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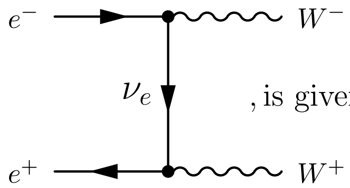
Exercise 1: W **pair production in the high-energy limit**

We consider the scattering process

$$e^-(p_1)e^+(p_2) \to W^-(p_3)W^+(p_4)$$

in the high-energy limit with $s := k^2 \gg m_W^2$, with $k = p_1 + p_2 = p_3 + p_4$. The gauge-boson masses cannot be neglected, in contrast to the fermion masses, which we set to zero.

(a) Assume that there is no three gauge-boson self interaction in the Standard Model. Show that in leading order in $\frac{m_W^2}{s}$, i.e. $m_W^2 \ll s, t, u$, the averaged squared amplitude, which only emerges from a t-channel neutrino exchange depicted by the Feynman diagram



, is given by $\overline{\sum}|\mathcal{M}|^2 \approx -\frac{e^4}{16 s_W^4 m_W^4} t(s+t) = \frac{\alpha^2\pi^2}{4 s_W^4}\frac{s^2}{m_W^4}\left(1-\cos^2\theta\right)$,

where s and $t = (p_1 - p_3)^2 = m_W^2 - 2p_1 \cdot p_3 = (p_2 - p_4)^2 = m_W^2 - 2p_2 \cdot p_4$ are Mandelstam variables and θ denotes the scattering angle between the incoming electron e^- and the outgoing W^- in the center-of-mass frame. The sine of the weak mixing angle is $s_W := \sin\theta_W$ and the fine structure constant is defined by $\alpha = \frac{e^2}{4\pi}$.

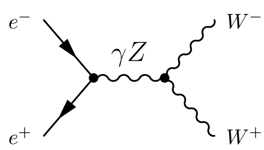
Hint: Argue why the polarisation sum of the W bosons can be approximated by $\sum_\lambda \epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) \approx \frac{p_\mu p_\nu}{m_W^2}$. Neglect all non-leading terms in $\frac{m_W^2}{s}$ in the calculation of the trace. A collection of relevant Feynman rules is given at the end of this exercise.

(b) Determine the total cross section in leading order in $\frac{m_W^2}{s}$ in the center-of-mass system of the incoming particles, based on the calculation of the squared amplitude in the previous exercise. What happens to the cross section in the high-energy limit, i.e. $s \to \infty$?

(c) The correct high-energy behaviour is obtained, when the three gauge-boson self-interaction is added. The calculation of the total cross section for this case is, however, quite lengthy, such that we consider the high-energy limit only for one helicity combination. Examine the amplitude for $e_R^- e_L^+ \to W_L^- W_L^+$ for an incoming right-handed electron and left-handed positron and two outgoing longitudinally polarised W bosons and show that in leading order in $\frac{m_W^2}{s}$ the sum of all diagrams yields

$$\mathcal{M}(e_R^- e_L^+ \to W_L^- W_L^+) \approx \frac{ie^2}{2c_W^2}\frac{1}{s}\overline{v}_R(p_2)(\not{p}_4 - \not{p}_3)u_R(p_1)\,.$$

Hint: Use the equation of motions for massless fermions being $\overline{v}(p_2)\not{p}_2 = 0$ and $\not{p}_1 u(p_1) = 0$ as well as the approximation, that the polarisation vector of the longitudinal W boson in the high-energy limit is given by $\epsilon_L^\mu(p) \approx \frac{p^\mu}{m_W}$. If you failed in (a) or (b), you can also restart here.



(d) Determine the total cross section for $e_R^- e_L^+ \to W_L^- W_L^+$ and examine the behaviour in the high-energy limit, i.e. $s \to \infty$.

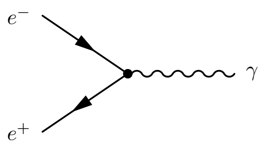
(e) The Goldstone-boson equivalence theorem states that the amplitudes of longitudinally polarised gauge bosons in the high-energy limit equal the amplitudes, in which the gauge bosons are replaced by their Goldstone bosons (except from an unobservable phase). Show the equivalence of

$$\mathcal{M}(e_R^- e_L^+ \to \phi^- \phi^+) = \mathcal{M}(e_R^- e_L^+ \to W_L^- W_L^+)$$

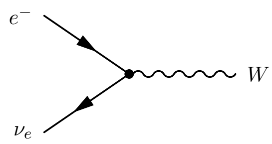
in leading order in $\frac{m_W^2}{s}$.

