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Exercise 1: The generating functional $\Gamma[\phi]$

For n-point functions with $n \leq 3$ the vertex functions $\Gamma^{(n)}$ and the truncated Green functions $G^{(n)}_{\text{trunc}}$ are essentially the same, as shown in the lecture. The first nontrivial case is $n = 4$.

(a) Derive the relation between

$$\Gamma^{(4)}(x_1, x_2, x_3, x_4; \phi) = \frac{\delta^4 \Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)\delta\phi(x_4)}$$

and

$$G^{(4)}_{\text{trunc}}(y_1, y_2, y_3, y_4; j)$$

by functional differentation of the relevant generating functionals.

(b) For the example of the previous subexercise give a graphic representation of the relation with $\Gamma^{(n)}$'s represented by 'shaded blobs' and $G^{(n)}$'s represented by 'white blobs' as in the lecture.

Exercise 2: Dilogarithm

A special function, that very often appears in the calculation of loop diagrams, is the dilogarithm

$$\text{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} dt = -\int_0^1 \frac{\ln(1-zt)}{t} dt\,.$$

On the main branch, which is determined by the main branch of $\ln(z)$, the dilogarithm $\text{Li}_2(z)$ is a well-defined analytic function for $z \in \mathbb{C}$, except for $z > 1$ on the real axis. For $|z| < 1$ the integral representations can be rewritten in a series of the form

$$\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}\,.$$

From this series we can determine the value $\text{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$. To determine $\text{Li}_2(z)$ outside the unit circle, we need transformation rules for analytic continuation. For this purpose prove the following identities:

(a)
$$\text{Li}_2(1-z) = -\text{Li}_2(z) - \ln(1-z)\ln z + \frac{\pi^2}{6}\,.$$

(b)
$$\text{Li}_2\left(\frac{z}{z-1}\right) = -\text{Li}_2(z) - \frac{1}{2}\ln^2(1-z)\,.$$

(c)
$$\text{Li}_2\left(\frac{z-1}{z}\right) = \text{Li}_2(z) + \ln(1-z)\ln(z) - \frac{1}{2}\ln^2(z) - \frac{\pi^2}{6}\,.$$

(d)
$$\text{Li}_2\left(\frac{1}{z}\right) = -\text{Li}_2(z) - \frac{1}{2}\ln^2(-z) - \frac{\pi^2}{6}\,.$$

(e)
$$\text{Li}_2\left(\frac{1}{1-z}\right) = \text{Li}_2(z) - \frac{1}{2}\ln^2(1-z) + \ln(1-z)\ln(-z) + \frac{\pi^2}{6}\,.$$