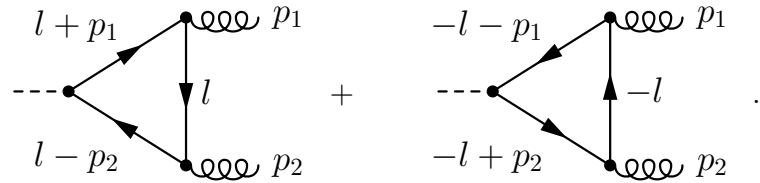


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Exercise 1: Higgs boson decay into gluons - Part 1

The aim of this exercise is to calculate the partial decay width of the Standard Model Higgs boson into a pair of gluons, $h^0 \rightarrow gg$, in the first non-vanishing order. The decay is loop-mediated, i.e. the Higgs boson couples to two gluons through quark loops. The quark running in the loop with mass m couples to the Higgs boson with the Yukawa coupling $y_q = \frac{m}{v}$ with the vacuum expectation value $v = 1/\sqrt{\sqrt{2}G_F}$. The relevant two Feynman diagrams for each quark, depicting the four-momenta, are given by



The two final-state gluons have outgoing momenta p_1 and p_2 as well as Lorentz indices μ and ν and colors a and b , respectively. Accordingly, the initial-state Higgs boson has momentum $p_1 + p_2$. We want all particles to be on-shell, i.e. $(p_1 + p_2)^2 = m_{h^0}^2$, $p_1^2 = p_2^2 = 0$. Since there are no tree-level diagrams and thus no counterterms, the final result of the loop diagrams cannot develop an ultraviolet divergence.

- (a) Show that the amplitude in dimensional regularisation ($d = 4 - 2\epsilon$) involving one quark q with mass m is of the form

$$\mathcal{M}_q = \epsilon_{1,\mu}^* \epsilon_{2,\nu}^* (ig_s)^2 (-iy_q) (-1) i^3 \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{T_{ij}^a T_{ji}^b \text{Tr}[S^{\mu\nu}]}{(l^2 - m^2)((l + p_1)^2 - m^2)((l - p_2)^2 - m^2)}$$

with $S^{\mu\nu} = \gamma^\mu (\not{l} + \not{p}_1 + m)(\not{l} - \not{p}_2 + m)\gamma^\nu (\not{l} + m) + (-\not{l} + m)\gamma^\nu (-\not{l} + \not{p}_2 + m)(-\not{l} - \not{p}_1 + m)\gamma^\mu$.

- (b) Show that $\text{Tr}[S^{\mu\nu}] = 8m(g^{\mu\nu}(m^2 - l^2 - p_1 \cdot p_2) + 4l^\mu l^\nu + p_2^\mu p_1^\nu)$. Argue why the second term in $S^{\mu\nu}$ yields the same contribution as the first term.
- (c) Introduce Feynman parameters in the form

$$\frac{1}{abc} = 2 \int dx dy dz \frac{\delta(1-x-y-z)}{(xa+yb+zc)^3} = 2 \int_0^1 dy \int_0^{1-y} dz \frac{1}{((1-y-z)a+yb+zc)^3}$$

and shift the loop momentum l such that the denominator takes the form $(l^2 - (zp_2 - yp_1)^2 - m^2)^3 = (l^2 + yzm_{h^0}^2 - m^2)^3 =: (l^2 - M^2)^3$. Transform the numerator accordingly.

- (d) Use the tensor integrals from sheet 7 to show that

$$\mathcal{M}_q = -\epsilon_{1,\mu}^* \epsilon_{2,\nu}^* g_s^2 y_q \frac{\delta^{ab}}{24\pi^2} \frac{m_{h^0}^2}{m} \epsilon_{1,\mu}^* \epsilon_{2,\nu}^* \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu \right) f \left(\frac{m_{h^0}^2}{m^2} \right)$$

with

$$f(x) = 3 \int_0^1 dy \int_0^{1-y} dz \frac{1 - 4yz}{1 - xyz}.$$

Hint: $g^{\mu\nu} I_d(0, 2, M^2) + 4I_d^{\mu\nu}(0, 3, M^2)$ might be a useful relation.