(f) Determine the amplitude for $e_L^- e_R^+ \to W_L^- W_L^+$ for an incoming left-handed electron and right-handed positron and two outgoing longitudinally polarised W bosons using the Goldstone-boson equivalence theorem. Lastly obtain the total cross section for $e^- e^+ \to W_L^- W_L^+$, which is the sum of $\sigma(e_R^- e_L^+ \to W_L^- W_L^+)$ and $\sigma(e_L^- e_R^+ \to W_L^- W_L^+)$, as other helicity combinations vanish and no interference terms appear.

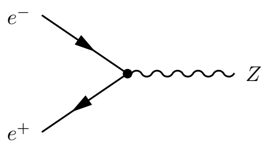
Relevant Feynman rules for the interactions are given by the following expressions:



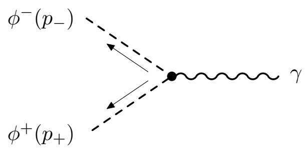
$= -ie\gamma^\mu(P_L + P_R)$



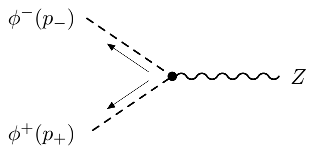
$= \frac{ie}{\sqrt{2}s_W}\gamma^\mu P_L$



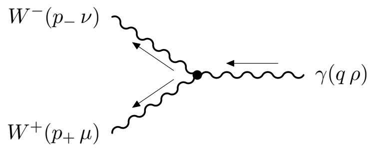
$= \frac{ie}{s_W c_W}\gamma^\mu \left((-\frac{1}{2} + s_W^2)P_L + (s_W^2)P_R\right)$



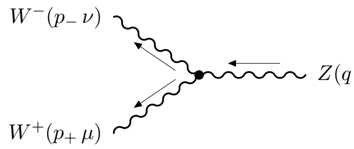
$= ie(p_- - p_+)^\mu$



$= \frac{ie(\frac{1}{2} - s_W^2)}{c_W s_W}(p_- - p_+)^\mu$



$= ief^{\mu\nu\rho}$



$= \frac{iec_W}{s_W}f^{\mu\nu\rho}$

Therein the momentum flow is indicated through the additional arrows. The left- and right-handed projection operators are given by $P_L = \frac{1-\gamma_5}{2}$ and $P_R = \frac{1+\gamma_5}{2}$. Moreover it yields $f^{\mu\nu\rho} = g^{\mu\nu}(p_- - p_+)^\rho + g^{\nu\rho}(-q - p_-)^\mu + g^{\rho\mu}(q + p_+)^\nu$. Again s_W and c_W are defined through $\sin\theta_W$ and $\cos\theta_W$, respectively. The propagators of the photon and the Z boson in Feynman gauge are given by

$$\frac{-ig_{\mu\nu}}{k^2} \quad \text{and} \quad \frac{-ig_{\mu\nu}}{k^2 - m_Z^2} \quad \text{, respectively.}$$

Goldstone bosons do not couple to massless fermions.

Solution of exercise 1

Before going into the details of this exercise you may elaborate on the polarisation vectors and the polarisation vector sum of massive gauge bosons, as there was some confusion in the lecture, apparently. I copy the results of an exercise on sheet 8 from TTP1:

Exercise: We introduce a Fourier decomposition for the massive vector boson in analogy to the photon field, i.e.

$$A^\mu(x) = \int d\tilde{k} \sum_{r=0}^{3} \left(\varepsilon_r^\mu(k) a_r(k) e^{-ik\cdot x} + \varepsilon_r^{\mu *}(k) a_r^\dagger(k) e^{ik\cdot x} \right) .$$

A priori, this includes four polarization vectors $\varepsilon_r^\mu(k)$. Due to $\partial_\mu A^\mu = 0$ and thus $\sum_r k_\mu \varepsilon_r^\mu(k) a_r(k) = 0$ only three polarization vectors are physical. Show that a convenient basis for these polarization vectors, in the reference frame with $\vec{k} = (0, 0, |\vec{k}|)$, is given by

$$\varepsilon_1 = (0, 1, 0, 0) , \qquad \varepsilon_2 = (0, 0, 1, 0) , \qquad \varepsilon_3 = \frac{1}{m}(|\vec{k}|, 0, 0, \omega_k)$$

for the three physical polarization vectors, which are orthogonal to the unphysical polarization vector $\varepsilon_0^\mu = k^\mu/m$. The physical polarization vectors obey the orthonormality condition $\varepsilon_r^\mu(k)\varepsilon_{\mu s}^*(k) = -\delta_{rs}$. Using these explicit expressions show that the completeness relation of the physical polarization vectors reads

$$\sum_{r=1}^{3} \varepsilon_r^\mu(k) \varepsilon_r^{\nu *}(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} .$$

Hint: In the rest frame of the particle, $k' = (m, 0, 0, 0)$, we e.g. choose $\varepsilon_0' = (1, \vec{0})$ and the three unit vectors $\varepsilon_i' = (0, \vec{e}_i)$. Then the vectors ε_i' automatically fulfill $k'_\mu \varepsilon_i'^\mu = 0$. Boost from the rest frame into the above reference frame. *Add-on:* We showed the completeness relation in a special frame, but it is actually Lorentz-covariant.

