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Exercise 2: Differential cross section for $q\bar{q} \to gg$

We continue to discuss the process $q(p_1)\bar{q}(p_2) \to g(k_1)g(k_2)$ for massless quarks. In the lecture we deduced the color factors, which allowed to write the squared amplitude in the form

$$\overline{\sum}|M|^2 = \frac{1}{9}\overline{\sum}_{\text{pol}}\left[c_+|M^{(+)}|^2 + c_-|M^{(-)}|^2\right]$$

with $c_+ = \frac{7}{3}$ and $c_- = 3$. We make the dependence of $M^{(\pm)}$ on the polarisation vectors explicit by rewriting

$$M^{(\pm)} = M^{(\pm)}_{\mu\nu}\epsilon_1^{*\mu}\epsilon_2^{*\nu}\,.$$

The two individual contributions were given by

$$\begin{aligned} M^{(+)}_{\mu\nu} &= g^2(-M^t_{\mu\nu} - M^u_{\mu\nu}) \\ M^{(-)}_{\mu\nu} &= g^2(-M^t_{\mu\nu} + M^u_{\mu\nu} - 2M^s_{\mu\nu}) \end{aligned}$$

with

$$\begin{aligned} M^t_{\mu\nu} &= \overline{v}(p_2)\gamma_\nu\frac{1}{\not{p}_1 - \not{k}_1}\gamma_\mu u(p_1) \\ M^u_{\mu\nu} &= \overline{v}(p_2)\gamma_\mu\frac{1}{\not{p}_1 - \not{k}_2}\gamma_\nu u(p_1) \\ M^s_{\mu\nu} &= \overline{v}(p_2)\frac{1}{s}\left\{g_{\mu\nu}(\not{k}_1 - \not{k}_2) + \gamma_\nu(k_2 - p)_\mu + \gamma_\mu(p - k_1)_\nu\right\}u(p_1)\,. \end{aligned}$$

(a) Show that $M^{(-)}_{\mu\nu}$ can be rewritten in the form

$$\begin{aligned} M^{(-)}_{\mu\nu} =& g^2\overline{v}(p_2)\left\{-\gamma_\nu\frac{1}{\not{p}_1 - \not{k}_1}\gamma_\mu + \gamma_\mu\frac{1}{\not{k}_1 - \not{p}_2}\gamma_\nu\right. \\ &\left. -\frac{2}{s}[g_{\mu\nu}(\not{k}_1 - \not{k}_2) + 2k_{2\mu}\gamma_\nu - 2k_{1\nu}\gamma_\mu]\right\}u(p_1)\,. \end{aligned}$$

(b) Show that in this notation we obtain current conservation, i.e.

$$k_1^\mu M^{(-)}_{\mu\nu} = k_2^\nu M^{(-)}_{\mu\nu} = 0\,,$$

like for $M^{(+)}_{\mu\nu}$. This simplifies the gluon polarisation sums substantially.

(c) Determine the polarisation averaged squared amplitudes

$$\overline{\sum}_{\text{pol}}|M^{(+)}|^2 \quad \text{and} \quad \overline{\sum}_{\text{pol}}|M^{(-)}|^2\,.$$

Reexpress your result in terms of the Mandelstam variables $s = (p_1 + p_2)^2, t = (p_1 - k_1)^2$ and $u = (p_1 - k_2)^2$.

Note: In contrast to the lecture perform the averaged sum over the quark spins for all terms, which is quite painful though. If you have experience with a code that handles such problems, feel free to prepare a short demonstration. If your code does not allow to split the amplitude into $M^{(\pm)}$, but handles only the full amplitude, this is completely fine. Among such codes are `Form`, `FeynArts`, `FormCalc`, `FeynCalc`, `Maple` or `Reduce`. The final result is

$$\overline{\sum}|M|^2 = g^4 \left(\frac{32}{27} \frac{u^2+t^2}{ut} - \frac{8}{3} \frac{u^2+t^2}{s^2} \right) .$$

(d) Finally express all momenta in the center-of-mass frame through s and the scattering angle θ. Replace t and u through s and θ and determine the differential cross section with the help of the previous exercise.

