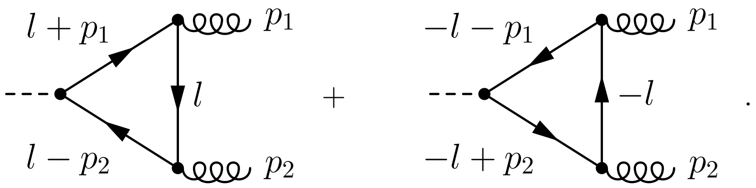
Sheet 12

Release: 16.01.19
Tutorial: 23.01.19

Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)
Shruti Patel (shruti.patel@kit.edu) (Office 12/14 - Build. 30.23)

Exercise 1: Higgs boson decay into gluons - Part 1

The aim of this exercise is to calculate the partial decay width of the Standard Model Higgs boson into a pair of gluons, $h^0 \to gg$, in the first non-vanishing order. The decay is loop-mediated, i.e. the Higgs boson couples to two gluons through quark loops. The quark running in the loop with mass m couples to the Higgs boson with the Yukawa coupling $y_q = \frac{m}{v}$ with the vacuum expectation value $v = 1/\sqrt{\sqrt{2}G_F}$. The relevant two Feynman diagrams for each quark, depicting the four-momenta, are given by



The two final-state gluons have outgoing momenta p_1 and p_2 as well as Lorentz indices μ and ν and colors a and b, respectively. Accordingly, the initial-state Higgs boson has momentum $p_1 + p_2$. We want all particles to be on-shell, i.e. $(p_1 + p_2)^2 = m_{h^0}^2$, $p_1^2 = p_2^2 = 0$. Since there are no tree-level diagrams and thus no counterterms, the final result of the loop diagrams cannot develop an ultraviolet divergence.

(a) Show that the amplitude in dimensional regularisation ($d = 4 - 2\epsilon$) involving one quark q with mass m is of the form

$$\mathcal{M}_q = \epsilon^*_{1,\mu}\epsilon^*_{2,\nu}(ig_s)^2(-iy_q)(-1)i^3\mu^{4-d}\int \frac{d^dl}{(2\pi)^d}\frac{T^a_{ij}T^b_{ji}\mathrm{Tr}[S^{\mu\nu}]}{(l^2 - m^2)((l + p_1)^2 - m^2)((l - p_2)^2 - m^2)}$$

with $S^{\mu\nu} = \gamma^\mu(\not{l} + \not{p}_1 + m)(\not{l} - \not{p}_2 + m)\gamma^\nu(\not{l} + m) + (-\not{l} + m)\gamma^\nu(-\not{l} + \not{p}_2 + m)(-\not{l} - \not{p}_1 + m)\gamma^\mu$.

(b) Show that $\mathrm{Tr}[S^{\mu\nu}] = 8m(g^{\mu\nu}(m^2 - l^2 - p_1 \cdot p_2) + 4l^\mu l^\nu + p_2^\mu p_1^\nu)$. Argue why the second term in $S^{\mu\nu}$ yields the same contribution as the first term.

(c) Introduce Feynman parameters in the form

$$\frac{1}{abc} = 2\int dxdydz \frac{\delta(1 - x - y - z)}{(xa + yb + zc)^3} = 2\int_0^1 dy \int_0^{1-y} dz \frac{1}{((1 - y - z)a + yb + zc)^3}$$

and shift the loop momentum l such that the denominator takes the form $(l^2 - (zp_2 - yp_1)^2 - m^2)^3 = (l^2 + yzm_{h^0}^2 - m^2)^3 =: (l^2 - M^2)^3$. Transform the numerator accordingly.

(d) Use the tensor integrals from sheet 7 to show that

$$\mathcal{M}_q = -\epsilon^*_{1,\mu}\epsilon^*_{2,\nu}g_s^2 y_q \frac{\delta^{ab}}{24\pi^2}\frac{m_{h^0}^2}{m}\epsilon^*_{1,\mu}\epsilon^*_{2,\nu}\left(g^{\mu\nu} - \frac{2}{m_{h^0}^2}p_1^\nu p_2^\mu\right) f\left(\frac{m_{h^0}^2}{m^2}\right)$$

with

$$f(x) = 3\int_0^1 dy \int_0^{1-y} dz \frac{1-4yz}{1-xyz}\,.$$

Hint: $g^{\mu\nu} I_d(0,2,M^2) + 4I_d^{\mu\nu}(0,3,M^2)$ might be a useful relation.