Solution: Since A^μ is a real massive field which satisifes a Klein-Gordon equation, the solution has the same form of the one for a real scalar field (and the one of the photon), i.e.

$$A^\mu(x) = \int d\tilde{k} \sum_{r=0}^{3} \left(\varepsilon_r^\mu(k) a_r(k) e^{-ik\cdot x} + \varepsilon_r^{\mu *}(k) a_r^\dagger(k) e^{ik\cdot x} \right) .$$

Therein we decomposed the vector boson $A^\mu(x)$ onto a basis of polarization vectors $\varepsilon_r^\mu(k)$. We choose the basis such that the following orthonormality condition is satisifed:

$$\varepsilon_r^\mu(k) \varepsilon_{\mu s}^* = g_{rs} ,$$

which for $r = s = \{1, 2, 3\}$ results in $-\delta_{rs}$. An example of such a basis is given in the exercise and reads

$$\varepsilon_0'(k) = (1, \vec{0}) , \qquad \varepsilon_i'(k) = (0, \vec{e}_i) , i = \{1, 2, 3\} ,$$

where $\vec{e}_i$ is the set of the three unit vectors along the directions of the euclidean space. Imposing $\partial_\mu A^\mu = 0$ we obtain the relation $\sum_r k'_\mu \varepsilon_r'^\mu(k') a_r(k') = 0$, which implies that only three of the four polarization vectors are physical. If the vector boson is at rest, we can write the momentum as $k'^\mu = (m, 0, 0, 0)$, from which we obtain

$$k'^\mu \varepsilon_{0\mu} \neq 0 \qquad k'^\mu \varepsilon_{i\mu} = 0\,,$$

such that we discard the non-physical polarization vector ε_0, i.e. in practice the Fourier coefficients have to vanish. Thus in practice we get

$$A^\mu(x) = \int d\tilde{k} \sum_{r=0}^{3} \left(\varepsilon_r^\mu(k) a_r(k) e^{-ik\cdot x} + \varepsilon_r^{\mu *}(k) a_r^\dagger(k) e^{ik\cdot x} \right) .$$

We now perform a boost along the z direction by applying

$$k^\mu = \Lambda^\mu_\nu k'^\nu = (\omega_k, 0, 0, |\vec{k}|)\,,$$

for which the boost takes the form

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \qquad \gamma = \frac{\omega_k}{m}\,, \qquad \beta = \frac{|\vec{k}|}{\omega_k}\,.$$

As the boost is orthogonal to $\vec{e}_1$ and $\vec{e}_2$ the two polarization vectors $\varepsilon_1 = \varepsilon'_1$ and $\varepsilon_2 = \varepsilon'_2$ remain identical. For the longitudinal polarization we obtain

$$\varepsilon_3^\mu = \Lambda^\mu_\nu \varepsilon_3^\nu = \frac{1}{m}(|\vec{k}|, 0, 0, \omega_k)\,.$$

Similarly we can confirm that $\varepsilon_0^\mu = k^\mu/m$. Using the explicit expression we can write down the sum over the physical polarizations, which is

$$\sum_{r=1}^{3} \varepsilon_r^\mu(k)\varepsilon_r^\nu(k) = \begin{pmatrix} \frac{|\vec{k}|^2}{m^2} & 0 & 0 & \frac{\omega_k|\vec{k}|}{m^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\omega_k|\vec{k}|}{m^2} & 0 & 0 & \frac{\omega_k^2}{m^2} \end{pmatrix}$$

In the same way we can just write down the right-hand side of the expression using $k^\mu = (\omega_k, 0, 0, |\vec{k}|)$ and obtain

$$-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} = \begin{pmatrix} -1 + \frac{\omega_k^2}{m^2} & 0 & 0 & \frac{\omega_k|\vec{k}|}{m^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\omega_k|\vec{k}|}{m^2} & 0 & 0 & 1 + \frac{|\vec{k}|^2}{m^2} \end{pmatrix} = \begin{pmatrix} \frac{|\vec{k}|^2}{m^2} & 0 & 0 & \frac{\omega_k|\vec{k}|}{m^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\omega_k|\vec{k}|}{m^2} & 0 & 0 & \frac{\omega_k^2}{m^2} \end{pmatrix}.$$

Although we derived the relation in a special frame, it holds in any other reference frame as well, since both sides transform equally under Lorentz transformations. One can even make a generic ansatz $Ag^{\mu\nu} + Bk^\mu k^\nu$ and obtains the same form as these are the only two Lorentz structures that can appear.