- (e) Check gauge invariance explicitly by replacing the polarisation vector of each gluon through the corresponding momentum. Introduce a sum over different quarks $\mathcal{M} = \sum_q \mathcal{M}_q$ with masses m_q . Square the amplitude \mathcal{M} and perform the polarisation sum over the gluon polarisation. You should obtain

$$|\mathcal{M}|^2 = \alpha_s^2 \sqrt{2} G_F \frac{4m_{h^0}^4}{9\pi^2} \left| \sum_q f \left(\frac{m_{h^0}^2}{m_q^2} \right) \right|^2.$$

- (f) Finally calculate the partial decay width, which is given by

$$\Gamma(h^0 \rightarrow gg) = \frac{\alpha_s^2 G_F m_{h^0}^3}{36\pi^3 \sqrt{2}} \left| \sum_q f \left(\frac{m_{h^0}^2}{m_q^2} \right) \right|^2.$$

Solution of exercise 1

- (a) The form of the amplitude follows by writing down step by step the different propagators and couplings

$$\mathcal{M}_q = \epsilon_{1,\mu}^* \epsilon_{2,\nu}^* (ig_s)^2 (-iy_q) (-1) i^3 \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{T_{ij}^a T_{ji}^b \text{Tr}[S^{\mu\nu}]}{(l^2 - m^2)((l + p_1)^2 - m^2)((l - p_2)^2 - m^2)}$$

with $S^{\mu\nu} = \gamma^\mu (\not{l} + \not{p}_1 + m)(\not{l} - \not{p}_2 + m)\gamma^\nu (\not{l} + m) + (-\not{l} + m)\gamma^\nu (-\not{l} + \not{p}_2 + m)(-\not{l} - \not{p}_1 + m)\gamma^\mu$. Some comments are on order: The fermionic propagators are $\frac{i(\not{p} + m)}{p^2 - m^2}$. The factor of $(-1)i^3$ thus originates from the i of the propagators and (-1) for a fermion loop. The couplings of the quark to gluons are $ig_s \gamma^\mu T^a$ and the coupling of the quark to the Higgs boson is $-iy_q$. The color factor yields $T_{ij}^a T_{ji}^b = \frac{1}{2} \delta^{ab}$.

- (b) The two terms of $\text{Tr}[S^{\mu\nu}]$ yield identical results, since the second term can be rewritten as follows:

$$(-\not{l} + m)\gamma^\nu (-\not{l} + \not{p}_2 + m)(-\not{l} - \not{p}_1 + m)\gamma^\mu = -\gamma^\mu (\not{l} + \not{p}_1 - m)(\not{l} - \not{p}_2 - m)\gamma^\nu (\not{l} - m).$$

In the trace only even numbers of γ matrices survive, which implies that only terms proportional to m or to m^3 remain. Those terms do however have the same sign as the terms in the first term of $S^{\mu\nu}$.

Add-on: If we were to add a gluon rather than a Higgs boson externally, i.e. consider the three gluon vertex, then we would have another γ matrix from the corresponding vertex and we would be left with terms m^0 and m^2 , but those have a different sign and cancel!

This is Furry's theorem!

