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Exercise 1: One-loop calculation in ϕ^4 theory - Part 3

In Part 2 we deduced the renormalization constants δZ_λ and δZ_m in the $\overline{\text{MS}}$ renormalization scheme, which is e.g. often applied in QCD calculations. However, e.g. in electroweak theory the masses of the particles are measurable quantities, such that we want them to be constants. In a sloppy formulation we thus demand that one-loop corrections should vanish at physical masses for the self-energy corrections and similarly at a certain momentum for the scattering amplitude. This leads to the concept of on-shell renormalization.

(a) We obtained the correction $-iM(p^2)$ in Part 2 of this exercise. Combine this correction to the infinite series of propagator corrections

and use the geometric series to get the all-order loop-corrected propagator

$$\frac{i}{p^2 - m^2 - M(p^2)} \approx \frac{i}{p^2 - m^2}\left(1 + \frac{M(p^2)}{p^2 - m^2}\right),$$

where the latter expression corresponds to the pure one-loop result. Demand that the location of the pole and the residue of the propagator do not change between one-loop and tree-level through

$$\lim_{p^2 \to m^2} \frac{i}{p^2 - m^2}\left(1 + \frac{M(p^2)}{p^2 - m^2}\right) \stackrel{!}{=} \lim_{p^2 \to m^2} \frac{i}{p^2 - m^2}.$$

Perform a Taylor expansion of $M(p^2)$ around $p^2 = m^2$ and thus show $M(p^2)|_{p^2=m^2} = 0$ and $\frac{d^2}{dp^2} M(p^2)|_{p^2=m^2} = 0$. Show that the first equation leads to

$$\delta Z_m^{\text{OS}} = \frac{\lambda}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + 1 + \log\left(\frac{\mu^2}{m^2}\right)\right).$$

(b) Next we demand that the scattering amplitude fulfills $i\mathcal{M}(q_1 q_2 \to k_1 k_2) \stackrel{!}{=} -i\lambda$ for $q^2 = s = 4m^2$. Show that this equation leads to

$$\delta Z_\lambda^{\text{OS}} = i\lambda[iV(4m^2) + 2iV(0)].$$

Also provide the form of $i\mathcal{M}(q_1 q_2 \to k_1 k_2)$ for arbitrary q^2. Is there a dependence on the renormalization scale μ left in $i\mathcal{M}(q_1 q_2 \to k_1 k_2)$ for arbitrary q^2?
Hint: You don't have to perform the integration over the Feynman parameter.

(c) We know that the bare mass parameter m_0^2 is independent of the chosen scheme. We can thus obtain a relation between the $\overline{\text{MS}}$ mass and the on-shell mass, which yields

$$m_{\text{OS}}^2 = m_{\overline{\text{MS}}}^2(\mu) Z_m^{\overline{\text{MS}}} (Z_m^{\text{OS}})^{-1}\,.$$

Insert the expressions obtained for the renormalization constants and expand in λ to first order. You should obtain a finite expression, which for $\mu = m_{\overline{\text{MS}}}$ simplifies to

$$m_{\text{OS}}^2 = m_{\overline{\text{MS}}}^2(m_{\overline{\text{MS}}}) \left(1 - \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)\right)\,.$$

Thus, for the choice $\mu = m_{\overline{\text{MS}}}$ potentially large logarithms cancel out of the relation.

(d) We know that by definition the bare mass has no dependence on the renormalization scale. We want to show that also the on-shell mass is renormalization-scale independent, which we already anticipated in the previous subexercise. For this purpose consider

$$0 = \frac{d}{d\log\mu^2} m_0^2 = \frac{d}{d\log\mu^2}\left(Z_m^{\text{OS}} m_{\text{OS}}^2\right) = \left(\frac{dm_{\text{OS}}^2}{d\ln\mu^2}\right) Z_m^{\text{OS}} + m_{\text{OS}}^2 \left(\frac{\partial Z_m^{\text{OS}}}{\partial\log\mu^2} + \frac{\partial\lambda}{\partial\log\mu^2}\frac{\partial Z_m^{\text{OS}}}{\partial\lambda}\right)$$

and solve this expression for $\frac{dm_{\text{OS}}^2}{d\ln\mu^2}$ to first order in λ.
Hint: The β-function is given by $\beta(\lambda) = \frac{\partial\lambda}{\partial\log\mu^2} = -\epsilon\lambda + \frac{3}{16\pi^2}\lambda^2 + \mathcal{O}(\lambda^3)$.

Exercise 2: Higgs boson decay into bottom quarks with running quark mass

On sheet 9 we obtained the decay width of a Higgs boson into a pair of bottom quarks, being the dominant decay mode of the Standard Model Higgs boson below the W boson threshold, i.e. for $m_{h^0} < 2m_W$. For $m_{h^0} \gg m_b$ we cannot ignore the QCD evolution of the bottom-quark mass in the $\overline{\text{MS}}$ scheme. *Note*: $m_b(\mu)$ in this exercise is understood as $m_{b,\overline{\text{MS}}}(\mu)$.

(a) Calculate the running mass

$$m_b(t) = m_b(0)\exp\left\{\int_{g(1)}^{g(t)} \frac{\gamma_m(g)}{\beta(g)} dg\right\} \quad \text{with} \quad t = \log\mu$$

at the one-loop level with five active quarks, i.e. $n_f = 5$. The relevant mass counterterm for the bottom-quark discussed in the lecture is given by

$$\frac{\delta m_b}{m_b} = -\frac{4}{3}\frac{3g^2}{16\pi^2}\left(\frac{1}{\epsilon} + F_m\right)\,.$$

Hint: Explain $\gamma_m(g) = -\frac{g^2}{2\pi^2}$ and $\beta(g) = -\frac{g^3}{16\pi^2}\frac{23}{3}$. You should obtain

$$m_b(\mu) = m_b(\mu_0)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{12}{23}}\,.$$

(b) Calculate the bottom-quark mass $m_b(\mu_{h^0})$ at the scale $\mu_{h^0} = \frac{m_{h^0}}{2}$ starting from $m_b(m_b) = 4\,\text{GeV}$. For this purpose you need to get $\alpha_s(m_b)$ and $\alpha_s(\mu_{h^0})$ from $\alpha_s(m_Z = 91.2\,\text{GeV}) = 0.12$ according to the lecture. Insert the bottom-quark mass at a renormalization scale of μ_{h^0} into the tree-level partial decay width obtained on sheet 9 and compare the results.

We wish you a Merry Christmas, some relaxing days and a Happy New Year!
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