For the photon the Gupta-Bleuler formalism was used to remove another degree of freedom. We now continue with the main exercise:

(a) We start with a few necessary relations. The projection operators are inherited from the fact that the Standard Model is a chiral theory, in which left- and right-handed fermions transform differently under $SU(2)_L$. For the projection operators we obtain

$$P_{L/R}^2 = \left(\frac{1 \mp \gamma_5}{2}\right)^2 = \frac{1 + 1 \mp 2\gamma_5}{4} = \frac{1 \mp \gamma_5}{2} = P_{L/R}\,.$$

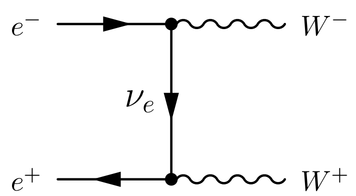
Moreover we note that

$$\begin{aligned}\overline{(\gamma^\mu P_{L/R})} &= \gamma^0 \left(\gamma^\mu \frac{1 \mp \gamma_5}{2}\right)^\dagger \gamma^0 = \gamma^0 \frac{1 \mp \gamma_5^\dagger}{2} \gamma^{\mu\dagger} \gamma^0 \\ &= \frac{1 \pm \gamma_5}{2} \gamma^\mu = \gamma^\mu \frac{1 \mp \gamma_5}{2} = \gamma^\mu P_{L/R}\,.\end{aligned} \tag{1}$$

Lastly we simplify the polarisation sum for the massive W boson as follows:

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2} \overset{k \gg m_W}{\approx} \frac{k_\mu k_\nu}{m_W^2}\,,$$

since $g_{\mu\nu}$ has entries of order 1. For the two fermions we can use $\sum_\lambda u(p_1,\lambda)\bar{u}(p_1,\lambda) = \not{p}_1$ and $\sum_\lambda v(p_2,\lambda)\bar{v}(p_2,\lambda) = \not{p}_2$. The matrix element for the process is based on the following Feynman diagrams



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if we neglect the corresponding three gauge-boson self-interactions. The matrix element yields

$$\mathcal{M} = \overline{v}(p_2)\frac{ie}{\sqrt{2}s_W}\gamma^\mu P_L \frac{i}{\not{p}_1 - \not{p}_3}\frac{ie}{\sqrt{2}s_W}\gamma^\nu P_L u(p_1)\epsilon_\nu^*(p_3)\epsilon_\mu^*(p_4)$$

This expression makes use of the Feynman rules provided on the sheet and Eq. (1). We can just square the expression, which results in

$$\overline{\sum}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{4s_W^4}\mathrm{Tr}\left[\not{p}_2\gamma^\mu P_L(\not{p}_1 - \not{p}_3)\gamma^\nu P_L \not{p}_1 \gamma^\rho P_L(\not{p}_1 - \not{p}_3)\gamma^\sigma P_L\right]\left(\frac{1}{p_1 - p_3}\right)^4 \frac{p_{4\mu}p_{4\sigma}}{m_W^2}\frac{p_{3\nu}p_{3\rho}}{m_W^2}$$

The factor $\frac{1}{4}$ averages the initial-state fermion spins. We move all P_L's to the very left, which due to the crossing of multiples of two γ matrices and $P_L^2 = P_L$ results in

$$\begin{aligned}\overline{\sum}|\mathcal{M}|^2 &= \frac{e^4}{16(p_1 - p_3)^4 m_W^4 s_W^4}\mathrm{Tr}\left[\not{p}_2\not{p}_4(\not{p}_1 - \not{p}_3)\not{p}_3\not{p}_1\not{p}_3(\not{p}_1 - \not{p}_3)\not{p}_4 P_L\right] \\ &= \frac{e^4}{16(p_1 - p_3)^4 m_W^4 s_W^4}\mathrm{Tr}\left[\not{p}_4\not{p}_2\not{p}_4(\not{p}_1 - \not{p}_3)\not{p}_3\not{p}_1\not{p}_3(\not{p}_1 - \not{p}_3) P_R\right]\end{aligned}$$

