Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)
Martin Gabelmann (martin.gabelmann@kit.edu) (Office 12/17 - Build. 30.23)
Jonas Müller (jonas.mueller@kit.edu) (Office 12/17 - Build. 30.23)

Exercise 1: Massive quantum electrodynamics - Stückelberg mechanism

We had shown on the previous exercise sheet that a naive mass term for the photon field A is not gauge invariant. Instead we can add a real scalar field to the Lagrangian, such that

$$\mathcal{L}_{\mathrm{S}} = -\frac{m_S^2}{2}\left(A^\mu + \frac{1}{m_S}\partial^\mu\sigma\right)\left(A_\mu + \frac{1}{m_S}\partial_\mu\sigma\right) .$$

This addition is a special case of the Higgs mechanism, that can be used for Abelian theories. In order to complete the theory we have to add an additional gauge fixing term (see "gauge fixing, Fadeev-Popov trick" later in the lecture), given by

$$\mathcal{L}_{\mathrm{G}} = -\frac{1}{2\xi}\left(\partial^\mu A_\mu + m_S\xi\sigma\right)^2 , \qquad \xi \in \mathbb{R} .$$

(a) How does $\sigma(x)$ need to transform under the gauge transformation provided on the previous sheet, such that $\mathcal{L}_{\mathrm{S}}$ remains gauge invariant?

(b) Show that α needs to fulfill the Proca equation $(\partial^\mu\partial_\mu - \xi m_S^2)\alpha = 0$ in order for $\mathcal{L}_{\mathrm{G}}$ to be gauge invariant.

(c) Add up $\mathcal{L}_{\mathrm{S}}$ and $\mathcal{L}_{\mathrm{G}}$ and expand all terms. Show that terms that mix A^μ and σ vanish as their sum is a total derivative only. *Note:* For Abelian theories gauge fixing allows to set $\sigma = 0$ which results in the full Stückelberg Lagrangian with a massive gauge field

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{m_S^2}{2}A^\mu A_\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 .$$

Exercise 2: Goldstone bosons in O(3)

We consider a model, which consists of a scalar field $\sigma = (\sigma_1, \sigma_2, \sigma_3)^T$, that transforms under the fundamental representation of O(3). The corresponding Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^\dagger(\partial^\mu\sigma) - \lambda(\sigma^2 + \mu^2)^2$$

with real parameters μ^2 and λ.

(a) Calculate the mass spectrum for the cases $\mu^2 > 0$ and $\mu^2 < 0$. For the latter case $\mu^2 < 0$ start with an ansatz $\sigma \to \sigma' + (0,0,v)^T$, i.e. introduce a vacuum expectation value along the third component. What is the remaining symmetry in the latter case? How many Goldstone bosons are part of the model then? *Hint:* The masses m_i can be read off from the Lagrangian directly by identifying the quadratic terms $-\frac{m_i^2}{2}\sigma_i\sigma_i$ and $-\frac{m_i^2}{2}\sigma_i'\sigma_i'$ for the two cases, respectively.

(b) We now “gauge” the model and introduce a gauge interaction with a gauge field W_μ^a ($a = 1, 2, 3$). For this purpose we replace the partial derivative with the covariant derivative in $\mathcal{L}$ through

$$\partial_\mu \to D_\mu = \partial_\mu + igt^a W_\mu^a \,,$$

where $t^a_{kl} = -i\epsilon_{akl}$ are the generators of O(3). Again deduce the masses of scalars and gauge bosons for the two cases $\mu^2 > 0$ and $\mu^2 < 0$. For the latter case expand σ as follows

$$\sigma = \exp\left(\frac{i}{v} t^a \theta^a\right)(\sigma_0 + \eta')$$

with $v = \sqrt{-\mu^2}$, $\sigma_0 = (0, 0, v')^T$, $\eta' = (0, 0, \eta)^T$ and $\theta = (\theta_1, \theta_2, 0)^T$.
Hint: A gauge transformation $\sigma \to e^{it^a\alpha^a}\sigma$ can simplify $\mathcal{L}$. Make an intelligent choice for the three-dimensional α. You also have to transform $W_\mu^a \to W_\mu'^a$, but for simplicity keep the latter in the Lagrangian as you will get the same mass terms.

Exercise 3: Subalgebras of SU(3)

For a Lie algebra $\mathfrak{g}$ the subspace $\mathfrak{h} \subseteq \mathfrak{g}$ forms a Lie subalgebra, if it is closed under the Lie product, i.e. for all $X, Y \in \mathfrak{h}$ it yields $-i[X, Y] \in \mathfrak{h}$. We now consider the Lie algebra of the Lie group SU(3), which in the fundamental representation is for example given by the Gell-Mann matrices as generators

$$\lambda_i = \begin{pmatrix} \multicolumn{2}{c}{\sigma_i} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

where σ_i are the three Pauli matrices.

(a) Choose a constant c in $T_a = c\lambda_a$ such, that T_1, T_2 and T_3 form a fundamental representation of the Lie algebra of the Lie group SU(2). Show that they fulfill the corresponding Lie algebra.

(b) Choose a constant c' in $T_a = c'\lambda_a$ such, that T_2, T_5 and T_7 constitute the generators of the adjoint representation of the Lie group SU(2). When exponentiated, these matrices generate the Lie group SO(3), since SO(3) is the adjoint group of SU(2).