We thus have to calculate

$$\begin{aligned}
\text{Tr}[S^{\mu\nu}] &= 2m\text{Tr}[\gamma^\mu(\not{l} + \not{p}_1)(\not{l} - \not{p}_2)\gamma^\nu + \gamma^\mu(\not{l} + \not{p}_1)\gamma^\nu\not{l} + \gamma^\mu(\not{l} - \not{p}_2)\gamma^\nu\not{l}] + 2m^3\text{Tr}[\gamma^\mu\gamma^\nu] \\
&= 8m [(l^\mu + p_1^\mu)(l^\nu - p_2^\nu) + g^{\mu\nu}(l + p_1) \cdot (l - p_2) - (l^\mu - p_2^\mu)(l^\nu + p_1^\nu) \\
&\quad + (l^\mu + p_1^\mu)l^\nu + l^\mu(l^\nu + p_1^\nu) - g^{\mu\nu}l \cdot (l + p_1) \\
&\quad + (l^\mu - p_2^\mu)l^\nu + l^\mu(l^\nu - p_2^\nu) - g^{\mu\nu}l \cdot (l - p_2)] \\
&\quad + 8m^3g^{\mu\nu} \\
&= 8m [l^\mu l^\nu + p_1^\mu l^\nu - l^\mu p_2^\nu - p_1^\mu p_2^\nu + g^{\mu\nu}(l^2 + (p_1 - p_2) \cdot l - p_1 \cdot p_2) \\
&\quad - l^\mu l^\nu - l^\mu p_1^\nu + p_2^\mu l^\nu + p_2^\mu p_1^\nu + 2l^\mu l^\nu + p_1^\mu l^\nu + l^\mu p_1^\nu - g^{\mu\nu}(l^2 + p_1 \cdot l) \\
&\quad + 2l^\mu l^\nu - p_2^\mu l^\nu - l^\mu p_2^\nu - g^{\mu\nu}(l^2 - p_2 \cdot l)] + 8m^3g^{\mu\nu} \\
&= 8m[4l^\mu l^\nu + 2 \underbrace{p_1^\mu}_{\rightarrow 0} l^\nu - 2l^\mu \underbrace{p_2^\nu}_{\rightarrow 0} - \underbrace{p_1^\mu p_2^\nu}_{\rightarrow 0} + p_2^\mu p_1^\nu + g^{\mu\nu}(-l^2 - p_1 \cdot p_2)] + 8m^3g^{\mu\nu}
\end{aligned}$$

Three terms in the last equation vanish due to $\epsilon_i \cdot p_i = 0$. We thus have

$$\text{Tr}[S^{\mu\nu}] = 8m(g^{\mu\nu}(m^2 - l^2 - p_1 \cdot p_2) + 4l^\mu l^\nu + p_2^\mu p_1^\nu).$$

(c) The matrix element now reads

$$\mathcal{M}_q = i\epsilon_{1,\mu}^* \epsilon_{2,\nu}^* g_s^2 y_q \frac{\delta^{ab}}{2} \underbrace{\mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{\text{Tr}[S^{\mu\nu}]}{(l^2 - m^2)((l + p_1)^2 - m^2)((l - p_2)^2 - m^2)}}_{=J^{\mu\nu}}$$

For $J^{\mu\nu}$ we introduce Feynman parameters following the formulas on the exercise sheet and obtain

$$\begin{aligned}
J^{\mu\nu} &= 2\mu^{4-d} \int_0^1 dy \int_0^{1-y} dz \\
&\quad \int \frac{d^d l}{(2\pi)^d} \frac{\text{Tr}[S^{\mu\nu}]}{[y((l + p_1)^2 - m^2) + z((l - p_2)^2 - m^2) + (1 - y - z)(l^2 - m^2)]^3}.
\end{aligned}$$

We transform the denominator further, which results in

$$\begin{aligned}
&y((l + p_1)^2 - m^2) + z((l - p_2)^2 - m^2) + (1 - y - z)(l^2 - m^2) \\
&= y(l^2 + 2l \cdot p_1 + p_1^2 - m^2) + z(l^2 - 2l \cdot p_2 + p_2^2 - m^2) + (1 - z - y)(l^2 - m^2) \\
&= l^2 - 2l \cdot (zp_2 - yp_1) - m^2
\end{aligned}$$

We now shift the loop momentum by $l \rightarrow l + (zp_2 - yp_1)$ and then obtain for the denominator

$$\begin{aligned}
&l^2 + 2l \cdot (zp_2 - yp_1) + (zp_2 - yp_1)^2 - 2l \cdot (zp_2 - yp_1) - 2(zp_2 - yp_1)^2 - m^2 \\
&= l^2 - (zp_2 - yp_1)^2 - m^2 = l^2 + 2zy p_2 \cdot p_1 - m^2 = l^2 + zy m_{h^0}^2 - m^2 = l^2 - M^2.
\end{aligned}$$

Here we used $m_{h^0}^2 = (p_1 + p_2)^2 = p_1^2 + 2p_1 \cdot p_2 + p_2^2 = 2p_1 \cdot p_2$ and defined $M^2 = m^2 - yzm_{h^0}^2$.