(e) Check gauge invariance explicitly by replacing the polarisation vector of each gluon through the corresponding momentum. Introduce a sum over different quarks $\mathcal{M} = \sum_q \mathcal{M}_q$ with masses m_q. Square the amplitude $\mathcal{M}$ and perform the polarisation sum over the gluon polarisation. You should obtain

$$|\mathcal{M}|^2 = \alpha_s^2 \sqrt{2} G_F \frac{4m_{h^0}^4}{9\pi^2} \left| \sum_q f\left(\frac{m_{h^0}^2}{m_q^2}\right) \right|^2 .$$

(f) Finally calculate the partial decay width, which is given by

$$\Gamma(h^0 \to gg) = \frac{\alpha_s^2 G_F m_{h^0}^3}{36\pi^3\sqrt{2}} \left| \sum_q f\left(\frac{m_{h^0}^2}{m_q^2}\right) \right|^2 .$$

Solution of exercise 1

(a) The form of the amplitude follows by writing down step by step the different propagators and couplings

$$\mathcal{M}_q = \epsilon_{1,\mu}^* \epsilon_{2,\nu}^* (ig_s)^2(-iy_q)(-1)i^3\mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{T_{ij}^a T_{ji}^b \mathrm{Tr}[S^{\mu\nu}]}{(l^2-m^2)((l+p_1)^2-m^2)((l-p_2)^2-m^2)}$$

with $S^{\mu\nu} = \gamma^\mu(\not{l}+\not{p}_1+m)(\not{l}-\not{p}_2+m)\gamma^\nu(\not{l}+m)+(-\not{l}+m)\gamma^\nu(-\not{l}+\not{p}_2+m)(-\not{l}-\not{p}_1+m)\gamma^\mu$. Some comments are on order: The fermionic propgators are $\frac{i(\not{p}+m)}{p^2-m^2}$. The factor of $(-1)i^3$ thus originates from the i of the propagators and (-1) for a fermion loop. The couplings of the quark to gluons are $ig_s\gamma^\mu T^a$ and the coupling of the quark to the Higgs boson is $-iy_q$. The color factor yields $T_{ij}^a T_{ji}^b = \frac{1}{2}\delta^{ab}$.

(b) The two terms of $\mathrm{Tr}[S^{\mu\nu}]$ yield identical results, since the second term can be rewritten as follows:

$$(-\not{l}+m)\gamma^\nu(-\not{l}+\not{p}_2+m)(-\not{l}-\not{p}_1+m)\gamma^\mu = -\gamma^\mu(\not{l}+\not{p}_1-m)(\not{l}-\not{p}_2-m)\gamma^\nu(\not{l}-m)\,.$$

In the trace only even numbers of γ matrices survive, which implies that only terms proportional to m or to m^3 remain. Those terms do however have the same sign as the terms in the first term of $S^{\mu\nu}$.

Add-on: If we were to replace all external legs with photons rather than having a Higgs boson and gluons, i.e. consider the three-photon vertex, then we would have another γ matrix from the corresponding vertex and we would be left with terms m^0 and m^2, but those have a different sign and cancel! This is Furry's theorem! It does not hold for three external gluons due to the color structure $T^aT^bT^c$ and $T^aT^cT^b$ for the two diagrams. We thus have to calculate

