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Exercise 1: Color algebra

We consider the generators $t^{a(R)}$ of SU(N) in the representation R, which obey

$$\left[t^{a(R)}, t^{b(R)}\right] = i f^{abc} t^{c(R)}\,, \quad \mathrm{tr}\left(t^{a(R)} t^{b(R)}\right) = T_R \delta_{ab}\,, \qquad \sum_a (t^{a(R)})^2 = C_R 1_R\,.$$

Therein f^{abc} are the totally antisymmetric structure constants and C_R is the quadratic Casimir operator in the representation R. Casimir operators allow to distinguish different representations.

(a) In "Theoretische Teilchenphysik I" we already showed that in the fundamental representation $R = F$ with generators T^a, which are normed by $T_F = \frac{1}{2}$, it yields

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$$

with the consequence that $T^a_{ik} T^a_{kl} = C_F \delta_{il}$ with $C_F = \frac{N^2-1}{2N}$. Use these results to show that

$$\left(T^a T^b T^a\right)_{il} = -\frac{1}{2N} T^b_{il} \qquad \text{and} \qquad \mathrm{tr}\left(T^a T^b T^c\right) = \frac{1}{4}(d_{abc} + i f_{abc})$$

with d_{abc} defined through $\{T^a, T^b\} = \frac{1}{N}\delta_{ab} 1_{N\times N} + d_{abc} T^c$, i.e. $d_{abc} = 2\mathrm{tr}(\{T^a, T^b\} T^c)$, such that d_{abc} is a totally symmetric tensor.

(b) Use the Jacobi identity for mixed commutation and anti-commutation relations

$$[A, \{B, C\}] + [B, \{C, A\}] + [C, \{A, B\}] = 0$$

to obtain a relation connecting f_{abc} and d_{abc} by inserting the generators in the fundamental representation. Use this result to show that

$$C_{3F} = d_{abc} T^a T^b T^c$$

is a Casimir invariant, i.e. $[C_{3F}, T^a] = 0$ for all generators T^a.[1]

(c) The generators in the adjoint representation are given by the structure constants themselves, i.e. $(F^a)_{bc} = -i f_{abc}$. Prove that $T_A = C_A = N$ by showing that $\mathrm{tr}(F^a F^b) = N\delta_{ab}$. *Hint:* Show that $\mathrm{tr}(F^a F^b) = -2\mathrm{tr}([T^a, T^c][T^b, T^c])$ and then use the previous results.

[1] Defining the $\overline{N}$ representation through generators $-T^{a*}$ yields a fundamental representation, under which anti-quarks transform. They differ from the fundamental N representation in the cubic Casimir as $C_{3N} = -C_{3\overline{N}}$.

Exercise 2: Higgs boson decays at tree level

In this exercise we want to discuss all $1 \to 2$ Higgs boson decays in the Standard Model of particle physics, which emerge at tree-level. The general formula for the decays is

$$d\Gamma(H \to XY) = \frac{1}{2m_H} d\Phi_2 \overline{\sum}_{\text{spins, colors}} |\mathcal{M}_{H\to XY}|^2 \,,$$

where $d\Phi_2$ is again the two-particle phase space as printed on the first exercise sheet and for $X = Y$ an additional symmetry factor $\frac{1}{2}$ has to be added. In TTP1 you already determined the decay width into fermions, which was found to be

$$\Gamma(H \to f\bar{f}) = N_c \frac{G_F m_f^2 m_H}{4\sqrt{2}\pi} \left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2}$$

with $G_F = \frac{1}{\sqrt{2}v^2} = 1.16638 \cdot 10^{-5}\,\text{GeV}$ and a color factor N_c, which is 3(1) for quarks(leptons).

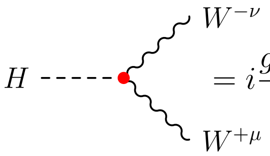
(a) Calculate the partial decay widths into a pair of gauge bosons, i.e. $\Gamma(H \to W^+W^-)$ and $\Gamma(H \to ZZ)$. Show that the result can be written in the form

$$\Gamma(H \to W^+W^-) = \frac{G_F m_H^3}{8\sqrt{2}\pi} \left(1 - \frac{4m_W^2}{m_H^2}\right)^{1/2} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right) .$$

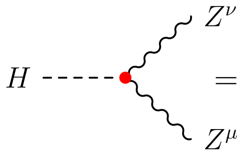
For the decays into a pair of Z bosons an additional factor of $\frac{1}{2}$ has to be added.
Hint: Make use of the polarisation sum for massive gauge bosons with mass m_V, being

$$\sum_{\lambda=1}^{3} \epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_V^2} \,.$$

The relevant Feynman rules are given by



$$= i\frac{g^2 v}{2} g^{\mu\nu} \,,$$



$$= i\frac{g^2 v}{2c_W^2} g^{\mu\nu} \,,$$

where we can identify $m_W = \frac{1}{2}gv$ and $m_Z = \frac{1}{2}\sqrt{g^2 + g'^2}v = \frac{m_W}{c_W}$ with the cosine of the weak mixing angle $c_W = \frac{m_W}{m_Z}$.

(b) For each final state $H \to XY$ we define its branching ratio through the ratio of the partial width through the total width of the Higgs boson, i.e.

$$\text{BR}(H \to XY) = \frac{\Gamma(H \to XY)}{\sum_{AB} \Gamma(H \to AB)} \,.$$

Consider the decays into the two massive gauge bosons as well as into the four heaviest fermions of the Standard Model, which have masses

Gauge bosons :	$m_W = 80.4\,\text{GeV}$,	$m_Z = 91.2\,\text{GeV}$	
Quarks with $N_c = 3$:	$m_t = 173.2\,\text{GeV}$,	$m_b = 4.8\,\text{GeV}$,	$m_c = 1.3\,\text{GeV}$
Leptons with $N_c = 1$:	$m_\tau = 1.8\,\text{GeV}$.		

Make a graphic, that shows the branching ratios of the Higgs boson as a function of the Higgs boson mass m_H between 10 and 1000 GeV.