Now we use $\not{p}_4\not{p}_2\not{p}_4 = -m_W^2\not{p}_2 + 2p_2\cdot p_4\not{p}_4$, $\not{p}_3\not{p}_1\not{p}_3 = -m_W^2\not{p}_1 + 2p_1\cdot p_3\not{p}_3$ and $\not{p}_1 - \not{p}_3 = -(\not{p}_2 - \not{p}_4)$ and get

$$\begin{aligned}\overline{\sum}|\mathcal{M}|^2 &= -\frac{e^4}{16(p_1-p_3)^4 m_W^4 s_W^4}\mathrm{Tr}\left[(-m_W^2 + 2p_2\cdot p_4\not{p}_4)(\not{p}_2-\not{p}_4)(-m_W^2 + 2p_1\cdot p_3\not{p}_3)(\not{p}_1-\not{p}_3)P_R\right]\\ &= -\frac{e^4}{16(p_1-p_3)^4 m_W^4 s_W^4}\mathrm{Tr}\left[(m_W^2\not{p}_2\not{p}_4 + 2p_2\cdot p_4\not{p}_4\not{p}_2 - 2m_W^2 p_2\cdot p_4)\right.\\ &\qquad\left.\times\,(m_W^2\not{p}_1\not{p}_3 + 2p_1\cdot p_3\not{p}_3\not{p}_1 - 2m_W^2 p_1\cdot p_3)P_R\right]\end{aligned}$$

We transform $\not{p}_2\not{p}_4 = -\not{p}_4\not{p}_2 + 2p_2\cdot p_4$ and $\not{p}_1\not{p}_3 = -\not{p}_3\not{p}_1 + 2p_1\cdot p_3$, which allows to cancel many terms, and — noting the definition of t in the exercise hint — are left with

$$\begin{aligned}\overline{\sum}|\mathcal{M}|^2 &= -\frac{e^4}{16t^2 m_W^4 s_W^4}\mathrm{Tr}\left[(-t\not{p}_4\not{p}_2)(-t\not{p}_3\not{p}_1)P_R\right]\\ &= -\frac{e^4}{16 m_W^4 s_W^4}\frac{1}{2}4\left[p_4\cdot p_2 p_3\cdot p_1 + p_4\cdot p_1 p_2\cdot p_3 - p_4\cdot p_3 p_2\cdot p_1\right]\\ &= -\frac{e^4}{32 m_W^4 s_W^4}\left[(m_W^2-t)^2 + (m_W^2-u)^2 - s(s-2m_W^2)\right]\\ &\approx -\frac{e^4}{32 m_W^4 s_W^4}\left[t^2 + (s+t)^2 - s^2\right] = -\frac{e^4}{32 m_W^4 s_W^4}\left[2t^2 + 2st\right]\\ &= -\frac{e^4}{16 m_W^4 s_W^4}t(s+t)\end{aligned}$$

Therein we used $m_W^2 \ll s, t, u$. The contribution proportional to γ_5 yields $\epsilon^{\mu\nu\rho\sigma}$, which cancels, since one of the vectors can be replaced through momentum conservation with the other three vectors. We finally consider the center-of-mass frame, in which we write four-momentum conservation as follows

$$\begin{pmatrix}\frac{\sqrt{s}}{2}\\0\\0\\\frac{\sqrt{s}}{2}\end{pmatrix} + \begin{pmatrix}\frac{\sqrt{s}}{2}\\0\\0\\\frac{\sqrt{s}}{2}\end{pmatrix} = \begin{pmatrix}\sqrt{q^2+m_W^2}\\q\sin\theta\\0\\q\cos\theta\end{pmatrix} + \begin{pmatrix}\sqrt{q^2+m_W^2}\\-q\sin\theta\\0\\-q\cos\theta\end{pmatrix}.$$

From this relation we deduce

$$\tfrac{s}{4} = q^2 + m_W^2 \qquad\rightarrow\qquad q^2 = \tfrac{s}{4} - m_W^2$$

We can then deduce t from $(p_1-p_3)^2 = (0, -q\sin\theta, 0, \frac{\sqrt{s}}{2} - q\cos\theta)^2$, which results in

$$\begin{aligned}t &= -q^2 s_\theta^2 - \left(\frac{\sqrt{s}}{2} - q\cos\theta\right)^2 = -q^2\sin^2\theta - \frac{s}{4} - q^2\cos^2\theta + 2\frac{\sqrt{s}}{2}q\cos\theta\\ &= -\frac{s}{4} - \frac{s}{4} + m_W^2 + \sqrt{s}\sqrt{\frac{s}{4} - m_W^2}\cos\theta \approx -\frac{s}{2} + \frac{s}{2}\cos\theta = -\frac{s}{2}(1-\cos\theta)\end{aligned}$$

Thus we obtain

$$\begin{aligned}\overline{\sum}|\mathcal{M}|^2 &= -\frac{e^4}{16 m_W^4 s_W^4}t(s+t) = -\frac{e^4}{16 m_W^4 s_W^4}\left(\frac{s^2}{4}(1 - 2\cos\theta + \cos^2\theta) - \frac{s^2}{2}(1-\cos\theta)\right)\\ &= \frac{e^4}{64 s_W^4}\frac{s^2}{m_W^4}(1-\cos^2\theta) = \frac{\alpha^2\pi^2}{4s_W^4}\frac{s^2}{m_W^4}(1-\cos^2\theta)\,.\end{aligned}$$

