Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)
Shruti Patel (shruti.patel@kit.edu) (Office 12/14 - Build. 30.23)

Exercise 1: ***W* pair production in the high-energy limit**

We consider the scattering process

$$e^-(p_1)e^+(p_2) \to W^-(p_3)W^+(p_4)$$

in the high-energy limit with $s := k^2 \gg m_W^2$, with $k = p_1 + p_2 = p_3 + p_4$. The gauge-boson masses cannot be neglected, in contrast to the fermion masses, which we set to zero.

(a) Assume that there is no three gauge-boson self interaction in the Standard Model. Show that in leading order in $\frac{m_W^2}{s}$, i.e. $m_W^2 \ll s, t, u$, the averaged squared amplitude, which only emerges from a t-channel neutrino exchange, is given by

$$\overline{\sum}|\mathcal{M}|^2 \approx -\frac{e^4}{16 s_W^4 m_W^4} t(s+t) = \frac{\alpha^2 \pi^2}{4 s_W^4} \frac{s^2}{m_W^4} \left(1 - \cos^2\theta\right) ,$$

where s and $t = (p_1 - p_3)^2 = m_W^2 - 2p_1 \cdot p_3 = (p_2 - p_4)^2 = m_W^2 - 2p_2 \cdot p_4$ are Mandelstam variables and θ denotes the scattering angle between the incoming electron e^- and the outgoing W^- in the center-of-mass frame. The sine of the weak mixing angle is $s_W := \sin\theta_W$ and the fine structure constant is defined by $\alpha = \frac{e^2}{4\pi}$.
Hint: Argue why the polarisation sum of the W bosons can be approximated by $\sum_\lambda \epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) \approx \frac{p_\mu p_\nu}{m_W^2}$. Neglect all non-leading terms in $\frac{m_W^2}{s}$ in the calculation of the trace. A collection of relevant Feynman rules is given at the end of this exercise.

(b) Determine the total cross section in leading order in $\frac{m_W^2}{s}$ in the center-of-mass system of the incoming particles, based on the calculation of the squared amplitude in the previous exercise. What happens to the cross section in the high-energy limit, i.e. $s \to \infty$?

(c) The correct high-energy behaviour is obtained, when the three gauge-boson self-interaction is added. The calculation of the total cross section for this case is however quite lengthy, such that we consider the high-energy limit only for one helicity combination. Examine the amplitude for $e_R^- e_L^+ \to W_L^- W_L^+$ for an incoming right-handed electron and left-handed positron and two outgoing longitudinally polarised W bosons and show that in leading order in $\frac{m_W^2}{s}$ the sum of all diagrams yields

$$\mathcal{M}(e_R^- e_L^+ \to W_L^- W_L^+) \approx \frac{ie^2}{2c_W^2} \frac{1}{s} \overline{v}_R(p_2)(\not{p}_4 - \not{p}_3) u_R(p_1) \,.$$

Hint: Use the equation of motions for massless fermions being $\overline{v}(p_2)\not{p}_2 = 0$ and $\not{p}_1 u(p_1) = 0$ as well as the approximation, that the polarisation vector of the longitudinal W boson in the high-energy limit is given by $\epsilon_L^\mu(p) \approx \frac{p^\mu}{m_W}$. If you failed in (a) or (b), you can also restart here.

(d) Determine the total cross section for $e^-_R e^+_L \to W^-_L W^+_L$ and examine the behaviour in the high-energy limit, i.e. $s \to \infty$.

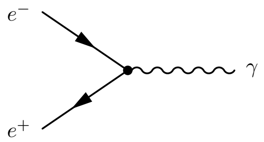
(e) The Goldstone-boson equivalence theorem states that the amplitudes of longitudinally polarised gauge bosons in the high-energy limit equal the amplitudes, in which the gauge bosons are replaced through their Goldstone bosons (except from an unobservable phase). Show the equivalence of

$$\mathcal{M}(e^-_R e^+_L \to \phi^- \phi^+) = \mathcal{M}(e^-_R e^+_L \to W^-_L W^+_L)$$

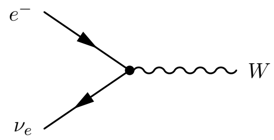
in leading order in $\frac{m_W^2}{s}$.

(f) Determine the amplitude for $e^-_L e^+_R \to W^-_L W^+_L$ for an incoming left-handed electron and right-handed positron and two outgoing longitudinally polarised W bosons using the Goldstone-boson equivalence theorem and lastly obtain the total cross section for $e^- e^+ \to W^-_L W^+_L$.

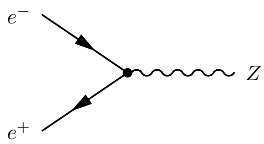
Relevant Feynman rules for the interactions are given by the following expressions:



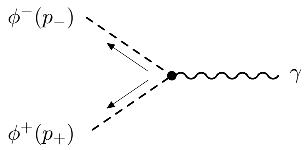
$= -ie\gamma^\mu(P_L + P_R)$



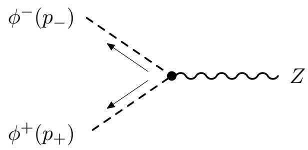
$= \frac{ie}{\sqrt{2}s_W}\gamma^\mu P_L$



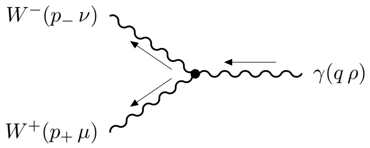
$= \frac{ie}{s_W c_W}\gamma^\mu \left((-\frac{1}{2} + s_W^2)P_L + (s_W^2)P_R\right)$



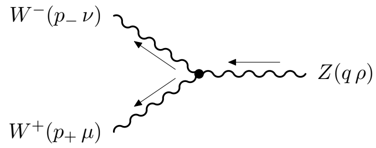
$= ie(p_- - p_+)^\mu$



$= \frac{ie(\frac{1}{2} - s_W^2)}{c_W s_W}(p_- - p_+)^\mu$



$= ief^{\mu\nu\rho}$



$= \frac{iec_W}{s_W}f^{\mu\nu\rho}$

Therein the momentum flow is indicated through the additional arrows. The left- and right-handed projection operators are given by $P_L = \frac{1-\gamma_5}{2}$ and $P_R = \frac{1+\gamma_5}{2}$. Moreover it yields $f^{\mu\nu\rho} = g^{\mu\nu}(p_- - p_+)^\rho + g^{\nu\rho}(-q - p_-)^\mu + g^{\rho\mu}(q + p_+)^\nu$. Again s_W and c_W are defined through $\sin\theta_W$ and $\cos\theta_W$, respectively. The propagators of the photon and the Z boson in R_ξ gauge (see sheet 5) are given by

$$\frac{-i\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)}{k^2} \quad \text{and} \quad \frac{-i\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi m_Z^2}\right)}{k^2 - m_Z^2}\ , \text{ respectively.}$$

Make a convenient gauge choice. Goldstone bosons do not couple to massless fermions.