Solution of exercise 2

(c) We move to the calculation of the helicity sums over the initial state quarks. We perform the calculation for $M^{(-)}$ only. The average over the initial helicities introduces a factor of 1/4. We get

$$\begin{aligned}\overline{\sum}_{\text{pol}}|M^{(-)}|^2 =& \frac{1}{4} g^4 \text{tr} \left\{ \not{p}_2 \left[-\gamma_\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma_\mu + \gamma_\mu \frac{\not{k}_1 - \not{p}_2}{u} \gamma_\nu - \frac{2}{s} \left(g_{\mu\nu} (\not{k}_1 - \not{k}_2 + 2k_{2\mu}\gamma_\nu - 2k_{1\nu}\gamma_\mu) \right) \right] \right. \\ & \left. \not{p}_1 \left[-\gamma^\mu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\nu + \gamma^\nu \frac{\not{k}_1 - \not{p}_2}{u} \gamma^\mu - \frac{2}{s} (g^{\mu\nu}(\not{k}_1 - \not{k}_2) + \gamma^\nu 2k_2^\mu - \gamma^\mu 2k_1^\nu) \right] \right\} .\end{aligned}$$

Therein we already made use of the Mandelstam variables, which for our purposes with massless external particles are given by

$$\begin{aligned} s &= (p_1+p_2)^2 = (k_1+k_2)^2 = 2p_1 \cdot p_2 = 2k_1 \cdot k_2 \\ t &= (p_1-k_1)^2 = (p_2-k_2)^2 = -2p_1 \cdot k_1 = -2p_2 \cdot k_2 \\ u &= (p_1-k_2)^2 = (p_2-k_1)^2 = -2p_1 \cdot k_2 = -2p_2 \cdot k_1 . \end{aligned}$$

We use the Mandelstam variables to sort the terms in the expression above. We start with terms proportional to $1/t^2$, for which we obtain

$$\begin{aligned} &\frac{g^4}{4t^2} \text{tr} \left\{ \not{p}_2 \gamma_\nu (\not{p}_1 - \not{k}_1) \gamma_\mu \not{p}_1 \gamma^\mu (\not{p}_1 - \not{k}_1) \gamma^\nu \right\} = \frac{g^4}{t^2} \text{tr} \left\{ \not{p}_2 (\not{p}_1 - \not{k}_1) \not{p}_1 (\not{p}_1 - \not{k}_1) \right\} \\ &= \frac{g^4}{t^2} \text{tr} (\not{p}_2 \not{p}_1 \not{p}_1 \not{p}_1 - \not{p}_2 \not{p}_1 \not{p}_1 \not{k}_1 - \not{p}_2 \not{k}_1 \not{p}_1 \not{p}_1 + \not{p}_2 \not{k}_1 \not{p}_1 \not{k}_1) \\ &= \frac{g^4}{t^2} 8 p_2 \cdot k_1 p_1 \cdot k_1 = \frac{g^4}{t^2} 2(-2p_2 \cdot k_1)(-2p_1 \cdot k_1) = 2g^4 \frac{u}{t} . \end{aligned}$$

Therein we used $\gamma^\nu \not{p}_i \gamma_\nu = -2\not{p}_i$ in the first line and $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$ from the second to the third line. Only the last term out of the four terms in the second line gives a contribution.

For all the other terms we just provide the final results:

$$\begin{aligned}
\frac{1}{u^2}: &\quad 2g^4\frac{t}{u}\\
\frac{1}{tu}: &\quad 0\\
\frac{1}{s^2}: &\quad 8g^4\frac{s^2-t^2-u^2}{s^2}\\
\frac{1}{st}: &\quad 8g^4\frac{u}{s}\\
\frac{1}{su}: &\quad 8g^4\frac{t}{s}
\end{aligned}$$

Summing up all terms yields

$$\overline{\sum}_{\text{pol}}|M^{(-)}|^2 = 2g^4\left[\frac{u}{t}+\frac{t}{u}+4\frac{s^2-t^2-u^2}{s^2}+4\frac{u+t}{s}\right] = 2g^4\left[\frac{u}{t}+\frac{t}{u}-\frac{t^2+u^2}{s^2}\right].$$

Very similar is the calculation of the polarisation sum $\overline{\sum}_{\text{pol}}|M^{(+)}|^2$, which yields

$$\overline{\sum}_{\text{pol}}|M^{(+)}|^2 = 2g^4\left[\frac{u}{t}+\frac{t}{u}\right].$$

