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Exercise 1: Running coupling in quantum electrodynamics (QED)

The aim of this exercise is to deduce the renormalization-scale dependence of the fine-structure constant $\alpha = \frac{e^2}{4\pi}$ in quantum electrodynamics (QED) in the $\overline{\text{MS}}$ renormalization scheme and thus the running of the coupling as a function of the energy.

(a) Calculate the photon self-energy $\Sigma^{\mu\nu}(k)$ for a photon with momentum k involving a fermion with mass m at one-loop. Express the loop integrals through the standard integrals A_0, B_0, B_1, B_{00} and B_{11} and show that

$$\Sigma^{\mu\nu}(k) = \frac{\alpha}{\pi}\left\{g^{\mu\nu}\left[2B_{00}(k,m,m) - A_0(m) - k^2 B_1(k,m,m)\right] \right. \\ \left. + k^\mu k^\nu\left[2B_{11}(k,m,m) + 2B_1(k,m,m)\right]\right\}\,.$$

(b) Split the self-energy of the photon into a transverse and longitudinal component

$$\Sigma^{\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\Sigma_T(k^2) + \frac{k^\mu k^\nu}{k^2}\Sigma_L(k^2)$$

to show that

$$\Sigma_L(k^2) = 0, \qquad \Sigma_T(k^2) = \frac{\alpha}{3\pi}\left[(k^2+2m^2)B_0(k,m,m) - \frac{1}{3}k^2 - 2m^2 B_0(0,m,m)\right]\,.$$

Use the relations $A_0(m) = m^2 B_0(0,m,m) + m^2$ and

$$\begin{aligned}
B_1(p,m,m) &= -\frac{1}{2}B_0(p,m,m)\\
B_{00}(p,m,m) &\overset{d\to 4}{=} \frac{1}{6}\left[A_0(m) + 2m^2 B_0(p,m,m) + p^2 B_1(p,m,m) + 2m^2 - \frac{p^2}{3}\right]\\
B_{11}(p,m,m) &\overset{d\to 4}{=} \frac{1}{6p^2}\left[2A_0(m) - 2m^2 B_0(p,m,m) - 4p^2 B_1(p,m,m) - 2m^2 + \frac{p^2}{3}\right]\,.
\end{aligned}$$

(c) The vacuum polarisation is defined through $\Pi(k^2) = \Sigma_T(k^2)/k^2$. Calculate $\Pi(0)$. For this purpose expand B_0 in small $|k^2| \ll m^2$ by using

$$B_0(k,m,m) = \Delta - \ln\left(\frac{m^2}{\mu^2}\right) + \frac{k^2}{6m^2} + \mathcal{O}\left(\frac{k^4}{m^4}\right)$$

with $\Delta = \frac{2}{4-d} - \gamma_E + \ln(4\pi) = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$ for $d = 4 - 2\epsilon$.

(d) We finally need the ultraviolet divergence and thus the scale dependence of the fermion-fermion-photon vertex. Show that the relevant ultraviolet contribution is given by

$$\mathcal{M}^\mu = \frac{\alpha}{4\pi}\Delta\cdot\mathcal{M}^\mu_{\text{born}}$$

with $\mathcal{M}^\mu_{\text{born}} = ie\gamma^\mu$. *Hint:* Write down the loop integral with two fermion and one photon propagator, which takes the form

$$\mathcal{M}^\mu = (ie)^3 i^3 \mu^{4-d}\int\frac{d^d l}{(2\pi)^d}\frac{\gamma^\rho(\not{l} + \not{p}_2 + m)\gamma^\mu(\not{l} + \not{p}_1 + m)\gamma^\sigma(-g_{\rho\sigma})}{l^2((l+p_1)^2 - m^2)((l+p_2)^2 - m^2)}\,.$$

The ultraviolet-divergent contribution is the one with two occurences of the loop momentum l in the numerator, i.e. $\gamma^\rho \not{l} \gamma^\mu \not{l} \gamma_\rho$. Why? This leads to the integrals B_0 and C_{00}, which have a divergence parametrised by Δ and $\frac{1}{4}\Delta$, respectively. All other C_{ij} are finite.

(e) Use multiplicative renormalization defined through

$$A_0^\mu = \sqrt{Z_A} A^\mu\,, \quad \psi_0 = \sqrt{Z_\psi}\psi\,, \quad e_0 = \mu^\epsilon Z_e e$$

and the expansion $Z_i = 1 + \delta Z_i$ at one-loop to deduce the counterterm of the fermion-fermion-photon vertex at one-loop.[4]

(f) In the $\overline{\text{MS}}$ renormalization scheme we obtained $\delta Z_A = -\frac{\alpha}{3\pi}\Delta$ from the vacuum polarisation $\Pi(0)$. The result for the fermionic wave-function renormalization constant can be derived in a similar way from the fermionic self-energy. It is given by $\delta Z_\psi = -\frac{\alpha}{4\pi}\Delta$. Use the previous two exercises to obtain δZ_e.

(g) In the previous subexercise we deduced the relation between the bare coupling e_0 and the renormalized coupling e, which is

$$e_0 = \left(1 + \frac{e^2}{24\pi^2}\Delta\right) e\mu^\epsilon\,.$$

Differentiate with respect to μ and thus prove the differential equation $\mu\frac{\partial e}{\partial \mu} = \frac{e^3}{12\pi^2}$. For this purpose expand in small e and derive the limit $\epsilon \to 0$. Integrate the relation to show that

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2}\ln\left(\frac{\mu}{\mu_0}\right)}\,.$$

Hint: You can neglect divergent contributions proportional to $\mathcal{O}(e^5)$, as we do not perform a two-loop calculation taking into account the the corresponding counterterms.
Add-on: In the lecture you obtain a similar expression for the strong-coupling constant α_s, however with a different sign (and factor) in the denominator. Thus, whereas with increasing energy the fine-structure constant α increases and eventually hits a Landau pole, the strong-coupling constant α_s is asymptotically free, i.e. it decreases.

Throughout the exercise you may use the following standard loop integrals

$$\begin{aligned}
A_0(m) &= \frac{16\pi^2\mu^{4-d}}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\\
B_0(p,m_0,m_1) &= \frac{16\pi^2\mu^{4-d}}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-m_0^2][(k+p)^2-m_1^2]}\\
B_\mu(p,m_0,m_1) &= \frac{16\pi^2\mu^{4-d}}{i}\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu}{[k^2-m_0^2][(k+p)^2-m_1^2]} = B_1 p_\mu\\
B_{\mu\nu}(p,m_0,m_1) &= \frac{16\pi^2\mu^{4-d}}{i}\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu}{[k^2-m_0^2][(k+p)^2-m_1^2]} = B_{00}g_{\mu\nu} + B_{11}p_\mu p_\nu\\
C_{\{0;\mu;\mu\nu\}}(p_1,p_2,m_0,m_1,m_2) &= \frac{16\pi^2\mu^{4-d}}{i}\int \frac{d^dk}{(2\pi)^d}\frac{\{1;k_\mu;k_\mu k_\nu\}}{[k^2-m_0^2][(k+p_1)^2-m_1^2][(k+p_2)^2-m_2^2]}
\end{aligned}$$

$$C_\mu = \sum_{i=1}^{2} C_i p_{i\mu}\,, \qquad C_{\mu\nu} = C_{00}g_{\mu\nu} + \sum_{i,j=1}^{2} C_{ij}p_{i\mu}p_{j\nu}\,.$$

[4] The replacement $g \to \mu^\epsilon g$ guarantees the right mass/energy dimension for all terms in the Lagrangian.