We now shift the numerator equally, which results in

$$\begin{aligned}
\text{Tr}[S^{\mu\nu}] &= 8m [g^{\mu\nu}(m^2 - l^2 - 2l \cdot (zp_2 - yp_1) - (zp_2 - yp_1)^2 - p_1 \cdot p_2) \\
&\quad + 4l^\mu l^\nu + 4(zp_2^\mu - yp_1^\mu)(zp_2^\nu - yp_1^\nu) + p_2^\mu p_1^\nu] \\
&= 8m \left[g^{\mu\nu}(m^2 - l^2 + yzm_{h^0}^2 - \frac{m_{h^0}^2}{2}) + 4l^\mu l^\nu - 4yzp_1^\nu p_2^\mu + p_1^\nu p_2^\mu \right].
\end{aligned}$$

(d) We combine the previous results and rewrite

$$\begin{aligned}
J^{\mu\nu} &= -16m\mu^{4-d} \int_0^1 dy \int_0^{1-y} dz \\
&\int \frac{d^d l}{(2\pi)^d} \frac{g^{\mu\nu}(-l^2 + M^2 + 2yzm_{h^0}^2 - \frac{m_{h^0}^2}{2}) + 4l^\mu l^\nu - p_1^\nu p_2^\mu (4yz - 1)}{(-l^2 + M^2)^3} \\
&= -16m\mu^{4-d} \int_0^1 dy \int_0^{1-y} dz \\
&\int \frac{d^d l}{(2\pi)^d} \left[\frac{g^{\mu\nu}}{(-l^2 + M^2)^2} + \frac{(\frac{1}{2}m_{h^0}^2 g^{\mu\nu} - p_1^\nu p_2^\mu)(4yz - 1) + 4l^\mu l^\nu}{(-l^2 + M^2)^3} \right]
\end{aligned}$$

We can now map all terms to the various tensor integrals discussed on sheet 7. Remember the definition

$$I_d(q, a, M^2) = \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{[-p^2 + 2p \cdot q + M^2]^a}$$

$I_d^\mu(q, a, M^2)$ and $I_d^{\mu\nu}(q, a, M^2)$ have numerators p^μ and $p^\mu p^\nu$ instead. We need the integrals $I_d(0, 3, M^2)$ and $I_d^{\mu\nu}(0, 3, M^2)$ and $I_d(0, 2, M^2)$ and obtain

$$\begin{aligned}
J^{\mu\nu} &= -16m \int_0^1 dy \int_0^{1-y} dz [g^{\mu\nu} I_d(0, 2, M^2) + (\frac{1}{2}m_{h^0}^2 g^{\mu\nu} - p_1^\nu p_2^\mu)(4yz - 1) I_d(0, 3, M^2) \\
&\quad + 4I_d^{\mu\nu}(0, 3, M^2)]
\end{aligned}$$

For the three integrals we perform an expansion in small ϵ and get

$$\begin{aligned}
I_d(0, 2, M^2) &= \frac{i}{16\pi^2} (4\pi\mu^2)^\epsilon \frac{\Gamma(\epsilon)}{\Gamma(2)} \frac{1}{(M^2)^\epsilon} = \frac{i}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) \right) - \frac{i}{16\pi^2} \log \left(\frac{M^2}{\mu^2} \right) \\
I_d(0, 3, M^2) &= \frac{i}{16\pi^2} (4\pi\mu^2)^\epsilon \frac{\Gamma(1+\epsilon)}{\Gamma(3)} \frac{1}{(M^2)^{1+\epsilon}} = \frac{i}{32\pi^2 M^2} \\
I_d^{\mu\nu}(0, 3, M^2) &= \frac{i}{16\pi^2} (4\pi\mu^2)^\epsilon \frac{\Gamma(1+\epsilon)}{\Gamma(3)} \frac{1}{(M^2)^\epsilon} \left(-\frac{1}{2\epsilon} g^{\mu\nu} \right) \\
&= \left[-\frac{i}{64\pi^2} \left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) \right) + \frac{i}{64\pi^2} \log \left(\frac{M^2}{\mu^2} \right) \right] g^{\mu\nu}
\end{aligned}$$