(b) In this subexercise we need to add the phase space to the result of the previous subexercise to obtain the total cross section. For this purpose we need the phase space discussed in TTP1 and mentioned already on the first exercise sheet of TTP2, which results in

$$\int d\Phi_2 = \int d\Omega \frac{\lambda(s, m_W^2, m_W^2)}{32\pi^2 s} = \int d\cos\theta \frac{\sqrt{s^2 - 4sm_W^2}}{16\pi s} = \int d\cos\theta \frac{1}{16\pi}\sqrt{1 - \frac{4m_W^2}{s}}\,.$$

If we were to neglect the W mass for the kinematics, we are just left with $\frac{1}{16\pi}\int d\cos\theta$. We obtain

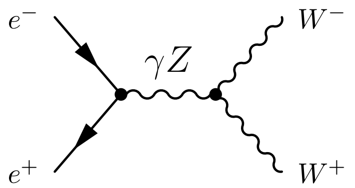
$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\frac{1}{16\pi}\overline{\sum}|\mathcal{M}|^2 = \frac{\alpha^2\pi}{128 s_W^4}\frac{s}{m_W^4}(1 - \cos^2\theta)\,.$$

Integration $\int_{-1}^{1} d\cos\theta(1 - \cos^2\theta) = \frac{4}{3}$ yields

$$\sigma = \frac{\alpha^2\pi}{96 s_W^4}\frac{s}{m_W^4}\,.$$

This result diverges for $s \to \infty$. The divergence is cured by the inclusion of the three gauge-boson self-interaction, which we consider in the next subexercise.

(c) Since taking into account the three gauge-boson self-interaction yields a quite lengthy result, we write down the amplitude for $e_R^- e_L^+ \to W_L^- W_L^+$ for one helicity combination. Due to the right-handed electron and the left-handed positron in the initial-state the t-channel contribution involving a neutrino is absent. For this combination we just have to add the two s-channel contributions involving the photon and the Z boson. The corresponding Feynman diagrams are



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The amplitude is given by

$$\mathcal{M} = \overline{v}(p_2)P_L\left((-ie\gamma^\mu P_R)\frac{-i(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2})}{k^2}(ief^{\sigma\rho\nu}) + \frac{ies_W}{c_W}\gamma^\mu P_R\frac{-i(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2-\xi m_Z^2})}{k^2 - m_Z^2}(ie\frac{c_W}{s_W}f^{\sigma\rho\nu})\right)P_R u(p_1)\epsilon_\rho^*(p_3)\epsilon_\sigma^*(p_4)$$

We first discuss the two terms proportional to $k_\mu k_\nu$, which due to $f^{\sigma\rho\nu} = g^{\sigma\rho}(-p_4+p_3)^\nu + g^{\rho\nu}(-p_3-k)^\sigma + g^{\nu\sigma}(k+p_4)^\rho$ and due to the replacement $\epsilon_L^\mu \approx p^\mu/m_W$ are proportional to

$$k_\nu p_{3\rho} p_{4\sigma} f^{\sigma\rho\nu} = p_3\cdot p_4(-p_4\cdot k + p_3\cdot k) + p_3\cdot k(-p_3\cdot p_4 - k\cdot p_4) + p_4\cdot k(k\cdot p_3 + p_4\cdot p_3) = 0\,.$$

The two contributions with $k_\mu k_\nu$ therefore vanish, such that also the gauge choice is not of relevance. We are thus left with

$$\begin{aligned}\mathcal{M} = \overline{v}(p_2)P_L \Bigg(&(-ie\gamma^\mu P_R)\frac{-ig_{\mu\nu}}{k^2}(ief^{\sigma\rho\nu}) \\ &+\frac{ies_W}{c_W}\gamma^\mu P_R \frac{-ig_{\mu\nu}}{k^2-m_Z^2}(ie\frac{c_W}{s_W}f^{\sigma\rho\nu})\Bigg) P_R u(p_1)\epsilon^*_\rho(p_3)\epsilon^*_\sigma(p_4) \\ = \overline{v}(p_2)P_L \left(ie^2 g_{\mu\nu}\left(\frac{1}{k^2-m_Z^2}-\frac{1}{k^2}\right)\gamma^\mu f^{\sigma\rho\nu}\right) &P_R u(p_1)\epsilon^*_\rho(p_3)\epsilon^*_\sigma(p_4)\end{aligned}$$