We combine all results including the color factors and the color average, which results in

$$\begin{aligned}
\overline{\sum}|M|^2 &= \frac{2g^4}{9}\left[(c_+ + c_-)\left(\frac{u}{t}+\frac{t}{u}\right) - 4c_-\frac{u^2+t^2}{s^2}\right]\\
&= g^4\left(\frac{32}{27}\frac{u^2+t^2}{ut} - \frac{8}{3}\frac{u^2+t^2}{s^2}\right).
\end{aligned}$$

(d) Lastly we want to deduce the differential cross section as a function of the scattering angle θ. We consider massless particles. Being in the center-of-mass frame we therefore obtain for the initial-state momenta

$$p_1^\mu = (\sqrt{s}/2, 0, 0, \sqrt{s}/2), \qquad p_2^\mu = (\sqrt{s}/2, 0, 0, -\sqrt{s}/2).$$

The outgoing momenta equally share the energy and the angle between $\vec{p}_1$ and $\vec{k}_1$ is denoted θ, i.e. $\vec{p}_1\cdot\vec{k}_1 = \frac{s}{4}\cos\theta$. Therefore $p_1\cdot k_1 = p_1^\mu k_{1\mu} = \frac{s}{4} - \vec{p}_1\cdot\vec{k}_1 = \frac{s}{4} - \frac{s}{4}\cos\theta$. We can thus rewrite t and u to be

$$\begin{aligned}
t &= -2p_1\cdot k_1 = -2\frac{s}{4}(1-\cos\theta) = -\frac{s}{2}(1-\cos\theta)\\
u &= -2p_2\cdot k_1 = -2\frac{s}{4}(1+\cos\theta) = -\frac{s}{2}(1+\cos\theta).
\end{aligned}$$

With these definitions we get

$$\begin{aligned}
u^2+t^2 = \frac{s^2}{4}\left[(1-\cos\theta)^2+(1+\cos\theta)^2\right] = \frac{s^2}{4}(2+2\cos^2\theta) &= \frac{s^2}{2}(1+\cos^2\theta)\\
ut = \frac{s^2}{4}(1+\cos\theta)(1-\cos\theta) &= \frac{s^2}{4}\sin^2\theta.
\end{aligned}$$

Our squared amplitude therefore turns into

$$\overline{\sum}|M|^2 = g^4 \left(\frac{64}{27} \frac{1 + \cos^2\theta}{\sin^2\theta} - \frac{4}{3}(1 + \cos^2\theta) \right) .$$

Finally we use the previous exercise to also add the two-particle phase space. For massless particles in the final state the Källen function turns into $\lambda(q^2, 0, 0) = q^2$, such that only the prefactor $\frac{1}{32\pi^2}$ remains, i.e. $\int \mathrm{d}\Phi_2 = \frac{1}{32\pi^2} d\cos\theta d\phi$. The cross section is thus given by

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2}} S d\cos\theta d\phi \frac{1}{32\pi^2} \overline{\sum}|M|^2 .$$

The factor S is a symmetry factor, which is $S = \frac{1}{2}$ due to the two identical gluons in the final state. Since our squared amplitude does not exhibit a dependence on the polar angle ϕ, we can perform its integration, which yields 2π. Due to $4p_1 \cdot p_2 = 2s$ we obtain

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \frac{1}{2s}\frac{1}{2}\frac{1}{16\pi}\overline{\sum}|M|^2 \\ &= \frac{\pi}{4s}\frac{g^4}{(4\pi)^2}\left[\frac{64}{27}\frac{1+\cos^2\theta}{\sin^2\theta} - \frac{4}{3}(1+\cos^2\theta)\right] \\ &= \frac{\pi}{3s}\alpha_s^2\left[\frac{16}{9\sin^2\theta} - 1\right]\left[1+\cos^2\theta\right] . \end{aligned}$$

This result is actually divergent for $\theta = 0$ or π. This is not surprising, since in the latter case the gluons travel in the direction of the initial state quarks. Such contributions are intrinsically part of the parton distribution functions, where they are resummed.
Note: The webpage allows to download a Mathematica Notebook with a calculation of the squared amplitude including all color, polarisation and helicity sums using `FeynArts` and `FormCalc`. Since therein the amplitude is not so nicely split up into $M^{(\pm)}$ the gluon polarisation sum also includes the unphysical vectors η_3 and η_4. Generally the dependence on η_i has to cancel analytically and numerically, but this is often not obvious. A clever replacement of the two vectors in `FormCalc` avoids to introduce new vectors and yields exactly the same result as the previous subexercise.