$$\begin{aligned}
\mathrm{Tr}[S^{\mu\nu}] &= 2m\mathrm{Tr}[\gamma^\mu(\not{l}+\not{p}_1)(\not{l}-\not{p}_2)\gamma^\nu + \gamma^\mu(\not{l}+\not{p}_1)\gamma^\nu\not{l} + \gamma^\mu(\not{l}-\not{p}_2)\gamma^\nu\not{l}] + 2m^3\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= 8m\,[(l^\mu+p_1^\mu)(l^\nu-p_2^\nu) + g^{\mu\nu}(l+p_1)\cdot(l-p_2) - (l^\mu-p_2^\mu)(l^\nu+p_1^\nu) \\
&\qquad + (l^\mu+p_1^\mu)l^\nu + l^\mu(l^\nu+p_1^\nu) - g^{\mu\nu}l\cdot(l+p_1) \\
&\qquad + (l^\mu-p_2^\mu)l^\nu + l^\mu(l^\nu-p_2^\nu) - g^{\mu\nu}l\cdot(l-p_2)] \\
&\quad + 8m^3g^{\mu\nu} \\
&= 8m\left[l^\mu l^\nu + p_1^\mu l^\nu - l^\mu p_2^\nu - p_1^\mu p_2^\nu + g^{\mu\nu}(l^2+(p_1-p_2)\cdot l - p_1\cdot p_2)\right. \\
&\qquad - l^\mu l^\nu - l^\mu p_1^\nu + p_2^\mu l^\nu + p_2^\mu p_1^\nu + 2l^\mu l^\nu + p_1^\mu l^\nu + l^\mu p_1^\nu - g^{\mu\nu}(l^2+p_1\cdot l) \\
&\qquad \left. +2l^\mu l^\nu - p_2^\mu l^\nu - l^\mu p_2^\nu - g^{\mu\nu}(l^2-p_2\cdot l)\right] + 8m^3g^{\mu\nu} \\
&= 8m[4l^\mu l^\nu + 2\underbrace{p_1^\mu}_{\to 0}l^\nu - 2l^\mu\underbrace{p_2^\nu}_{\to 0} - \underbrace{p_1^\mu p_2^\nu}_{\to 0} + p_2^\mu p_1^\nu + g^{\mu\nu}(-l^2-p_1\cdot p_2)] + 8m^3g^{\mu\nu}
\end{aligned}$$

Three terms in the last equation vanish due to $\epsilon_i\cdot p_i = 0$. We thus have

$$\mathrm{Tr}[S^{\mu\nu}] = 8m(g^{\mu\nu}(m^2-l^2-p_1\cdot p_2) + 4l^\mu l^\nu + p_2^\mu p_1^\nu)\,.$$

(c) The matrix element now reads

$$\mathcal{M}_q = i\epsilon_{1,\mu}^*\epsilon_{2,\nu}^*g_s^2y_q\frac{\delta^{ab}}{2}\underbrace{\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{\mathrm{Tr}[S^{\mu\nu}]}{(l^2-m^2)((l+p_1)^2-m^2)((l-p_2)^2-m^2)}}_{=J^{\mu\nu}}$$

For $J^{\mu\nu}$ we introduce Feynman parameters following the formulas on the exercise sheet and obtain

$$\begin{aligned}
J^{\mu\nu} =& 2\mu^{4-d}\int_0^1 dy\int_0^{1-y}dz \\
&\int\frac{d^dl}{(2\pi)^d}\frac{\mathrm{Tr}[S^{\mu\nu}]}{[y((l+p_1)^2-m^2)+z((l-p_2)^2-m^2)+(1-y-z)(l^2-m^2)]^3}\,.
\end{aligned}$$

We transform the denominator further, which results in

$$\begin{aligned}
&y((l+p_1)^2-m^2)+z((l-p_2)^2-m^2)+(1-y-z)(l^2-m^2) \\
&= y(l^2+2l\cdot p_1+p_1^2-m^2)+z(l^2-2l\cdot p_2+p_2^2-m^2)+(1-z-y)(l^2-m^2) \\
&= l^2-2l\cdot(zp_2-yp_1)-m^2
\end{aligned}$$

We now shift the loop momentum by $l\to l+(zp_2-yp_1)$ and then obtain for the denominator

$$\begin{aligned}
&l^2+2l\cdot(zp_2-yp_1)+(zp_2-yp_1)^2-2l\cdot(zp_2-yp_1)-2(zp_2-yp_1)^2-m^2 \\
&= l^2-(zp_2-yp_1)^2-m^2 = l^2+2zyp_2\cdot p_1-m^2 = l^2+zym_{h^0}^2-m^2 = l^2-M^2\,.
\end{aligned}$$