Again replacing the external polarisation vectors with the aligned momenta, we can transform

$$\begin{aligned} g_{\mu\nu}\gamma^\mu f^{\sigma\rho\nu}p_{3\rho}p_{4\sigma} &= p_3\cdot p_4(-\not{p}_4+\not{p}_3)+\not{p}_3(-p_3\cdot p_4 - k\cdot p_4)+\not{p}_4(k\cdot p_3+p_3\cdot p_4) \\ &= -\not{p}_3 k\cdot p_4 + \not{p}_4 k\cdot p_3 = (\not{p}_4-\not{p}_3)(m_W^2+p_3\cdot p_4)\end{aligned}$$

In the last step we replaced $k = p_3 + p_4$, such that $k\cdot p_4 = p_3\cdot p_4 + m_W^2 = k\cdot p_4$. We thus obtain

$$\begin{aligned}\mathcal{M} &= \overline{v}(p_2)P_L\left(ie^2\left(\frac{1}{k^2-m_Z^2}-\frac{1}{k^2}\right)\frac{1}{m_W^2}(\not{p}_4-\not{p}_3)(m_W^2+p_3\cdot p_4)\right)P_R u(p_1) \\ &= ie^2\overline{v}_R(p_2)\,(\not{p}_4-\not{p}_3)\,u_L(p_1)\underbrace{\left(\frac{1}{k^2-m_Z^2}-\frac{1}{k^2}\right)\frac{1}{m_W^2}(m_W^2+p_3\cdot p_4)}_{=:J}\end{aligned}$$

We transform J using $k^2 = s = (p_3+p_4)^2 = 2m_W^2 + 2p_3\cdot p_4$ and obtain

$$\begin{aligned} J &= \left(\frac{1}{s-m_Z^2}-\frac{1}{s}\right)\frac{1}{m_W^2}\left(m_W^2+\frac{s}{2}-m_W^2\right) \\ &= \frac{s-(s-m_Z^2)}{s(s-m_Z^2)}\frac{s}{2m_W^2} = \frac{m_Z^2}{(s-m_Z^2)2m_W^2} \approx \frac{m_Z^2}{2sm_W^2} = \frac{1}{2sc_W^2}\,.\end{aligned}$$

This results in the expression provided on the exercise sheet

$$\mathcal{M} = ie^2\overline{v}_R(p_2)\,(\not{p}_4-\not{p}_3)\,u_L(p_1)\frac{1}{2sc_W^2}\,.$$

(d) We square the expression and calculate the fermion trace. The polarisation sum was indirectly performed through the replacement with the momenta. Note that

$$\mathcal{M} = ie^2\overline{v}_R(p_2)\,(\not{p}_4-\not{p}_3)\,u_L(p_1)\frac{1}{2sc_W^2} = ie^2\overline{v}_R(p_2)\,(-2\not{p}_3)\,u_L(p_1)\frac{1}{2sc_W^2}\,,$$

since $\not{p}_4-\not{p}_3 = \not{p}_1+\not{p}_2-\not{p}_3-\not{p}_3$ and $\not{p}_1+\not{p}_2$ vanish when combined with the external spinors. We thus get

$$\begin{aligned}
\overline{\sum}|\mathcal{M}|^2 &= \frac{1}{4}\frac{e^4}{4s^2c_W^4}4\text{Tr}[\not{p}_2\not{p}_3\not{p}_1\not{p}_3P_R] \\
&= \frac{e^4}{4c_W^4s^2}\frac{1}{2}4\left(2p_2\cdot p_3p_1\cdot p_3 - p_1\cdot p_2m_W^2\right) \\
&= \frac{e^4}{4c_W^4s^2}\left((m_W^2-u)(m_W^2-t)-sm_W^2\right) \\
&\approx \frac{e^4}{4c_W^4}\frac{(-t)(s+t)}{s^2} = \frac{4\pi^2\alpha^2}{c_W^4}\frac{1}{s^2}\frac{s}{2}(1-\cos\theta)\left(s-\frac{s}{2}(1-\cos\theta)\right) \\
&= \frac{\pi^2\alpha^2}{c_W^4}(1-\cos^2\theta)
\end{aligned}$$

As before we can add the two-particle phase space and obtain the cross section

$$\begin{aligned}
\frac{d\sigma}{d\cos\theta} &= \frac{1}{2s}\frac{1}{16\pi}\overline{\sum}|\mathcal{M}|^2 = \frac{\alpha^2\pi}{32c_W^4}\frac{1}{s}(1-\cos^2\theta) \\
\sigma &= \frac{\alpha^2\pi}{24c_W^4}\frac{1}{s}\,.
\end{aligned}$$

In contrast to the result to the t-channel neutrino exchange in (b) we obtain $\sigma(s\to\infty)\to 0$.

