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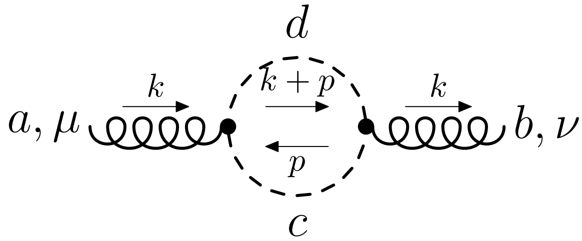
Exercise 1: One-loop ghost contribution to the gluon propagator

Calculate the ghost contribution $\omega_{\mu\nu}^{ab}(k)$ to the vacuum polarisation tensor of the gluon up to order $\mathcal{O}(\alpha_s)$ in a general SU(N) gauge theory. The corresponding Feynman diagram is given by



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(a) Construct $\omega_{\mu\nu}^{ab}$ and show that the color factor is given by $N\delta^{ab}$.

(b) Express your result with the help of the master integrals worked out on sheet 7.
Hint: Use the Feynman parametrization and shift the integration momentum such that linear terms in the denominator vanish. A useful relation might be

$$\int_0^1 dx x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,.$$

(c) Expand the result in ϵ with $d = 4 - 2\epsilon$ and neglect terms proportional to $\mathcal{O}(\epsilon)$.
Hint: Expanding the Γ function yields $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \frac{1}{12}(6\gamma_E^2 + \pi^2)\epsilon$ and $\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E$.
Your result for the ghost contribution should be

$$\omega_{\mu\nu}^{ab}(k) = i\frac{g^2}{32\pi^2}N\delta^{ab}\left\{\left(\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2}{-k^2}\right)\right)\left(\frac{1}{3}k_\mu k_\nu + \frac{1}{6}g_{\mu\nu}k^2\right) + \frac{5}{9}k_\mu k_\nu + \frac{4}{9}g_{\mu\nu}k^2\right\}\,.$$

(d) Show that the sum with the gluon contribution from the lecture only leaves transverse terms in the vacuum polarisation tensor, i.e. terms proportional to $(k^2 g_{\mu\nu} - k_\mu k_\nu)$.

Exercise 2: Optical theorem in ϕ^4 theory

On exercise sheet 7 we considered one-loop corrections to the self-interaction in ϕ^4 theory. For the scattering amplitude we identified three contributing Feynman diagrams, which sum up to $i\mathcal{M} = (-i\lambda)^2[iV(s) + iV(t) + iV(u)]$ using

$$V(q^2) = \frac{i}{2}\int_0^1 dx I_d(0, 2, m^2 - x(1-x)q^2)\,.$$

For this example we prove the optical theorem, which claims that $\operatorname{Im}\mathcal{M} = 2\sqrt{s}p_{\rm cm}\sigma(\phi\phi \to \phi\phi)$. Therein $\sigma(\phi\phi \to \phi\phi)$ is the tree-level cross section of the scattering of two massive scalars into two massive scalars. $\sqrt{s}$ is again the total center-of-mass energy and $p_{\rm cm}$ is the absolute value of the three-momentum of either particle in the center-of-mass frame.

(a) Express the scalar integral in $V(q^2)$ with the help of the master integrals worked out on sheet 7 and expand in ϵ. You should obtain

$$V(q^2) = \frac{i}{2}\int_0^1 dx \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)\right]. \qquad (1)$$

Hint: Use the expansion of Γ functions provided for the previous exercise.

(b) Perform the integration over x, e.g. by using Mathematica. You will encounter factors of $\sqrt{4m^2 - q^2}$, which develop an imaginary part for $q^2 > 4m^2$. Then $V(q^2)$ develops a branch cut for $q^2 \in [4m^2, \infty]$. This is intrinsically related to internal on-shell particles. Explain why only $V(s)$, but not $V(t)$ and $V(u)$, develops an imaginary part.
Hint: Which values do the Mandelstam variables t and u take?

(c) In order to identify the imaginary part of $\mathcal{M}$ go back to Eq. 1 and investigate, for which values of x the logarithm $\log(m^2 - x(1-x)q^2)$ has a negative argument. Replace $\operatorname{Im}\log(-y \pm i\epsilon) = \pm\pi$ for $y > 0$ and perform the integration over x only within the restricted interval of negative arguments. You should obtain

$$\operatorname{Im}\mathcal{M} = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{s}}\,.$$

(d) Prove the optical theorem: Show that the previous equation equals the cross section $\sigma(\phi\phi \to \phi\phi)$ multiplied with $2\sqrt{s}p_{\text{cm}}$. *Hint:* Work in the center-of-mass frame, where you get $p_{\text{cm}} = \sqrt{\frac{s}{4} - m^2}$ for the absolute value of the three-momentum $\vec{p}_{\text{cm}}$.