We note that $g^{\mu\nu} I_d(0, 2, M^2) + 4I_d^{\mu\nu}(0, 3, M^2) = 0$, which implies that the result is ultraviolet finite, and are left with

$$\begin{aligned}
J^{\mu\nu} &= -16m \int_0^1 dy \int_0^{1-y} dz \frac{m_{h^0}^2}{2} \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu \right) (4yz - 1) \frac{i}{32\pi^2 (m^2 - yzm_{h^0}^2)} \\
&= i \frac{1}{2\pi^2} \frac{m_{h^0}^2}{2m} \int_0^1 dy \int_0^{1-y} dz \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu \right) \frac{1 - 4yz}{1 - yzm_{h^0}^2/m^2}
\end{aligned}$$

The matrix element is thus given by

$$\mathcal{M}_q = -\epsilon_{1,\mu}^* \epsilon_{2,\nu}^* g_s^2 y_q \delta^{ab} \frac{1}{24\pi^2} \frac{m_{h^0}^2}{m} \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu \right) 3 \int_0^1 dy \int_0^{1-y} dz \frac{1 - 4yz}{1 - yzm_{h^0}^2/m^2}.$$

Identifying $f(x)$ this equals the result given on the exercise sheet.

(e) We consider the expression

$$\left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu \right) \epsilon_{1,\mu}^* \epsilon_{2,\nu}^* .$$

We replace the polarisation vectors with the corresponding momenta and obtain

$$\begin{aligned} \epsilon_{1,\mu}^* \rightarrow p_{1,\mu} : \quad & \left(p_1^\nu - \frac{2}{m_{h^0}^2} p_1^\nu (p_1 \cdot p_2) \right) = 0 \\ \epsilon_{2,\nu}^* \rightarrow p_{2,\nu} : \quad & \left(p_2^\mu - \frac{2}{m_{h^0}^2} p_2^\mu (p_1 \cdot p_2) \right) = 0 . \end{aligned}$$

We thus explicitly checked gauge invariance and can use the simplified polarisation sum $\sum_\lambda \epsilon_\mu^* \epsilon_\nu = -g_{\mu\nu}$. We add the sum over various quarks $\mathcal{M} = \sum_q \mathcal{M}_q$ and get

$$\mathcal{M} = -\epsilon_{1,\mu}^* \epsilon_{2,\nu}^* g_s^2 \delta^{ab} \frac{1}{24\pi^2} \frac{m_{h^0}^2}{v} \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu \right) \sum_q f \left(\frac{m_{h^0}^2}{m_q^2} \right) .$$

We can square the expression and obtain perform the polarisation sum

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{g_s^4}{(24\pi^2)^2} \frac{m_{h^0}^4}{v^2} \delta^{aa} \left| \sum_q f \left(\frac{m_{h^0}^2}{m_q^2} \right) \right|^2 \\ &\quad \times \left(g^{\mu\nu} - \frac{2}{m_{h^0}^2} p_1^\nu p_2^\mu \right) \left(g^{\rho\sigma} - \frac{2}{m_{h^0}^2} p_1^\sigma p_2^\rho \right) g_{\mu\rho} g_{\nu\sigma} . \end{aligned}$$

The last line equals

$$g_\mu^\mu - \frac{2}{m_{h^0}^2} p_1 \cdot p_2 - \frac{2}{m_{h^0}^2} p_1 \cdot p_2 + \frac{4}{m_{h^0}^4} p_1^2 p_2^2 = 4 - 1 - 1 + 0 = 2 .$$

We note that $\delta^{aa} = 8$ and also replace $g_s^2 = 4\pi\alpha_s$ and get the result

$$|\mathcal{M}|^2 = \alpha_s^2 \sqrt{2} G_F \frac{4m_{h^0}^4}{9\pi^2} \left| \sum_q f \left(\frac{m_{h^0}^2}{m_q^2} \right) \right|^2 .$$

(f) We finally need the phase space of the decay, which we can copy from sheet 9, such that

$$\Gamma = \frac{1}{16\pi m_{h^0}} |\mathcal{M}|^2$$

We need a symmetry factor $\frac{1}{2}$ due to the two identical gluons and thus obtain

$$\Gamma(h^0 \rightarrow gg) = \frac{\alpha_s^2 G_F m_{h^0}^3}{36\pi^3 \sqrt{2}} \left| \sum_q f \left(\frac{m_{h^0}^2}{m_q^2} \right) \right|^2 .$$

Exercise 2: Higgs boson decay into gluons - Part 2

We continue with the previous exercise. *Hint:* This exercise can be performed independently. All relevant results are given in the previous exercise.

