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Exercise 1: Optical theorem in ϕ^4 theory

On exercise sheet 10 we considered one-loop corrections to the self-interaction in ϕ^4 theory. For the scattering amplitude we identified three contributing Feynman diagrams, which sum up to $i\mathcal{M} = (-i\lambda)^2[iV(s) + iV(t) + iV(u)]$ using

$$V(q^2) = \frac{i}{2}\int_0^1 dx I_d(0, 2, m^2 - x(1-x)q^2)\,.$$

For this example we prove the optical theorem, which claims that $\mathrm{Im}\,\mathcal{M} = 2\sqrt{s}p_{\mathrm{cm}}\sigma(\phi\phi \to \phi\phi)$. Therein $\sigma(\phi\phi \to \phi\phi)$ is the tree-level cross section of the scattering of two massive scalars into two massive scalars. $\sqrt{s}$ is again the total center-of-mass energy and p_{cm} is the absolute value of the three-momentum of either particle in the center-of-mass frame.

(a) Express the scalar integral in $V(q^2)$ with the help of the master integrals worked out on sheet 10 and expand in ϵ. You should obtain

$$V(q^2) = \frac{i}{2}\int_0^1 dx \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)\right]\,. \tag{1}$$

Hint: Useful relations are $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \frac{1}{12}(6\gamma_E^2 + \pi^2)\epsilon + \mathcal{O}(\epsilon^2)$, $\Gamma(1+\epsilon) = 1 - \gamma_E\epsilon + \mathcal{O}(\epsilon^2)$, $\Gamma(-1+\epsilon) = -\frac{1}{\epsilon} - 1 + \gamma_E + \frac{1}{12}(-12 + 12\gamma_E - 6\gamma_E^2 - \pi^2)\epsilon + \mathcal{O}(\epsilon^2)$, $x^\epsilon = 1 + \epsilon\log x + \mathcal{O}(\epsilon^2)$.

(b) Perform the integration over x, e.g. by using Mathematica. You will encounter factors of $\sqrt{4m^2 - q^2}$, which develop an imaginary part for $q^2 > 4m^2$. Then $V(q^2)$ develops a branch cut for $q^2 \in [4m^2, \infty]$. This is intrinsically related to internal on-shell particles. Explain why only $V(s)$, but not $V(t)$ and $V(u)$, develops an imaginary part.
Hint: Which values do the Mandelstam variables t and u take?

(c) In order to identify the imaginary part of $\mathcal{M}$ go back to Eq. 1 and investigate, for which values of x the logarithm $\log(m^2 - x(1-x)q^2)$ has a negative argument. Replace $\mathrm{Im}\log(-y \pm i\epsilon) = \pm\pi$ for $y > 0$ and perform the integration over x only within the restricted interval of negative arguments. You should obtain

$$\mathrm{Im}\,\mathcal{M} = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{s}}\,.$$

(d) Prove the optical theorem: Show that the previous equation equals the cross section $\sigma(\phi\phi \to \phi\phi)$ multiplied with $2\sqrt{s}p_{\mathrm{cm}}$. *Hint:* Work in the center-of-mass frame, where you get $p_{\mathrm{cm}} = \sqrt{\frac{s}{4} - m^2}$ for the absolute value of the three-momentum $\vec{p}_{\mathrm{cm}}$.

Exercise 2: One-loop calculation in ϕ^4 theory - Part 2

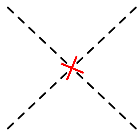
After the calculation of the one-loop corrections on sheet 10 and the exercise on the optical theorem we get to the renormalization of ϕ^4 theory. For this purpose we understand the parameters of the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

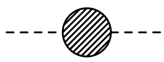
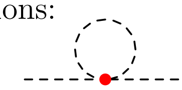
as bare parameters, which is indicated by the additional subscript 0. We introduce the subsequent renormalization constants that map to the physical parameters denoted without subscript

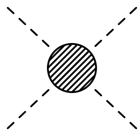
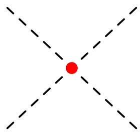
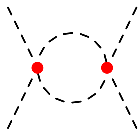
$$\begin{aligned} \phi_0 &= \sqrt{Z}\phi && \text{with} && Z = 1 + \delta Z\,, \\ m_0^2 &= Z_m m^2 && \text{with} && Z_m = 1 + \delta Z_m\,, \\ \lambda_0 &= Z_\lambda \lambda && \text{with} && Z_\lambda = 1 + \delta Z_\lambda\,. \end{aligned}$$

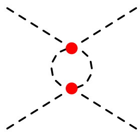
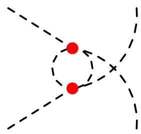
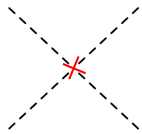
(a) Insert the above relations into the Lagrangian and split it into the physical Lagrangian and the counterterm Lagrangian that contains the renormalization constants linear in either δZ, δZ_m and δZ_λ. Higher-order terms in $\delta Z_{(\lambda,m)}$ are only needed for beyond one-loop calculations. Motivate that the Feynman rules for the counterterms at one-loop level are given by

 $= i((p^2 - m^2)\delta Z - m^2 \delta Z_m)\,,$  $= -i(\delta Z_\lambda + 2\delta Z)\lambda\,.$

(b) Combine the results from previous exercises (sheet 10 and subexercise (a)) to get the analytical form of the subsequent expressions:

$-iM(p^2) =$  $=$  $+$ 

$i\mathcal{M}(q_1 q_2 \to k_1 k_2) =$  $=$  $+$ 

$+$  $+$  $+$ 

Argue from $M(p^2)$ why $\delta Z = 0$ at the one-loop level by checking the momentum-dependence of the expression.

(c) In the previous formulas we can still shift finite parts between the master integrals I_d and the renormalization constants. A renormalization prescription resolves this ambiguity and defines which divergent and finite parts to absorb in the renormalization constants. We start with the simplest case of the minimal subtraction (MS) scheme, where the renormalization constants absorb only the divergent part $\frac{1}{\epsilon}$ (MS) or additionally associated constants $\frac{1}{\epsilon} - \gamma_E + \log(4\pi)$ ($\overline{\text{MS}}$). Expand the master integrals for $d = 4 - 2\epsilon$ in small ϵ and show that in the $\overline{\text{MS}}$ scheme the renormalization constants are given by

$$\delta Z_m^{\overline{\text{MS}}} = \frac{\lambda}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi)\right) \quad \text{and} \quad \delta Z_\lambda^{\overline{\text{MS}}} = \frac{3\lambda}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi)\right)\,.$$

Also provide the results for $-iM(p^2)$ and $i\mathcal{M}(q_1 q_2 \to k_1 k_2)$ in the $\overline{\text{MS}}$ scheme and check the dependence on the renormalization scale μ.
Hint: When expanding in ϵ use the relations for Γ provided in the previous exercise.