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Exercise 1: Propagator of a free particle

We want to evaluate the path integral of a free particle in one-dimension by parametrizing all potential paths through a variation of the coefficients of a Fourier series. The propagator is defined by

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q e^{\frac{i}{\hbar}S} \qquad \text{with} \qquad S = \int_{t_i}^{t_f} dt \frac{m}{2} \dot{q}^2$$

with the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$.

(a) Find the classical path of the particle $q_c(t)$ as a function of q_i, q_f, t_i and t_f.

(b) In order to evaluate the path integral we allow for quantum mechanical perturbations around the classical path, $q(t) = q_c(t) + \delta q(t)$, and expand $\delta q(t)$ in a Fourier series knowing that the fluctuations vanish at q_i and q_f. We therefore obtain

$$\delta q(t) = \sum_{n=1}^{\infty} a_n \sin \left[\frac{n\pi}{t_f - t_i} (t - t_i) \right] .$$

Using the orthogonality of different modes, show that the action S turns into

$$S = \frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i} + \frac{m}{2} \sum_{n=1}^{\infty} \frac{1}{2} \frac{(n\pi)^2}{t_f - t_i} a_n^2 .$$

(c) In this example the path integral can be rewritten in terms of integrals over the spectrum of the Fourier coefficients, such that $\int \mathcal{D}q = c \int \prod_{n=1}^{\infty} da_n$ with a normalization constant c. Rearrange the integrals and show

$$\langle q_f, t_f | q_i, t_i \rangle = \tilde{c}(t_f - t_i) \exp \left[\frac{i}{\hbar} \frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i} \right] ,$$

where the normalization constant $\tilde{c}(t_f - t_i)$ can only be a function of the time difference. *Hint:* You can use $\int_{-\infty}^{\infty} da \exp \left[\frac{i}{\hbar} \frac{m}{2} \frac{1}{2} \frac{(n\pi)^2}{t_f - t_i} a^2 \right] = \left(-\frac{imn^2\pi}{2\hbar(t_f - t_i)} \right)^{-1/2}$.

(d) Show that the normalization constant $\tilde{c}(t)$ is given by

$$\tilde{c}(t) = \sqrt{\frac{m}{2\pi i \hbar t}} \quad \text{from the requirement} \quad \int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle = \langle q_f, t_f | q_i, t_i \rangle .$$

Hint: You can use $\int_{-\infty}^{\infty} dq \exp \left[\frac{i}{\hbar} \frac{m}{2} \left(\frac{(q_f - q)^2}{(t_f - t)} + \frac{(q - q_i)^2}{(t - t_i)} \right) \right] = \sqrt{\frac{2\pi i \hbar (t_f - t)(t - t_i)}{m(t_f - t_i)}} \exp \left[\frac{i}{\hbar} \frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i} \right]$.

(e) The wave-function of the particle thus evolves according to

$$\psi(q, t) = \int dq_i \langle q, t | q_i, 0 \rangle \psi(q_i, 0) .$$

Show that $\psi(q, t)$ satisfies the Schrödinger equation and that $\psi(q, t) \to \psi(q, 0)$ for $t \to 0$. *Hint:* Fresnel representation of the δ distribution.

Exercise 2: Path integral formalism for the harmonic oscillator

We investigate the propagator $\langle q_f, t_f | q_i, t_i \rangle$ of the one-dimensional harmonic oscillator in the path integral formalism of quantum mechanics. In contrast to the previous exercise we use $\hbar = 1$.

(a) Show that due to the completeness relation of the energy eigenstates $|n\rangle$ with energy E_n we can write

$$\langle q_f, T | q_i, 0 \rangle = \sum_n \Phi_n(q_f)\Phi_n^*(q_i)e^{-iE_nT} \tag{1}$$

with $\Phi_n(q) = \langle q | n \rangle$.

(b) The propagator of the harmonic oscillator can be determined from the path integral formalism and yields

$$\langle q_f, T | q_i, 0 \rangle = \sqrt{\frac{m\omega}{2\pi i \sin\omega T}} \exp\left\{ \frac{im\omega}{2\sin\omega T}\left[(q_f^2 + q_i^2)\cos\omega T - 2q_i q_f\right]\right\}. \tag{2}$$

Determine the wave function of the ground state $\Phi_0(q)$. In order to do so, set the time to an imaginary value $T = -it, t \to \infty$, rewrite sin and cos in terms of exponentials and compare Eq. 1 and Eq. 2. For this purpose rewrite Eq. 2 in the form $e^{-\omega/2t} f(e^{-\omega t})$ and expand around the argument of $f(x)$. The first term corresponds to the ground state. *Note:* We refer to the literature for the derivation of the path integral representation of the harmonic oscillator.

(c) Calculate the wave function of the first excited state $\Phi_1(q)$.

(d) Show that the energy levels are given by $E_n = \omega(n + \frac{1}{2})$.
Hint: Again compare Eq. 1 and Eq. 2.

(e) We finally allow the harmonic oscillator to be influenced by an external driving force $J(t)$. The propagator then takes the form

$$\langle q_f, t_f | q_i, t_i \rangle_J = \int \mathcal{D}q \exp\left[i \int_{t_i}^{t_f} dt \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + J(t)q(t) \right) \right] \tag{3}$$

with the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$.

(f) Show that for $\omega^2 \to \omega^2 - i\epsilon$ the propagator in Eq. 3 can be rewritten in the form

$$Z[J] = \langle q_f, \infty | q_i, -\infty \rangle_J = \langle q_f, \infty | q_i, -\infty \rangle_0 \exp\left[-\frac{i}{2} \int dE \frac{\tilde{J}(E)\tilde{J}(-E)}{E^2 - \omega^2 + i\epsilon} \right], \tag{4}$$

where we have introduced

$$\tilde{J}(E) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-iEt} J(t) \quad \leftrightarrow \quad J(t) = \int_{-\infty}^{\infty} \frac{dE}{\sqrt{2\pi}} e^{iEt} \tilde{J}(E).$$

Hints: Make use of a Fourier transform ($t \leftrightarrow E$) in the exponent of Eq. 3 for both $J(t)$ and $q(t)$, write $Jq = \frac{1}{2}[Jq + qJ]$, use $\int dt \exp[i(E + E')t] = 2\pi\delta(E + E')$ and motivate a shift $\tilde{q}(E) \to \tilde{q}(E) - \frac{\tilde{J}(E)}{E^2 - \omega^2}$.

(g) Transform the exponent in Eq. 4 into a time integral, i.e.

$$-\frac{i}{2} \int dt dt' J(t)\Delta(t - t')J(t') \quad \text{with} \quad \Delta(t - t') = \int \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{E^2 - \omega^2 + i\epsilon},$$

and discuss its physical meaning.