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Exercise 1: Tensor integrals

While performing higher-order calculations we have to integrate over internal momenta. In the lecture you first considered the simplest case, the scalar integral A. We want to evaluate a very similar integral in d dimensions, but with a higher power in the denominator. The calculation in d dimensions allows to make the ultraviolet-divergent pieces of the integrals explicit, such that they can be reabsorbed into the definition of the parameter of the theory (renormalization).

(a) Show that the integral

$$I_d(q,a) = \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{f(p)}{[-p^2 + 2p\cdot q + M^2]^a} \tag{1}$$

for $f(p) = 1$ is given by

$$I_d(q,a) = \frac{i}{16\pi^2} (4\pi\mu^2)^{\frac{4-d}{2}} \frac{\Gamma(a - \frac{d}{2})}{\Gamma(a)} (q^2 + M^2)^{\frac{d}{2}-a} .$$

Therein a is a positive integer, μ is the renormalization scale, p is the momentum to be integrated over, d is the number of dimensions, M is the mass of the internal particle and q is another external momentum. *Hint:* Follow the calculation of A in the lecture. For $f(p) = 1$ the integral can be shifted in $p \to p + q$ to read

$$I_d(q,a) = \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\left[-p^2 + \tilde{M}^2\right]^a} \quad \text{with} \quad \tilde{M}^2 = q^2 + M^2 .$$

After a substitution of the Euclidean momentum you may use the following identity

$$\int_0^\infty dt t^{x-1} (1+t)^{-x-y} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} .$$

(b) Determine the loop integrals $I_d^\mu(q,a)$ and $I_d^{\mu\nu}(q,a)$, which you obtain by setting $f(p) = p^\mu$ and $f(p) = p^\mu p^\nu$ in the numerator in Eq. 1, respectively. For this purpose differentiate with respect to q_μ. *Hint:* Since after integration over p^μ the only remaining four vector is q^μ, the final result for I_d^μ has to be proportional to q^μ. For $I_d^{\mu\nu}$ both a term proportional to $g^{\mu\nu}$ and a term proportional to $q^\mu q^\nu$ is possible. Perform shifts in a by ± 1.

(c) The contraction of $I_d^{\mu\nu}(q,a)$ with the metric tensor $g_{\mu\nu}$ in d dimensions yields a relation between the three previously discussed loop integrals, namely

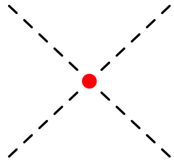
$$g_{\mu\nu} I_d^{\mu\nu}(q,a) = -I_d(q,a-1) + 2q_\mu I_d^\mu(q,a) + M^2 I_d(q,a) .$$

Prove this relation by using the original definition of the three integrals. Verify the relation with your explicit results from the previous subexercise and thus confirm, that the trace of the metric tensor in d dimensions is given by $g^\mu{}_\mu = d$.

Exercise 2: One-loop calculation in ϕ^4 theory - Part 1

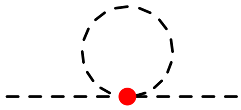
We want to learn how to perform a one-loop calculation, where we regularize our loop integrals through dimensional regularisation and absorb remaining divergences in the renormalization of the underlying parameters. For simplicity we will carry out this calculation in a toy model being ϕ^4 theory. We start with the calculation of the relevant one-loop Feynman diagrams that appear in the calculation of the one-loop propagator and the one-loop vertex, which can all be mapped onto the previously discussed integral $I_d(0, a, M^2)$, where we added the argument M^2 in contrast to the previous exercise.[2] The relevant Feynman rules are

$$= \frac{i}{q^2 - m^2 + i\epsilon},$$



$$= -i\lambda\,.$$

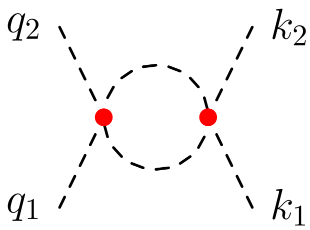
(a) Show that the one-loop correction to the propagator is given by



$$= -\frac{\lambda}{2} I_d(0, 1, m^2)\,.$$

Hint: Try to motivate the symmetry factor $\frac{1}{2}$ of the Feynman diagram. You may have a look in your favorite text book.

(b) Show that the "s-channel" correction to the four-point interaction, being the Feynman diagram



$$= (-i\lambda)^2 iV(s) \quad \text{with} \quad s = (q_1 + q_2)^2\,,$$

is given by

$$V(q^2) = \frac{i}{2} \int_0^1 dx I_d(0, 2, m^2 - x(1-x)q^2)\,.$$

Hint: Perform a Feynman parametrization in the form

$$\frac{1}{A \cdot B} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}\,.$$

We will discuss this parametrization in more detail later. Shift the loop momentum p wisely. Also here motivate the symmetry factor $\frac{1}{2}$.

(c) Draw the other two one-loop corrections to the four-point interaction and show that their sum is given by $(-i\lambda)^2 [iV(s) + iV(t) + iV(u)]$.

We wish you a Merry Christmas, some relaxing days and a Happy New Year!

[2] For the scalar integral $I_d(q, a, M^2)$ it is clear from the denominator in Eq. 1, that contributions can be shifted between q^2 and M^2. This is not the case for the tensor integrals, which is why we keep separated arguments q and M^2.