- (a) Perform the integrations in the definition of $f(x)$ by using the results obtained for the dilogarithm on sheet 11. *Hint:* Perform the integration over z and determine the roots of the argument of the remaining logarithm named y_{\pm} , such that $1 - xy + xy^2 = (y_+ - y)(y_- - y)x$. Split the logarithm, use sheet 11 and use

$$\arcsin(z) = -i \ln(iz \pm \sqrt{1 - z^2}) = -i \ln \left[\left(\frac{\sqrt{z^2 - 1} + z}{\sqrt{z^2 - 1} - z} \right)^2 \right].$$

If you succeed, you should get for $x < 4$

$$f(x) = \frac{6}{x} - \frac{6(4-x)}{x^2} \arcsin^2 \left(\frac{\sqrt{x}}{2} \right).$$

- (b) What value does $f(x)$ take for heavy quarks (heavy-top limit), i.e. $m_q \rightarrow \infty$ and $m_{h^0}^2/m_q^2 \rightarrow 0$? Why does the measurement of the decay of a Higgs boson to gluons or the production of a Higgs boson from gluons allow to make a statement on the number of heavy quark generations?
- (c) If you still don't have enough, try to obtain the squared amplitude using a computer, e.g. with **FeynArts** and **FormCalc**. You might encounter the Passarino-Veltman representation of the loop integrals, for which computer codes exist that allow their numerical evaluation.

Solution of exercise 2

- (a) We proceed with the calculation of $f(x)$, for which we obtain

$$\begin{aligned} f(x) &= 3 \int_0^1 dy \int_0^{1-y} dz \frac{1 - 4yz}{1 - xyz} \\ &= 3 \int_0^1 dy \left[\frac{-4yz}{-xy} + \frac{-xy + 4y}{x^2 y^2} \ln(1 - xzy) \right]_0^{1-y} \\ &= 3 \int_0^1 dy \left[\frac{4z}{x} + \frac{4-x}{x^2 y} \ln(1 - xzy) \right]_0^{1-y} \\ &= 3 \int_0^1 dy \left[\frac{4(1-y)}{x} + \frac{4-x}{x^2 y} \ln(1 - xy + xy^2) \right] \\ &= \frac{12}{x} \left[-\frac{1}{2}(1-y)^2 \right]_0^1 + \frac{3(4-x)}{x^2} \int_0^1 dy \frac{\ln(1 - xy + xy^2)}{y} \\ &= \frac{6}{x} + \frac{3(4-x)}{x^2} \underbrace{\int_0^1 dy \frac{\ln(1 - xy + xy^2)}{y}}_{=: J}. \end{aligned}$$

In order to calculate J , we determine the roots of the argument of the logarithms, which results in

$$1 - xy + xy^2 = 0 \quad \rightarrow \quad y_{\pm} = \frac{x \pm \sqrt{x^2 - 4x}}{2x} = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4}{x}} \right).$$

We note that $y_+y_- = \frac{1}{4}(1 - (1 - \frac{4}{x})) = \frac{1}{x}$ and $y_- = 1 - y_+$. Thus we can rewrite $1 - xy + xy^2 = (y_+ - y)(y_- - y)x$. This results in:

$$\begin{aligned}
J &= \int_0^1 dy \frac{\ln((y_+ - y)(y_- - y)x)}{y} = \int_0^1 dy \frac{\ln(xy_+y_-) + \ln(1 - \frac{y_+}{y}) + \ln(1 - \frac{y_-}{y})}{y} \\
&= 0 - \text{Li}_2\left(\frac{1}{y_+}\right) - \text{Li}_2\left(\frac{1}{y_-}\right) = -\text{Li}_2\left(\frac{1}{y_+}\right) - \text{Li}_2\left(\frac{1}{1 - y_+}\right) \\
&= \text{Li}_2(y_+) + \frac{1}{2} \ln^2(-y_+) + \frac{\pi^2}{6} - \text{Li}_2(y_+) + \frac{1}{2} \ln^2(1 - y_+) - \ln(-y_+) \ln(1 - y_+) - \frac{\pi^2}{6} \\
&= \frac{1}{2} (\ln^2(-y_+) - 2 \ln(-y_+) \ln(y_-) + \ln^2(y_-)) \\
&= \frac{1}{2} (\ln(-y_+) - \ln(y_-))^2 = \frac{1}{2} \ln^2\left(-\frac{y_+}{y_-}\right) \\
&= \frac{1}{2} \ln^2\left(\frac{\sqrt{1 - \frac{4}{x}} + 1}{\sqrt{1 - \frac{4}{x}} - 1}\right)
\end{aligned}$$

