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Exercise 1: Tensor integrals

In the lecture you considered the loop integral

$$I_d(q,a) = \mu^{4-d}\int \frac{d^dp}{(2\pi)^d}\frac{f(p)}{[-p^2+2p\cdot q+M^2]^a} = \mu^{4-d}\int \frac{d^dp}{(2\pi)^d}\frac{f(p)}{[-p^2+q^2+M^2]^a} \tag{1}$$

for $f(p)=1$, for which you obtained

$$I_d(q,a) = \frac{i}{16\pi^2}(4\pi\mu^2)^{\frac{4-d}{2}}\frac{\Gamma(a-\frac{d}{2})}{\Gamma(a)}(q^2+M^2)^{\frac{d}{2}-a}\,.$$

(a) Determine the loop integrals $I_d^\mu(q,a)$ and $I_d^{\mu\nu}(q,a)$, which you obtain by setting $f(p)=p^\mu$ and $f(p)=p^\mu p^\nu$ in the numerator, respectively. For this purpose differentiate with respect to q_μ.
Hint: Since after integration over p^μ the only remaining four vector is q^μ, the final result for I_d^μ has to be proportional to q^μ. For $I_d^{\mu\nu}$ instead both a term proportional to $g^{\mu\nu}$ and a term proportional to $q^\mu q^\nu$ is possible. Perform shifts in a by ±1.

(b) The contraction of $I_d^{\mu\nu}(q,a)$ with the metric tensor $g_{\mu\nu}$ in d dimensions yields a relation between the three previously discussed loop integrals, namely

$$g_{\mu\nu}I_d^{\mu\nu}(q,a) = -I_d(q,a-1) + 2q_\mu I_d^\mu(q,a) + M^2 I_d(q,a)\,.$$

Prove this relation by using the original definition of the three integrals. Verify the relation with your explicit results from the previous subexercise and thus confirm, that the trace of the metric tensor in d dimensions is given by

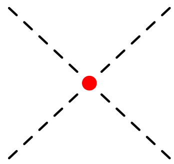
$$g^\mu{}_\mu = d\,.$$

Exercise 2: One-loop calculation in ϕ^4 theory - Part 1

We want to learn how to perform a one-loop calculation, where we regularize our loop integrals through dimensional regularisation and absorb remaining divergences in the renormalization of the underlying parameters. For simplicity we will carry out this calculation in a toy model being ϕ^4 theory. We start with the calculation of the relevant one-loop Feynman diagrams that appear in the calculation of the one-loop propagator and the one-loop vertex, which can all be mapped onto the previously discussed integral $I_d(0,a,M^2)$, where we added the argument M^2

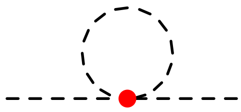
in contrast to the previous exercise.[3] The relevant Feynman rules are

$$= \frac{i}{q^2 - m^2 + i\epsilon},$$



$$= -i\lambda\,.$$

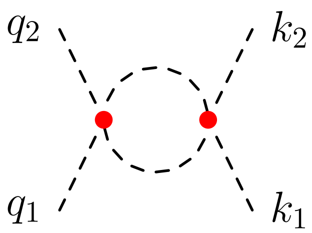
(a) Show that the one-loop correction to the propagator is given by



$$= -\frac{\lambda}{2} I_d(0, 1, m^2)\,.$$

Hint: Try to motivate the symmetry factor $\frac{1}{2}$ of the Feynman diagram. You may have a look in your favorite text book.

(b) Show that the "s-channel" correction to the four-point interaction, being the Feynman diagram



$$= (-i\lambda)^2 iV(s) \quad \text{with} \quad s = (q_1 + q_2)^2\,,$$

is given by

$$V(q^2) = \frac{i}{2}\int_0^1 dx I_d(0, 2, m^2 - x(1-x)q^2)\,.$$

Hint: You need to introduce the Feynman parametrization discussed in the lecture. Shift the loop momentum p wisely. Also here motivate the symmetry factor $\frac{1}{2}$.

(c) Draw the other two one-loop corrections to the four-point interaction and show that their sum is given by $(-i\lambda)^2\,[iV(s) + iV(t) + iV(u)]$.

[3]For the scalar integral $I_d(q, a, M^2)$ it is clear from the last denominator in Eq. 1, that contributions can be shifted between q^2 and M^2. This is not the case for the tensor integrals, which is why we keep separated arguments q and M^2.