(e) In this subexercise we show that the amplitude obtained in (c) equals the amplitude when we replace the two W bosons with Goldstone bosons. Also the Feynman diagrams are the same as in (c) with the two W bosons replaced by the charged Goldstone bosons. The amplitude is of the form

$$\mathcal{M} = \overline{v}_R(p_2)\left(-ie^2\frac{g_{\mu\nu}}{k^2}(p_3-p_4)^\nu + ie^2\frac{s_W(c_W^2-s_W^2)}{2s_Wc_W^2}\frac{g_{\mu\nu}}{k^2-m_Z^2}(p_3-p_4)^\nu\right)\gamma^\mu u_R(p_1)\,.$$

We have dropped the contributions proportional to $k_\mu k_\nu$ in the propagators right from the start, since they yield contributions proportional to $k_\mu k_\nu(p_3-p_4)^\nu\gamma^\mu = \not{k}k\cdot(p_3-p_4) = \not{k}(p_3+p_4)\cdot(p_3-p_4) = \not{k}(m_W^2-m_W^2) = 0$. We are therefore left with

$$\begin{aligned}
\mathcal{M} &= ie^2\overline{v}_R(p_2)\gamma^\mu u_R(p_1)\left(-\frac{(p_3-p_4)_\mu}{k^2}+\frac{c_W^2-s_W^2}{2c_W^2}\frac{(p_3-p_4)_\mu}{k^2-m_Z^2}\right) \\
&= ie^2\overline{v}_R(p_3)(\not{p}_3-\not{p}_4)u_R(p_1)\underbrace{\left(\frac{-2c_W^2(s-m_Z^2)+sc_W^2-s(1-c_W^2)}{s(s-m_Z^2)2c_W^2}\right)}_{=:J}
\end{aligned}$$

We transform J as follows

$$J = \frac{2c_W^2m_Z^2-s}{2c_W^2s(s-m_Z^2)} \approx -\frac{1}{2sc_W^2}\,,$$

such that

$$\mathcal{M} = \overline{v}_R(p_2)(\not{p}_4-\not{p}_3)u_R(p_1)\frac{ie^2}{2c_W^2}\frac{1}{s}\,.$$

This is indeed identical to the result we obtained in the high-energy limit for the external gauge bosons.

(f) We repeat the calculation for a different helicity combination of the initial state, namely $e_L^- e_R^+ \to \phi^- \phi^+$, which results in

$$\mathcal{M} = \overline{v}_L(p_2)\gamma^\mu u_L(p_1)\left((-ie)\frac{-ig_{\mu\nu}}{k^2} ie(p_3-p_4)^\nu + \left(-ie\frac{1}{c_w s_w}(\frac{1}{2}-s_W^2)\right)\frac{-ig_{\mu\nu}}{k^2-m_Z^2}\left(ie\frac{c_W^2-s_W^2}{2c_W s_W}\right)\right).$$

Again we dropped $k_\mu k_\nu$ contributions in the propagator, see the previous subexercise. There is no t-channel neutrino contribution because of the massless initial state. We can transform the matrix element further and obtain

$$\mathcal{M} = ie^2\overline{v}_L(p_2)(\not{p}_3 - \not{p}_4)u_L(p_1)\underbrace{\left(-\frac{1}{s} - \frac{(c_W^2-s_W^2)^2}{4c_W^2 s_W^2}\frac{1}{s-m_Z^2}\right)}_{:=J}.$$

Again we transform J and get

$$\begin{aligned} J &= \frac{4c_W^2 s_W^2 s - 4c_W^2 s_W^2 m_Z^2 + s c_W^4 + s s_W^4 - 2 s_W^2 c_W^2 s}{4c_W^2 s_W^2 s(s-m_Z^2)} \\ &\approx \frac{c_W^4 + s_W^4 + 2c_W^2 s_W^2}{4c_W^2 s_W^2 s} = \frac{1}{4 s c_W^2 s_W^2}. \end{aligned}$$

We are thus left with

$$\mathcal{M} = \frac{ie^2}{4c_W^2 s_W^2}\frac{1}{s}\overline{v}_L(p_2)(\not{p}_3 - \not{p}_4)u_L(p_1).$$

We therefore conclude

$$\mathcal{M}(e_L^- e_R^+ \to \phi^-\phi^+) = \mathcal{M}(e_R^- e_L^+ \to \phi^-\phi^+)\frac{1}{2s_W^2}.$$

We can now build up the total cross section for $e^+e^- \to W_L^+ W_L^-$ in the high-energy limit, since there are no further interference terms. Also the LL and RR contributions vanish. We thus get

$$\sigma = \frac{\alpha^2\pi}{24c_W^4}\frac{1}{s}\left(1+\frac{1}{4s_W^4}\right) = \frac{\alpha^2\pi}{96c_W^4 s_W^4}\frac{1}{s}\left(1+4s_W^4\right).$$

We provide a notebook, which indeed shows that taking into account all diagrams, there is no divergence in the high-energy limit.