Here we used $m_{h^0}^2 = (p_1+p_2)^2 = p_1^2+2p_1\cdot p_2+p_2^2 = 2p_1\cdot p_2$ and defined $M^2 = m^2-yzm_{h^0}^2$. We now shift the numerator equally, which results in

$$\begin{aligned}
\mathrm{Tr}[S^{\mu\nu}] =& 8m\left[g^{\mu\nu}(m^2-l^2-2l\cdot(zp_2-yp_1)-(zp_2-yp_1)^2-p_1\cdot p_2)\right. \\
&\left. +4l^\mu l^\nu + 4(zp_2^\mu-yp_1^\mu)(zp_2^\nu-yp_1^\nu)+p_2^\mu p_1^\nu\right] \\
=& 8m\left[g^{\mu\nu}(m^2-l^2+yzm_{h^0}^2-\tfrac{m_{h^0}^2}{2})+4l^\mu l^\nu-4yzp_1^\nu p_2^\mu+p_1^\nu p_2^\mu\right]\,.
\end{aligned}$$

(d) We combine the previous results and rewrite

$$\begin{aligned}
J^{\mu\nu} =& -16m\mu^{4-d}\int_0^1 dy \int_0^{1-y} dz \\
& \int \frac{d^dl}{(2\pi)^d}\frac{g^{\mu\nu}(-l^2+M^2+2yzm_{h^0}^2-\frac{m_{h^0}^2}{2})+4l^\mu l^\nu - p_1^\nu p_2^\mu(4yz-1)}{(-l^2+M^2)^3} \\
=& -16m\mu^{4-d}\int_0^1 dy \int_0^{1-y} dz \\
& \int \frac{d^dl}{(2\pi)^d}\left[\frac{g^{\mu\nu}}{(-l^2+M^2)^2}+\frac{(\frac{1}{2}m_{h^0}^2 g^{\mu\nu}-p_1^\nu p_2^\mu)(4yz-1)+4l^\mu l^\nu}{(-l^2+M^2)^3}\right]
\end{aligned}$$

We can now map all terms to the various tensor integrals discussed on sheet 7. Remember the definition

$$I_d(q,a,M^2) = \mu^{4-d}\int \frac{d^dp}{(2\pi)^d}\frac{1}{[-p^2+2p\cdot q+M^2]^a}$$

$I_d^\mu(q,a,M^2)$ and $I_d^{\mu\nu}(q,a,M^2)$ have numerators p^μ and $p^\mu p^\nu$ instead. We need the integrals $I_d(0,3,M^2)$ and $I_d^{\mu\nu}(0,3,M^2)$ and $I_d(0,2,M^2)$ and obtain

$$\begin{aligned}
J^{\mu\nu} = -16m\int_0^1 dy\int_0^{1-y} dz \,\big[& g^{\mu\nu}I_d(0,2,M^2)+(\tfrac{1}{2}m_{h^0}^2 g^{\mu\nu}-p_1^\nu p_2^\mu)(4yz-1)I_d(0,3,M^2) \\
& +4I_d^{\mu\nu}(0,3,M^2)\big]
\end{aligned}$$

For the three integrals we perform an expansion in small ϵ and get

$$\begin{aligned}
I_d(0,2,M^2) &= \frac{i}{16\pi^2}(4\pi\mu^2)^\epsilon\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{1}{(M^2)^\epsilon} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\log(4\pi)\right)-\frac{i}{16\pi^2}\log\left(\frac{M^2}{\mu^2}\right) \\
I_d(0,3,M^2) &= \frac{i}{16\pi^2}(4\pi\mu^2)^\epsilon\frac{\Gamma(1+\epsilon)}{\Gamma(3)}\frac{1}{(M^2)^{1+\epsilon}} = \frac{i}{32\pi^2M^2} \\
I_d^{\mu\nu}(0,3,M^2) &= \frac{i}{16\pi^2}(4\pi\mu^2)^\epsilon\frac{\Gamma(1+\epsilon)}{\Gamma(3)}\frac{1}{(M^2)^\epsilon}\left(-\frac{1}{2\epsilon}g^{\mu\nu}\right) \\
&= \left[-\frac{i}{64\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\log(4\pi)\right)+\frac{i}{64\pi^2}\log\left(\frac{M^2}{\mu^2}\right)\right]g^{\mu\nu}
\end{aligned}$$

