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Exercise 1: Saddle-point approximation for a path integral - continued

Remember the last exercise on sheet 3, in which we showed in Euclidean space that in the classical limit with $\hbar \to 0$ the path integral was just given by the classical action

$$S_E[\phi_0, J] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_0)^2 + \frac{1}{2}m^2\phi_0^2 + V(\phi_0) - J(x)\phi_0\right] ,$$

where ϕ_0 fulfills the relation

$$-\partial_\mu^2 \phi_0(x) + m^2\phi_0(x) + V'(\phi_0(x)) - J(x) = 0 \,. \tag{1}$$

We consider ϕ^4 theory with

$$V(\phi) = \frac{\lambda}{4!}\phi^4 \,.$$

We want to derive the connected Euclidean Green functions $\tau_c^E(x_1, x_2)$ and $\tau_c^E(x_1, x_2, x_3, x_4)$.

(a) Show that the classical action can be rewritten in the form

$$S_E[\phi_0, J] = -\frac{1}{2}\int d^4x \left(J\phi_0 + 2\frac{\lambda}{4!}\phi_0^4\right) \,. \tag{2}$$

(b) We expand $\phi_0 = \phi^{[0]} + \lambda\phi^{[1]} + \lambda^2\phi^{[2]} + \ldots$. In Eq. 1 we sort by orders in λ and for the lowest order, λ^0, obtain $-(\partial^2 - m^2)_x\phi^{[0]}(x) = J(x) =: J_x$. The Euclidean two-point function $G^E(x, y)$ is defined through

$$(\partial^2 - m^2)_x G^E(x, y) = -\delta^{(4)}(x - y) \quad \leftrightarrow \quad G^E_{xy} := G^E(x, y) = \int \frac{d^4q_E}{(2\pi)^4}\frac{e^{iq_E(x-y)}}{q_E^2 + m^2} \,.$$

Show that it yields

$$\phi^{[0]}(x) = \int d^4a G^E_{xa} J_a \,.$$

(c) Consider the orders λ^1 and λ^2 in Eq. 1 to prove

$$\phi^{[1]}(x) = -\frac{1}{6}\int d^4a d^4b d^4c d^4d G^E_{xa}G^E_{ab}G^E_{ac}G^E_{ad}J_bJ_cJ_d$$

$$\phi^{[2]}(x) = \frac{1}{12}\int d^4a d^4b d^4c d^4d d^4e d^4f d^4y G^E_{xa}G^E_{ab}G^E_{ac}G^E_{ay}G^E_{yd}G^E_{ye}G^E_{yf}J_bJ_cJ_dJ_eJ_f \,.$$

(d) Finally expand also Eq. 2 in powers of λ up to λ^1 and insert the results from the previous two subexercises. According to the lecture the generating functional of the connected Green functions is just given by the classical action $W_E[J] = S_E[\phi_0, J]$, such that

$$\tau_c^E(x_1, x_2) = -\left.\frac{\delta^2 W_E[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} , \quad \tau_c^E(x_1, x_2, x_3, x_4) = -\left.\frac{\delta^4 W_E[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} .$$

Derive the connected Green functions starting from the expanded version of Eq. 2. Explain the physical meaning of the connected Green functions by identifying the corresponding Feynman diagrams.

Exercise 2: Grassmann variables

In the formulation of the path integral formalism for fermions we came across Grassmann variables. In this exercise we want to derive a few useful relations for them.

(a) Consider the Gaussian integral in the space of N real Grassmann variables θ_i with $i \in \{1, \ldots, N\}$, which is of the form

$$I_N(M, \chi) = \int d\theta_1 \ldots d\theta_N \exp\left[-\frac{1}{2}\theta^T M\theta + \chi^T\theta\right],$$

where M is an arbitrary antisymmetric matrix and $\chi = (\chi_1, \ldots, \chi_N)^T$ is a vector of N independent Grassmann variables. Show that for $N = 4$ the identity

$$I_4(M, \chi = 0) = \sqrt{\det M}$$

holds by explicitly evaluating the integral.
Hint: The determinant is given by $\det M = (M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23})^2$ with the matrix elements M_{ij} of M.

(b) Show for non-vanishing χ and $N = 4$ the identity

$$I_4(M, \chi) = \sqrt{\det M} \exp\left[c\chi^T M^{-1}\chi\right].$$

Determine the real constant c.
Hint: The inverse of the matrix M is given by

$$M^{-1} = \frac{1}{\sqrt{\det M}} \begin{pmatrix} 0 & -M_{34} & M_{24} & -M_{23} \\ M_{34} & 0 & -M_{14} & M_{13} \\ -M_{24} & M_{14} & 0 & -M_{12} \\ M_{23} & -M_{13} & M_{12} & 0 \end{pmatrix}.$$

(c) We consider integrals in the space of N Grassmann variables $\eta_1, \ldots, \eta_N$. How does the integration measure transform under a linear variable transformation of the form

$$\eta'_i = B_{ij}\eta_j\,?$$

Hint: Consider

$$\int d\eta_2 d\eta_1 \eta_1\eta_2 = \int d\eta'_2 d\eta'_1 \eta'_1\eta'_2 = 1$$

and replace $\eta'_1\eta'_2$ with $\eta_1\eta_2$. Try to generalize your findings for $N > 2$.

(d) Use the result of the previous subexercise to determine the complex, N-dimensional Gaussian integral

$$\int d\overline{\eta}_N \ldots d\overline{\eta}_1 d\eta_N \ldots d\eta_1 \exp\left[-\overline{\eta}B\eta\right]$$

for a hermitian matrix B.