In the last step we need

$$\arcsin(z) = -i \ln\left(iz \pm \sqrt{1 - z^2}\right) \quad \text{and} \quad \arcsin^2(z) = -\ln^2\left(iz \pm \sqrt{1 - z^2}\right)$$

and then transform the previous expression as follows:

$$J = -2 \left(\frac{(-i)^2}{4} \ln^2\left(\frac{(1 + \sqrt{1 - \frac{4}{x}})}{1 - \frac{4}{x} - 1}\right) \right) = -2 \left(-i \ln \sqrt{\frac{(1 + \sqrt{1 - \frac{4}{x}})^2}{-\frac{4}{x}}} \right)^2.$$

We can transform

$$\sqrt{\frac{(1 + \sqrt{1 - \frac{4}{x}})^2}{-\frac{4}{x}}} = \frac{i\sqrt{x}}{2} \left(1 + \sqrt{1 - \frac{4}{x}}\right) = i\frac{\sqrt{x}}{2} + i\sqrt{\frac{x}{4} - 1} = i\frac{\sqrt{x}}{2} - \sqrt{1 - \left(\frac{\sqrt{x}}{2}\right)^2}.$$

We thus have $J = -2 \arcsin^2\left(\frac{\sqrt{x}}{2}\right)$. Alternatively one may use the second expression for $\arcsin(z)$ on the exercise sheet and write

$$J = \frac{1}{2} \ln^2\left(\frac{\sqrt{\frac{x}{4} - 1} + \frac{\sqrt{x}}{2}}{\sqrt{\frac{x}{4} - 1} + \frac{\sqrt{x}}{2}}\right) = -2 \left(-i \ln \sqrt{\frac{\sqrt{\frac{x}{4} - 1} + \frac{\sqrt{x}}{2}}{\sqrt{\frac{x}{4} - 1} + \frac{\sqrt{x}}{2}}} \right)^2 = -2 \arcsin^2\left(\frac{\sqrt{x}}{2}\right).$$

When using the real version of $\arcsin(x)$ we have to restrict ourselves to the region of $x < 4$, which is the region where $f(x)$ does not develop an imaginary part and is of the form

$$f(x) = \frac{6}{x} - \frac{6(4 - x)}{x^2} \arcsin^2\left(\frac{\sqrt{x}}{2}\right).$$

For $x > 4$ on the other hand, the loop particles can be on-shell, an $f(x)$ has an imaginary part. We refrain from depicting the result.

- (b) For large quark masses we have to consider the limit $x \rightarrow 0$. We expand the $\arcsin(z)$ as follows

$$\arcsin(z) \approx z + \frac{z^3}{6} + \mathcal{O}(z^5), \quad \arcsin^2(z) \approx z^2 + \frac{z^4}{3} + \mathcal{O}(z^6).$$

We insert the result in $f(x)$ and get

$$f(x) \approx \frac{6}{x} - \frac{6(4-x)}{x^2} \left(\frac{x}{4} + \frac{x^2}{3 \cdot 16} + \dots \right) = \frac{6}{x} - \frac{6}{x} + \frac{3}{2} - \frac{24}{48} + \dots = 1 + \dots$$

We conclude that the contribution of a heavy quark is independent from the quark mass! This implies that in addition to the top-quark another fourth generation involving a heavy bottom and a heavy top would change the decay width or cross section by a factor of $|1 + 1 + 1|^2 = 9!$ Thus, the measurement of the gluon fusion cross section to be compatible with the SM expectation allows to exclude heavy quark generations.

- (c) This solution is not depicted here.