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Information regarding the exercise course:
The exercise sheets are published each Wednesday on the webpage of the course, see bottom of this page, and handed out in the lecture on Thursday.
Scheinkriterium: In order to successfully pass "Theoretische Teilchenphysik II" we request you to

- visit at least 10 exercise classes (We collect signatures!).
- present parts of/one/two exercises at the blackboard (in total $\sim 45-60$ minutes).
- register by writing an email to "stefan.liebler@kit.edu" before 26.10.2018.

Exercise 1: SU(N) **representations**

The Lie groups SU(N) are of particular importance in particle physics as they for example form the gauge structures of the Weinberg-Salam model and quantum chromodynamics, which are associated with the Lie groups SU(2) and SU(3), respectively. Also SU(3) explains the flavour structure in the quark model (Eightfold Way). A representation of a group is a mapping on a set of matrices that fulfill the group axioms. A particularly important representation of SU(N) groups is the *fundamental* representation, an irreducible matrix representation of dimension N, which is formed by the generators

$$T^a_{ij}\,, \qquad\qquad a = 1, \ldots, N^2 - 1\,, \qquad\qquad i, j = 1, \ldots, N\,,$$

which are hermitian, i.e. $T^{a\dagger} = T^a$, and traceless, which implies $\mathrm{Tr}(T^a) = 0$. They are normalized through the relation $\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$. For $N = 2$ and 3 the generators of the *fundamental* representation are given by

- the Pauli matrices $T^a = \frac{\sigma^a}{2}$ for SU(2)
- the Gell-Mann matrices $T^a = \frac{\lambda^a}{2}$ for SU(3).

For the *fundamental* representation the generators fulfill the following commutation and anti-commutation relations

$$\left[T^a, T^b\right] = i f_{abc} T^c\,, \tag{1}$$

$$\left\{T^a, T^b\right\} = \frac{1}{N}\,\delta^{ab}\,\mathbb{1}_N + d_{abc} T^c\,, \tag{2}$$

which in turn defines the totally antisymmetric structure constants f_{abc} and the totally symmetric symbols d_{abc}. The commutation relation in Eq. (1) is actually valid for any representation of SU(N) and is the so-called Lie algebra of SU(N). [1] In the *fundamental* representation each

[1] Another matrix representation is given by the structure constants f_{abc} themselves, the *adjoint* representation.

complex $(N \times N)$-matrix M can be written in terms of the $N^2 - 1$ generators and the unit matrix in the form

$$M = c_0 \mathbb{1} + \sum_{a=1}^{N^2-1} c_a T^a \,. \tag{3}$$

(a) Show that the orthogonality relation in combination with Eq. (3) and the tracelessness of the generators in the *fundamental* representation yield the Fierz identity of SU(N) given by

$$T^a_{ij} T^a_{kl} \equiv \sum_{a=1}^{N^2-1} T^a_{ij} T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl} \,. \tag{4}$$

(b) Show that for any representation of SU(N)

$$C_2 = T^a T^a$$

with a sum over a is a Casimir invariant, i.e. $[C_2, T^a] = 0$ for all generators T^a.

(c) Show that the structure constants f_{abc} and d_{abc} are real by using the hermiticity of the generators of the *fundamental* representation.

(d) Show that the $(N \times N)$ matrices

$$\mathcal{T}^a = -T^{a*}$$

are also a representation of SU(N), i.e. they fulfill the Lie algebra given in Eq. (1). These $\mathcal{T}^a$ are the generators of the so-called $\bar{N}$ ("N–bar") representation of SU(N).
Hint: T^{a*} is the complex conjugated, not the hermitian matrix!

(e) Calculate the value of C_2 for the two N-dimensional representations, the (N) as well as the $(\bar{N})$ representation.

(f) Use the Jacobi identity for mixed commutation and anti-commutation relations

$$[A, \{B, C\}] + [B, \{C, A\}] + [C, \{A, B\}] = 0$$

to obtain a relation connecting f_{abc} and d_{abc}. *Hint:* Insert the generators.

(g) Use the previous result to show that

$$C_3 = d_{abc} T^a T^b T^c$$

is a Casimir invariant of SU(N).

(h) Use Eq. (4), to calculate the sum over a in

$$T^a T^b T^a \,.$$

(i) Determine the value of C_3 for both the (N) and the $(\bar{N})$ representation. The result shows, that (N) and $(\bar{N})$ are not equivalent representations for $N > 2$.[2]

[2] As indicated in the lecture those two representations are identified with quarks and anti-quarks, respectively.