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Exercise 1: Dilogarithm

A special function, that very often appears in the calculation of loop diagrams, is the dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} dt = -\int_0^1 \frac{\ln(1-zt)}{t} dt \,.$$

On the main branch, which is determined by the main branch of $\ln(z)$, the dilogarithm $\mathrm{Li}_2(z)$ is a well-defined analytic function for $z \in \mathbb{C}$, except for $z > 1$ on the real axis. For $|z| < 1$ the integral representations can be rewritten in a series of the form $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$. From this series we can determine the value $\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$. To determine $\mathrm{Li}_2(z)$ outside the unit circle, we need transformation rules for analytic continuation. For this purpose prove the following identities:

(a)
$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) - \ln(1-z)\ln z + \frac{\pi^2}{6} \,.$$

(b)
$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) = -\mathrm{Li}_2(z) - \frac{1}{2}\ln^2(1-z) \,.$$

(c)
$$\mathrm{Li}_2\left(\frac{z-1}{z}\right) = \mathrm{Li}_2(z) + \ln(1-z)\ln(z) - \frac{1}{2}\ln^2(z) - \frac{\pi^2}{6} \,.$$

(d)
$$\mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{1}{2}\ln^2(-z) - \frac{\pi^2}{6} \,.$$

(e)
$$\mathrm{Li}_2\left(\frac{1}{1-z}\right) = \mathrm{Li}_2(z) - \frac{1}{2}\ln^2(1-z) + \ln(1-z)\ln(-z) + \frac{\pi^2}{6} \,.$$

Exercise 2: Higgs boson decay into gluons - Part 2

We continue with the exercise on the decay of the Higgs boson in a pair of gluons started on sheet 12. *Hint:* All relevant results are given on sheet 12.

(a) Perform the integrations in the definition of $f(x)$ by using the results obtained for the dilogarithm in the previous exercise. *Hint:* Perform the integration over z and determine the roots of the argument of the remaining logarithm named $y_\pm$, such that $1 - xy + xy^2 = (y_+ - y)(y_- - y)x$. Split the logarithm, use (d) and (e) of the previous exercise and use

$$\arcsin(z) = -i\ln(iz \pm \sqrt{1-z^2}) = -i\ln\left[\left(\frac{\sqrt{z^2-1}+z}{\sqrt{z^2-1}-z}\right)^{\frac{1}{2}}\right] \,.$$

If you succeed, you should get $f(x) = \frac{6}{x} - \frac{6(4-x)}{x^2}\arcsin^2\left(\frac{\sqrt{x}}{2}\right)$ for $x < 4$.

(b) What value does $f(x)$ take for heavy quarks (heavy-top limit), i.e. $m_q \to \infty$ and $m_{h^0}^2/m_q^2 \to 0$? Why does the measurement of the decay of a Higgs boson to gluons or the production of a Higgs boson from gluons allow to make a statement on the number of heavy quark generations?

Exercise 3: Emission of a collinear photon

The focus of this exercise is on the emission of a collinear photon from an initial-state electron and its "absorbtion" into a distribution function associated with the electron. This collinear splitting is also the basis of the parton distribution functions employed at hadron colliders, which map from an initial-state proton to the proton ingredients, the quarks and the gluon. For simplicity we consider a massive gauge boson B with mass M, which couples to the left- and right-handed component of the electron equally, i.e. the Feynman rule of the e^+e^-B coupling is $-ig\gamma^\mu$.[3] Since we couple B to a conserved current, we can also use the simplified polarisation sum $\sum_i \epsilon_\mu^{(i)}\epsilon_\nu^{(i)*} = -g_{\mu\nu}$. In the first subexercise we consider the electron to be massless.

(a) Compute the cross section for $e^+e^- \to B$. You should obtain

$$\sigma(e^+e^- \to B) = \frac{\pi g^2}{2M}\delta(\sqrt{s}-M) = \pi g^2 \delta(s-M^2)\,.$$

(b) Compute the differential cross section for $e^+e^- \to \gamma B$, for which you can draw a t- and a u-channel diagram. The Feynman rule for the photon coupling to the electron is $-ie\gamma^\mu$, as discussed on sheet 13. The result for the squared amplitude should be

$$\frac{1}{4}\sum |\mathcal{M}|^2 = 2e^2g^2\left[\frac{u}{t}+\frac{t}{u}+\frac{2sM^2}{tu}\right]\,.$$

Go to the center-of-mass frame and introduce the scattering angle θ between the initial-state electron and the photon. The result is of the form

$$\frac{d\sigma}{d\cos\theta} = \frac{\alpha g^2(1-M^2/s)}{2s\sin^2\theta}\left[1+\cos\theta^2+\frac{4sM^2}{(s-M^2)^2}\right]\,.$$

(c) The result is divergent in the limit $\theta \to 0$ and $\theta \to \pi$, which are the collinear and anticollinear limit, respectively. These are the regions where the massless photon is close to the incoming massless electron or positron, respectively. We consider the limit $\theta \to 0$ in more detail. We want to rearrange the formula to understand the physical origin of this (infrared) divergence. The divergence can be cut off by an electron mass m: Let the electron momentum be $p^\mu = (E,0,0,\sqrt{E^2-m^2})$, and let the photon momentum carry away a fraction of it, $k^\mu = (xE, xE\sin\theta, 0, xE\cos\theta)$. Determine the Mandelstam variables for this case. The denominator of the propagator then never becomes smaller than $\mathcal{O}(m^2/s)$. Finally integrate the cross section over forward angles only, cutting off the θ integral at $\theta^2 \sim (m^2/s)$ keeping only the logarithmic term proportional to $\log(s/m^2)$. Show that in this approximation the cross section for forward photon emission can be written as

$$\sigma(e^+e^- \to \gamma B) \approx \int dx\, f(x)\sigma(e^+e^- \to B \quad \text{at} \quad E_{\text{cm}}^2 = (1-x)s)\,, \tag{1}$$

where the annihilation cross section is evaluated for the collision of a positron of energy E and an electron of energy $(1-x)E$. The function $f(x)$ is the Weizsäcker-Williams distribution function and given by

$$f(x) = \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\log\left(\frac{s}{m^2}\right)\,.$$

It describes the collinear splitting of a photon from the initial-state electron, independent of the subsequent dynamics. *Hint:* Integrate over x in Eq. 1 by just employing $\delta(M^2-(1-x)s)$ and show that the result equals the one with a limited integration range in θ.

[3] Considering instead a Z boson does not introduce new phenomena.