hat{A}) = \vec{\nabla}(\vec{\nabla}\cdot\hat{A}) - \vec{\nabla}^2 A$)

$$\Rightarrow \partial_\mu \tilde{F}'^{\mu 0} = \vec{\nabla}\cdot\left(-\vec{\nabla}\varphi' - \frac{\partial \vec{A}'}{\partial t}\right) = \rho' \quad — (1)$$

$$\partial_\mu \tilde{F}'^{\mu i} = \frac{\partial}{\partial t}\left(\frac{\partial \vec{A}'}{\partial t} + \vec{\nabla}\varphi'\right) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A}') = \vec{j}' \quad — (2)$$

Define $\quad -\vec{\nabla}\varphi' - \dfrac{\partial \vec{A}'}{\partial t} + \vec{\nabla}\times\vec{A} = \vec{B}'$

and $\quad -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{\partial t} + \vec{\nabla}\times\vec{A}' = \vec{E}'$

We know that $\quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

and $\quad \vec{\nabla}\times\underbrace{\left(-\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{\partial t}\right)}_{\vec{E}} + \dfrac{\partial}{\partial t}\underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{B}} = 0$

From ① and ②

$\Rightarrow \quad \vec{\nabla}\cdot\vec{B}' = \rho'$ $\quad$ and $\quad -\dfrac{\partial \vec{B}'}{\partial t} - \vec{\nabla}\times\vec{E}' = \vec{j}'$

ρ' and $\vec{j}'$ denote the magnetic charge density and magnetic current density, defined analogously to electric charge and electric current densities.

Notice the complementarity to $\vec{\nabla}\cdot\vec{E} = \rho$ and

$\cancel{\partial \vec{B}}$ $\quad \vec{\nabla}\times\vec{B} = \vec{j} + \dfrac{\partial \vec{E}}{\partial t}$.

Auxiliary Material

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$