We note that $g^{\mu\nu}I_d(0,2,M^2)+4I_d^{\mu\nu}(0,3,M^2) = 0$, which implies that the result is ultraviolet finite, and are left with

$$\begin{aligned}
J^{\mu\nu} &= -16m\int_0^1 dy\int_0^{1-y} dz\frac{m_{h^0}^2}{2}\left(g^{\mu\nu}-\frac{2}{m_{h^0}^2}p_1^\nu p_2^\mu\right)(4yz-1)\frac{i}{32\pi^2(m^2-yzm_{h^0}^2)} \\
&= i\frac{1}{2\pi^2}\frac{m_{h^0}^2}{2m}\int_0^1 dy\int_0^{1-y} dz\left(g^{\mu\nu}-\frac{2}{m_{h^0}^2}p_1^\nu p_2^\mu\right)\frac{1-4yz}{1-yzm_{h^0}^2/m^2}
\end{aligned}$$

The matrix element is thus given by

$$\mathcal{M}_q = -\epsilon_{1,\mu}^*\epsilon_{2,\nu}^* g_s^2 y_q\delta^{ab}\frac{1}{24\pi^2}\frac{m_{h^0}^2}{m}\left(g^{\mu\nu}-\frac{2}{m_{h^0}^2}p_1^\nu p_2^\mu\right)3\int_0^1 dy\int_0^{1-y} dz\frac{1-4yz}{1-yzm_{h^0}^2/m^2}\,.$$

Identifiying $f(x)$ this equals the result given on the exercise sheet.

(e) We consider the expression

$$\left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu\right) \epsilon_{1,\mu}^* \epsilon_{2,\nu}^* \,.$$

We replace the polarisation vectors with the corresponding momenta and obtain

$$\begin{aligned} \epsilon_{1,\mu}^* \to p_{1,\mu} : \qquad & \left(p_1^\nu - \frac{2}{m_{h^0}^2} p_1^\nu (p_1 \cdot p_2)\right) = 0 \\ \epsilon_{2,\nu}^* \to p_{2,\nu} : \qquad & \left(p_2^\mu - \frac{2}{m_{h^0}^2} p_2^\mu (p_1 \cdot p_2)\right) = 0 \,. \end{aligned}$$

We thus explicitly checked gauge invariance and can use the simplifed polarisation sum $\sum_\lambda \epsilon_\mu^* \epsilon_\nu = -g_{\mu\nu}$. We add the sum over various quarks $\mathcal{M} = \sum_q \mathcal{M}_q$ and get

$$\mathcal{M} = -\epsilon_{1,\mu}^* \epsilon_{2,\nu}^* g_s^2 \delta^{ab} \frac{1}{24\pi^2} \frac{m_{h^0}^2}{v} \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu\right) \sum_q f\left(\frac{m_{h^0}^2}{m_q^2}\right) \,.$$

We can square the expression and obtain perform the polarisation sum

$$\begin{aligned} |\mathcal{M}|^2 =& \frac{g_s^4}{(24\pi^2)^2} \frac{m_{h^0}^4}{v^2} \delta^{aa} \left|\sum_q f\left(\frac{m_{h^0}^2}{m_q^2}\right)\right|^2 \\ & \times \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu\right) \left(g^{\rho\sigma} - \frac{2}{m_{h^0}^2} p_1^\sigma p_2^\rho\right) g_{\mu\rho} g_{\nu\sigma} \,. \end{aligned}$$

The last line equals

$$g_\mu^\mu - \frac{2}{m_{h^0}^2} p_1 \cdot p_2 - \frac{2}{m_{h^0}^2} p_1 \cdot p_2 + \frac{4}{m_{h^0}^4} p_1^2 p_2^2 = 4 - 1 - 1 + 0 = 2 \,.$$

We note that $\delta^{aa} = 8$ and also replace $g_s^2 = 4\pi\alpha_s$ and get the result

$$|\mathcal{M}|^2 = \alpha_s^2 \sqrt{2} G_F \frac{4 m_{h^0}^4}{9\pi^2} \left|\sum_q f\left(\frac{m_{h^0}^2}{m_q^2}\right)\right|^2 \,.$$

(f) We finally need the phase space of the decay, which we can copy from sheet 9, such that

$$\Gamma = \frac{1}{16\pi m_{h^0}} |\mathcal{M}|^2$$

We need a symmetry factor $\frac{1}{2}$ due to the two identical gluons and thus obtain

$$\Gamma(h^0 \to gg) = \frac{\alpha_s^2 G_F m_{h^0}^3}{36\pi^3\sqrt{2}} \left|\sum_q f\left(\frac{m_{h^0}^2}{m_q^2}\right)\right|^2 \,.$$

Exercise 2: Higgs boson decay into gluons - Part 2

We continue with the previous exercise. *Hint:* This exercise can be performed independently. All relevant results are given in the previous exercise.

