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Exercise 1: Harmonic oscillator with external force

We again consider the one-dimensional harmonic oscillator in the path integral formalism of quantum mechanics and allow it to be influenced by an external driving force $J(t)$. The propagator then takes the form

$$\langle q_f, t_f | q_i, t_i \rangle_J = \int \mathcal{D}q \exp\left[i \int_{t_i}^{t_f} dt \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + J(t) q(t) \right) \right] \tag{1}$$

with the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$. We set $\hbar = 1$.

(a) Show that for $\omega^2 \to \omega^2 - i\epsilon$ the propagator in Eq. 1 can be rewritten in the form

$$Z[J] = \langle q_f, \infty | q_i, -\infty \rangle_J = \langle q_f, \infty | q_i, -\infty \rangle_0 \exp\left[-\frac{i}{2} \int dE \frac{\tilde{J}(E)\tilde{J}(-E)}{E^2 - \omega^2 + i\epsilon} \right], \tag{2}$$

where we have introduced

$$\tilde{J}(E) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-iEt} J(t) \quad \leftrightarrow \quad J(t) = \int_{-\infty}^{\infty} \frac{dE}{\sqrt{2\pi}} e^{iEt} \tilde{J}(E) \,.$$

Hints: Make use of a Fourier transform ($t \leftrightarrow E$) in the exponent of Eq. 1 for both $J(t)$ and $q(t)$, write $Jq = \frac{1}{2}[Jq + qJ]$, use $\int dt \exp[i(E+E')t] = 2\pi\delta(E+E')$ and motivate a shift $\tilde{q}(E) \to \tilde{q}(E) - \frac{\tilde{J}(E)}{E^2-\omega^2}$.

(b) Transform the exponent in Eq. 2 into a time integral, i.e.

$$-\frac{i}{2} \int dt dt' J(t) \Delta(t-t') J(t') \quad \text{with} \quad \Delta(t-t') = \int \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{E^2 - \omega^2 + i\epsilon},$$

and discuss its physical meaning.

(c) We define the functional in Euclidean space through

$$Z_E[J] = \int \mathcal{D}q \exp\left[-\int d\tau \left(\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + \frac{1}{2}\omega^2 q^2 + J(\tau) q(\tau) \right) \right].$$

Show again that

$$Z_E[J] = Z_E[0] \exp\left[\frac{1}{2} \int d\tau d\tau' J(\tau) \Delta_E(\tau - \tau') J(\tau') \right].$$

and compare it with $Z[J]$. *Hint:* Discuss the two options $\tau = \mp it$.

Exercise 2: Gaussian integral

We define

$$G(A) = \int \prod_i dx_i e^{-x^T A x} = \int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_n e^{-x_i A_{ij} x_j}$$

with A being a symmetric, real $(n \times n)$ matrix with positive eigenvalues. Show that

$$G(A) = \pi^{n/2} \det(A)^{-1/2} \,.$$

Hint: Diagonalize the matrix through an orthogonal rotation.
Note: It yields $\int \prod_i dx_i \exp\left[-x^T A x + \omega^T x\right] = \pi^{n/2} \exp\left[\omega^T A^{-1} \omega\right] \det(A)^{-1/2}$. For a hermitian, non-singular matrix C one can use $\int \prod_i dz_i dz_i^* \exp\left[-z^\dagger C z\right] = \pi^n \det(C)^{-1}$.

Exercise 3: Saddle-point approximation for a path integral

Commonly a path integral is evaluated by expanding around a stationary phase for real times. As an alternative we consider again an imaginary time and then perform a saddle point approximation instead. The real-time propagator can afterwards be obtained from the imaginary-time propagator through analytic continuation. In order to perform the saddle-point approximation we will work with the Euclidean path integral throughout this exercise.
We consider the generating functional in Euclidean space with $x^2 = x_0^2 + \sum_i x_i^2$ of a generic scalar field theory, which is given by

$$Z_E[J] = \mathcal{N} \int \mathcal{D}\phi \exp\left[-\frac{1}{\hbar} S_E[\phi, J]\right] \,.$$

Therein the action reads

$$S_E[\phi, J] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + V(\phi) - J(x)\phi(x)\right] \,.$$

The classical field configuration ϕ_0 is determined by

$$\left.\frac{\delta S_E[\phi, J]}{\delta \phi}\right|_{\phi = \phi_0} = 0 \,,$$

such that we can expand $S_E[\phi, J]$ around ϕ_0 in the form

$$\begin{aligned} S_E[\phi, J] = S_E[\phi_0, J] &+ \int d^4x \Delta S_J^{(1)}(x) \left(\phi(x) - \phi_0(x)\right) \\ &+ \frac{1}{2} \int d^4x d^4y \Delta S_J^{(2)}(x, y) \left(\phi(x) - \phi_0(x)\right)\left(\phi(y) - \phi_0(y)\right) + \dots \end{aligned}$$

Show that in the limit in which we neglect the dots in the previous equation the functional $Z_E[J]$ is given by

$$Z_E[J] \approx \mathcal{N}' \exp\left[-\frac{1}{\hbar} S_E[\phi_0, J]\right] \left(\det \hat{K}\right)^{-1/2} \quad \text{with} \quad \hat{K} = \int d^4x \left[-\partial_\mu^2 + m^2 + V''(\phi_0)\right] \,.$$

Discuss the physical meaning of this approximation.
Hint: Define $\delta\phi = \phi - \phi_0$ and thus replace $\mathcal{D}\phi = \mathcal{D}\delta\phi$. Lastly define $\delta\phi' = \frac{1}{\sqrt{\hbar}}\delta\phi$ and count the orders in $\hbar$. Use the previous exercise to replace $\int \mathcal{D}\delta\phi' \exp[-\int d^4x \delta\phi' [-\partial_\mu^2 + m^2 + V''(\phi_0)]\delta\phi']$.