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Exercise 1: Propagator of the gauge field in the Stueckelberg Lagrangian

We consider the Stueckelberg Lagrangian of a single free massive gauge field given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \frac{m^2}{2}A^\mu A_\mu\,.$$

Therein we use the Abelian field strength tensor defined by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ as well as a covariant gauge fixing term employing the free parameter ξ and a mass term with mass m.

(a) Derive the equation of motion for the gauge field, for which you should obtain

$$\left[(\Box + m^2)g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\nu = 0\,.$$

Hint: You can use the functional derivative of the action $S = i\int d^4x\mathcal{L}$ with respect to A_ρ or use the Euler-Lagrange equation for the pair A_ρ and $\partial_\rho A_\sigma$. Be careful to perform the derivatives with respect to a new index A_ρ and thus add $\delta^\rho_{\mu,\nu,\ldots}$ where appropriate.

(b) Fourier transformation ($\partial_\mu \to ik_\mu$) of this equation allows to determine the Green's function $\Delta_{\nu\rho}(k)$ in momentum space

$$\left[(-k^2 + m^2)g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]\Delta_{\nu\rho}(k) = \delta^\mu_\rho\,.$$

Make the ansatz $\Delta_{\nu\rho}(k) = A(k^2)g_{\nu\rho} + B(k^2)k_\nu k_\rho$ and determine $A(k^2)$ and $B(k^2)$ by equating the coefficients. You should obtain

$$\Delta_{\mu\nu}(k) = \frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2} - \frac{\frac{k_\mu k_\nu}{m^2}}{k^2 - \xi m^2}\,.$$

(c) Discuss the cases $\xi \to 0$ (Landau gauge), $\xi \to 1$ (Feynman gauge) and $\xi \to \infty$ (unitary gauge) as well as $m \to 0$. Compare the latter result with the result of the gluon propagator obtained in the lecture.

Exercise 2: Propagator of the gluon field in axial gauge

Rather than choosing the gauge fixing term in the previous exercise an alternative choice is the axial gauge (Arnowitt-Fickler gauge), which in covariant form can be written in the form $n_\mu A^{\mu,a} = 0$ with an arbitrary constant vector n_μ. The for our purposes relevant part of the action therefore takes the form

$$S = i\int d^4x\left[-\frac{1}{4}F^{\mu\nu,a}F^a_{\mu\nu} - \frac{1}{2\xi}(n_\mu A^{\mu,a})^2 - \overline{\eta}^a n^\mu (D_\mu)_{ab}\eta^b\right]\,. \qquad (1)$$

We now work in a non-Abelian theory, which explains the additional Roman color indices. The last term of Eq. 1, which due to the Abelian structure was omitted in the previous exercise, includes ghost fields η and $\overline{\eta}$ and the covariant derivative $(D_\mu)_{bc} = \partial_\mu \delta_{bc} - igT^a_{bc}A^a_\mu$ in the adjoint representation.

(a) Derive the propagator of the gluon field from the first two terms in Eq. 1 in the same way it was done in the previous exercise. Use the ansatz

$$\Delta_{\mu\nu} = Ag_{\mu\nu} + Bk_\mu k_\nu + C(k_\mu n_\nu + k_\nu n_\mu) + Dn_\mu n_\nu$$

though. Take the limit $\xi \to 0$. The result might remind you of the polarization sum for massless gauge fields. Show that $n^\mu \Delta_{\mu\nu} = 0$ for $\xi \to 0$.
Hint: The term $+gf^{abc}A^b_\mu A^c_\nu$ in $F^{\mu\nu,a}$ is not of relevance for the propagator, since it leads to terms with three gauge fields.

(b) Motivate the form of the last term in Eq. 1 by looking in your lecture notes. Derive the equation of motion for the ghost field $\overline{\eta}$.
Note: Due to $n^\mu \Delta_{\mu\nu} = 0$ for $\xi \to 0$ ghosts decouple, i.e. they don't interact, which is why the axial gauge for $\xi \to 0$ is also called "physical" gauge. For practical calculations however terms proportional to $\frac{1}{n \cdot k}$ lead to unpleasant spurious divergences.