(a) Perform the integrations in the definition of $f(x)$ by using the results obtained for the dilogarithm on sheet 11. *Hint:* Perform the integration over z and determine the roots of the argument of the remaining logarithm named $y_\pm$, such that $1 - xy + xy^2 = (y_+ - y)(y_- - y)x$. Split the logarithm, use sheet 11 and use

$$\arcsin(z) = -i \ln(iz \pm \sqrt{1-z^2}) = -i \ln\left[\left(\frac{\sqrt{z^2-1}+z}{\sqrt{z^2-1}-z}\right)^2\right] .$$

If you succeed, you should get for $x < 4$

$$f(x) = \frac{6}{x} - \frac{6(4-x)}{x^2} \arcsin^2\left(\frac{\sqrt{x}}{2}\right) .$$

(b) What value does $f(x)$ take for heavy quarks (heavy-top limit), i.e. $m_q \to \infty$ and $m_{h^0}^2/m_q^2 \to 0$? Why does the measurement of the decay of a Higgs boson to gluons or the production of a Higgs boson from gluons allow to make a statement on the number of heavy quark generations?

(c) If you still don't have enough, try to obtain the squared amplitude using a computer, e.g. with `FeynArts` and `FormCalc`. You might encounter the Passarino-Veltman representation of the loop integrals, for which computer codes exist that allow their numerical evaluation.

Solution of exercise 2

(a) We proceed with the calculation of $f(x)$, for which we obtain

$$\begin{aligned}
f(x) &= 3\int_0^1 dy \int_0^{1-y} dz \frac{1-4yz}{1-xyz} \\
&= 3\int_0^1 dy \left[\frac{-4yz}{-xy} + \frac{-xy+4y}{x^2y^2}\ln(1-xzy)\right]_0^{1-y} \\
&= 3\int_0^1 dy \left[\frac{4z}{x} + \frac{4-x}{x^2y}\ln(1-xyz)\right]_0^{1-y} \\
&= 3\int_0^1 dy \left[\frac{4(1-y)}{x} + \frac{4-x}{x^2y}\ln(1-xy+xy^2)\right] \\
&= \frac{12}{x}\left[-\frac{1}{2}(1-y)^2\right]_0^1 + \frac{3(4-x)}{x^2}\int_0^1 dy \frac{\ln(1-xy+xy^2)}{y} \\
&= \frac{6}{x} + \frac{3(4-x)}{x^2}\underbrace{\int_0^1 dy \frac{\ln(1-xy+xy^2)}{y}}_{=:J} .
\end{aligned}$$

In order to calculate J, we determine the roots of the argument of the logarithms, which results in

$$1 - xy + xy^2 = 0 \quad \to \quad y_\pm = \frac{x \pm \sqrt{x^2-4x}}{2x} = \frac{1}{2}\left(1 \pm \sqrt{1-\tfrac{4}{x}}\right) .$$

We note that $y_+y_- = \frac{1}{4}(1-(1-\frac{4}{x})) = \frac{1}{x}$ and $y_- = 1-y_+$. Thus we can rewrite $1-xy+xy^2 = (y_+-y)(y_--y)x$. This results in:

