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Exercise 1: Two-particle phase space

For the calculation of both decay widths and cross sections we need the integration over the phase space of the particles in the final state. For a general process with two particles with momenta p_1 and p_2 and masses m_1 and m_2 in the final state this phase space integral (more precisely "phase space measure") is given by

$$\int \mathrm{d}\Phi_2 = \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2)\,,$$

where q is the four momentum of the incoming particle. For two incoming particles q equals the sum of the momenta of the two incoming particles. This integrals acts upon the squared matrix element as well as Heaviside step functions, which represent (potential) cuts on the phase space of the final state particles.

(a) Show that in the center-of-mass frame of the two final-state particles, the absolute value of the outgoing three-momenta is given by

$$|\vec{p}_1| = |\vec{p}_2| = \frac{\lambda(q^2, m_1^2, m_2^2)}{2\sqrt{q^2}}\,,$$

where λ is the *Källén function* given by

$$\lambda(a^2, b^2, c^2) \equiv \sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}\,.$$

(b) Show that

$$\int \mathrm{d}\Phi_2 = \int \mathrm{d}\Omega\, \frac{1}{32\pi^2 q^2} \lambda(q^2, m_1^2, m_2^2)\Theta(q_0)\Theta(q^2 - (m_1 + m_2)^2)$$

with the Heaviside step function Θ. Therein $\mathrm{d}\Omega = d\cos\theta_1 d\phi_1$ corresponds to the integration over the two angles describing particle 1 in the center-of-mass frame.
Note: Choose the center-of-mass frame of the two outgoing particles. You can use the relation

$$\frac{\mathrm{d}^3 p}{2E} = \mathrm{d}^4 p\, \Theta(p_0)\delta(p^2 - m^2)\,.$$

Exercise 2: Differential cross section for $q\bar{q} \to gg$

We continue to discuss the process $q(p_1)\bar{q}(p_2) \to g(k_1)g(k_2)$ for massless quarks. In the lecture we deduced the color factors, which allowed to write the squared amplitude in the form

$$\overline{\sum}|M|^2 = \frac{1}{9}\overline{\sum}_{\mathrm{pol}} \left[c_+|M^{(+)}|^2 + c_-|M^{(-)}|^2\right]$$

with $c_+ = \frac{7}{3}$ and $c_- = 3$. We make the dependence of $M^{(\pm)}$ on the polarisation vectors explicit by rewriting

$$M^{(\pm)} = M^{(\pm)}_{\mu\nu} \epsilon_1^{*\mu} \epsilon_2^{*\nu} \,.$$

The two individual contributions were given by

$$\begin{aligned} M^{(+)}_{\mu\nu} &= g^2(-M^t_{\mu\nu} - M^u_{\mu\nu}) \\ M^{(-)}_{\mu\nu} &= g^2(-M^t_{\mu\nu} + M^u_{\mu\nu} - 2M^s_{\mu\nu}) \end{aligned}$$

with

$$\begin{aligned} M^t_{\mu\nu} &= \overline{v}(p_2)\gamma_\nu \frac{1}{\not{p}_1 - \not{k}_1} \gamma_\mu u(p_1) \\ M^u_{\mu\nu} &= \overline{v}(p_2)\gamma_\mu \frac{1}{\not{p}_1 - \not{k}_2} \gamma_\nu u(p_1) \\ M^s_{\mu\nu} &= \overline{v}(p_2)\frac{1}{s}\left\{g_{\mu\nu}(\not{k}_1 - \not{k}_2) + \gamma_\nu(k_2 - p)_\mu + \gamma_\mu(p - k_1)_\nu\right\} u(p_1) \,. \end{aligned}$$

(a) Show that $M^{(-)}_{\mu\nu}$ can be rewritten in the form

$$\begin{aligned} M^{(-)}_{\mu\nu} = &g^2 \overline{v}(p_2) \Bigg\{ -\gamma_\nu \frac{1}{\not{p}_1 - \not{k}_1} \gamma_\mu + \gamma_\mu \frac{1}{\not{k}_1 - \not{p}_2} \gamma_\nu \\ &- \frac{2}{s}[g_{\mu\nu}(\not{k}_1 - \not{k}_2) + 2k_{2\mu}\gamma_\nu - 2k_{1\nu}\gamma_\mu] \Bigg\} u(p_1) \,. \end{aligned}$$

(b) Show that in this notation we obtain current conservation, i.e.

$$k_1^\mu M^{(-)}_{\mu\nu} = k_2^\nu M^{(-)}_{\mu\nu} = 0 \,,$$

like for $M^{(+)}_{\mu\nu}$. This simplifies the gluon polarisation sums substantially.

(c) Determine the polarisation averaged squared amplitudes

$$\overline{\sum}_{\text{pol}} |M^{(+)}|^2 \quad \text{and} \quad \overline{\sum}_{\text{pol}} |M^{(-)}|^2 \,.$$

Reexpress your result in terms of the Mandelstam variables $s = (p_1 + p_2)^2, t = (p_1 - k_1)^2$ and $u = (p_1 - k_2)^2$.
Note: In contrast to the lecture perform the averaged sum over the quark spins for all terms, which is quite painful though. If you have experience with a code that handles such problems, feel free to prepare a short demonstration. If your code does not allow to split the amplitude into $M^{(\pm)}$, but handles only the full amplitude, this is completely fine. Among such codes are `Form, FeynArts, FormCalc, FeynCalc, Maple` or `Reduce`. The final result is

$$\overline{\sum} |M|^2 = g^4 \left(\frac{32}{27} \frac{u^2 + t^2}{ut} - \frac{8}{3} \frac{u^2 + t^2}{s^2} \right) \,.$$

(d) Finally express all momenta in the center-of-mass frame through s and the scattering angle θ. Replace t and u through s and θ and determine the differential cross section with the help of the previous exercise.