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Exercise 1: Propagator of a free particle

We want to evaluate the path integral of a free particle in one-dimension by parametrizing all potential paths through a variation of the coefficients of a Fourier series. The propagator is defined by

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q e^{\frac{i}{\hbar} S} \quad \text{with} \quad S = \int_{t_i}^{t_f} dt \frac{m}{2} \dot{q}^2$$

with the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$.

- Find the classical path of the particle $q_c(t)$ as a function of q_i, q_f, t_i and t_f .
- In order to evaluate the path integral we allow for quantum mechanical perturbations around the classical path, $q(t) = q_c(t) + \delta q(t)$, and expand $\delta q(t)$ in a Fourier series knowing that the fluctuations vanish at q_i and q_f . We therefore obtain

$$\delta q(t) = \sum_{n=1}^{\infty} a_n \sin \left[\frac{n\pi}{t_f - t_i} (t - t_i) \right].$$

Using the orthogonality of different modes, show that the action S turns into

$$S = \frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i} + \frac{m}{2} \sum_{n=1}^{\infty} \frac{1}{2} \frac{(n\pi)^2}{t_f - t_i} a_n^2.$$

- In this example the path integral can be rewritten in terms of integrals over the spectrum of the Fourier coefficients, such that

$$\int \mathcal{D}q = c \int \prod_{n=1}^{\infty} da_n$$

with a normalization constant c . Rearrange the integrals and show

$$\langle q_f, t_f | q_i, t_i \rangle = \tilde{c}(t_f - t_i) \exp \left[\frac{i}{\hbar} \frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i} \right],$$

where the normalization constant $\tilde{c}(t_f - t_i)$ can only be a function of the time difference.

Hint: You can use

$$\int_{-\infty}^{\infty} da \exp \left[\frac{i}{\hbar} \frac{m}{2} \frac{(n\pi)^2}{t_f - t_i} a^2 \right] = \left(-\frac{imn^2\pi}{2\hbar(t_f - t_i)} \right)^{-1/2}.$$

(d) Show that the normalization constant $\tilde{c}(t)$ is given by

$$\tilde{c}(t) = \sqrt{\frac{m}{2\pi i \hbar t}} \quad \text{from the requirement} \quad \int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle = \langle q_f, t_f | q_i, t_i \rangle.$$

Hint: You can use

$$\int_{-\infty}^{\infty} dq \exp \left[\frac{i m}{\hbar} \left(\frac{(q_f - q)^2}{(t_f - t)} + \frac{(q - q_i)^2}{(t - t_i)} \right) \right] = \sqrt{\frac{2\pi i \hbar (t_f - t)(t - t_i)}{m(t_f - t_i)}} \exp \left[\frac{i m (q_f - q_i)^2}{\hbar} \frac{1}{t_f - t_i} \right].$$

(e) The wave-function of the particle thus evolves according to

$$\psi(q, t) = \int dq_i \langle q, t | q_i, 0 \rangle \psi(q_i, 0).$$

Show that $\psi(q, t)$ satisfies the Schrödinger equation and that $\psi(q, t) \rightarrow \psi(q, 0)$ for $t \rightarrow 0$.

Hint: Fresnel representation of the δ distribution.

Exercise 2: Propagator of the harmonic oscillator

We investigate the propagator $\langle q_f, t | q_i, 0 \rangle$ of a one-dimensional system. In contrast to the previous exercise we use $\hbar = 1$.

(a) Show that due to the completeness relation of the energy eigenstates $|n\rangle$ with energy E_n we can write

$$\langle q_f, T | q_i, 0 \rangle = \sum_n \Phi_n(q_f) \Phi_n^*(q_i) e^{-iE_n T} \quad (1)$$

with $\Phi_n(q) = \langle q | n \rangle$.

(b) The propagator of the harmonic oscillator can be determined from the path integral formalism and yields

$$\langle q_f, T | q_i, 0 \rangle = \sqrt{\frac{m\omega}{2\pi i \sin \omega T}} \exp \left\{ \frac{i m \omega}{2 \sin \omega T} [(q_f^2 + q_i^2) \cos \omega T - 2q_i q_f] \right\}. \quad (2)$$

Determine the wave function of the ground state $\Phi_0(q)$. In order to do so, set the time to an imaginary value $T = -it, t \rightarrow \infty$, rewrite sin and cos in terms of exponentials and compare Eq. 1 and Eq. 2. For this purpose rewrite Eq. 2 in the form $e^{-\omega/2t} f(e^{-\omega t})$ and expand around the argument of $f(x)$. The first term corresponds to the ground state.

Hint: We refer to the literature for the derivation of the path integral representation of the harmonic oscillator.

(c) Calculate the wave function of the first excited state $\Phi_1(q)$.

(d) Show that the energy levels are given by $E_n = \omega(n + \frac{1}{2})$.

Hint: Again compare Eq. 1 and Eq. 2.