$$\begin{aligned}
J &= \int_0^1 dy \frac{\ln((y_+-y)(y_--y)x)}{y} = \int_0^1 dy \frac{\ln(xy_+y_-) + \ln(1-\frac{y_+}{y}) + \ln(1-\frac{y_-}{y})}{y} \\
&= 0 - \mathrm{Li}_2\left(\frac{1}{y_+}\right) - \mathrm{Li}_2\left(\frac{1}{y_-}\right) = -\mathrm{Li}_2\left(\frac{1}{y_+}\right) - \mathrm{Li}_2\left(\frac{1}{1-y_+}\right) \\
&= \mathrm{Li}_2(y_+) + \frac{1}{2}\ln^2(-y_+) + \frac{\pi^2}{6} - \mathrm{Li}_2(y_+) + \frac{1}{2}\ln^2(1-y_+) - \ln(-y_+)\ln(1-y_+) - \frac{\pi^2}{6} \\
&= \frac{1}{2}\left(\ln^2(-y_+) - 2\ln(-y_+)\ln(y_-) + \ln^2(y_-)\right) \\
&= \frac{1}{2}\left(\ln(-y_+) - \ln(y_-)\right)^2 = \frac{1}{2}\ln^2\left(-\frac{y_+}{y_-}\right) \\
&= \frac{1}{2}\ln^2\left(\frac{\sqrt{1-\frac{4}{x}}+1}{\sqrt{1-\frac{4}{x}}-1}\right)
\end{aligned}$$

In the last step we need

$$\arcsin(z) = -i\ln\left(iz \pm \sqrt{1-z^2}\right) \quad \text{and} \quad \arcsin^2(z) = -\ln^2\left(iz \pm \sqrt{1-z^2}\right)$$

and then transform the previous expression as follows:

$$J = -2\left(\frac{(-i)^2}{4}\ln^2\left(\frac{(1+\sqrt{1-\frac{4}{x}})}{1-\frac{4}{x}-1}\right)\right) = -2\left(-i\ln\sqrt{\frac{(1+\sqrt{1-\frac{4}{x}})^2}{-\frac{4}{x}}}\right)^2 .$$

We can transform

$$\sqrt{\frac{(1+\sqrt{1-\frac{4}{x}})^2}{-\frac{4}{x}}} = \frac{i\sqrt{x}}{2}\left(1+\sqrt{1-\frac{4}{x}}\right) = i\frac{\sqrt{x}}{2} + i\sqrt{\frac{x}{4}-1} = i\frac{\sqrt{x}}{2} - \sqrt{1-\left(\frac{\sqrt{x}}{2}\right)^2} .$$

We thus have $J = -2\arcsin^2\left(\frac{\sqrt{x}}{2}\right)$. Alternatively one may use the second expression for $\arcsin(z)$ on the exercise sheet and write

$$J = \frac{1}{2}\ln^2\left(\frac{\sqrt{\frac{x}{4}-1}+\frac{\sqrt{x}}{2}}{\sqrt{\frac{x}{4}-1}+\frac{\sqrt{x}}{2}}\right) = -2\left(-i\ln\sqrt{\frac{\sqrt{\frac{x}{4}-1}+\frac{\sqrt{x}}{2}}{\sqrt{\frac{x}{4}-1}+\frac{\sqrt{x}}{2}}}\right)^2 = -2\arcsin^2\left(\frac{\sqrt{2}}{x}\right) .$$

When using the real version of $\arcsin(x)$ we have to restrict ourselves to the region of $x < 4$, which is the region where $f(x)$ does not develop an imaginary part and is of the form

$$f(x) = \frac{6}{x} - \frac{6(4-x)}{x^2}\arcsin^2\left(\frac{\sqrt{x}}{2}\right) .$$

For $x > 4$ on the other hand, the loop particles can be on-shell, an $f(x)$ has an imaginary part. We refrain from depicting the result.

(b) For large quark masses we have to consider the limit $x \to 0$. We expand the $\arcsin(z)$ as follows

$$\arcsin(z) \approx z + \frac{z^3}{6} + \mathcal{O}(z^5), \qquad \arcsin^2(z) \approx z^2 + \tfrac{z^4}{3} + \mathcal{O}(z^6)\,.$$

We insert the result in $f(x)$ and get

$$f(x) \approx \frac{6}{x} - \frac{6(4-x)}{x^2}\left(\frac{x}{4} + \frac{x^2}{3\cdot 16} + \dots\right) = \frac{6}{x} - \frac{6}{x} + \frac{3}{2} - \frac{24}{48} + \dots = 1 + \dots$$

We conclude that the contribution of a heavy quark is independent from the quark mass! This implies that in addition to the top-quark another fourth generation involving a heavy bottom and a heavy top would change the decay width or cross section by a factor of $|1+1+1|^2 = 9$! Thus, the measurement of the gluon fusion cross section to be compatible with the SM expectation allows to exclude heavy quark generations.

(c) This solution is not depicted here.