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Exercise 1: Functional derivatives

Functional derivatives for a functional $F[f]$, which maps a function $C^n(M)$ with $M \in \{\mathbb{R}, \mathbb{C}\}$ on the complex numbers $\mathbb{C}$, are defined by

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\kappa \to 0} \frac{F[f(x) + \kappa \delta(x-y)] - F[f(x)]}{\kappa} .$$

In four dimensions it follows $\frac{\delta}{\delta f(x)} f(y) = \delta^{(4)}(x-y)$, which is a generalization of the rule $\partial_i x_j = \delta_{ij}$. Use the above definitions to prove the following identities:

$$(a) \quad \frac{\delta}{\delta f(x)} \int d^4y f(y) g(y) = g(x) , \qquad (b) \quad \frac{\delta}{\delta f(x)} \int d^4y g(y) \partial_\mu f(y) = -\partial_\mu g(x) ,$$

$$(c) \quad \frac{\delta}{\delta f(x)} \int d^4y \frac{1}{2} \partial_\mu f(y) \partial^\mu f(y) = -\Box f(x) ,$$

$$(d) \quad \frac{\delta}{\delta f(x)} \exp\left[i \int d^4y g(y) f(y)\right] = i g(x) \exp\left[i \int d^4y g(y) f(y)\right] .$$

Exercise 2: Saddle-point approximation for a path integral

Commonly a path integral is evaluated by expanding around a stationary phase for real times. As an alternative we consider an imaginary time and then perform a saddle point approximation instead. The real-time propagator can afterwards be obtained from the imaginary-time propagator through analytic continuation. In order to perform the saddle-point approximation we will work with the Euclidean path integral throughout this exercise.
We consider the generating functional in Euclidean space with $x^2 = x_0^2 + \sum_i x_i^2$ of a generic scalar field theory, which is given by

$$Z_E[J] = \mathcal{N} \int \mathcal{D}\phi \exp\left[-\frac{1}{\hbar} S_E[\phi, J]\right] .$$

Therein the action reads

$$S_E[\phi, J] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + V(\phi) - J(x)\phi(x)\right] .$$

The classical field configuration ϕ_0 is determined by $\left.\frac{\delta S_E[\phi,J]}{\delta \phi}\right|_{\phi=\phi_0} = 0$, such that we can expand $S_E[\phi, J]$ around ϕ_0 in the form

$$S_E[\phi, J] = S_E[\phi_0, J] + \int d^4x \Delta S_J^{(1)}(x) \left(\phi(x) - \phi_0(x)\right) + \frac{1}{2} \int d^4x d^4y \Delta S_J^{(2)}(x, y) \left(\phi(x) - \phi_0(x)\right)\left(\phi(y) - \phi_0(y)\right) + \dots$$

Show that in the limit in which we neglect the dots in the previous equation the functional $Z_E[J]$ is given by

$$Z_E[J] \approx \mathcal{N}' \exp\left[-\frac{1}{\hbar} S_E[\phi_0, J]\right] \left(\det \hat{K}\right)^{-1/2} \quad \text{with} \quad \hat{K} = \int d^4x \left[-\partial_\mu^2 + m^2 + V''(\phi_0)\right] .$$

Discuss the physical meaning of this approximation.
Hint: Define $\delta\phi = \phi - \phi_0$ and thus replace $\mathcal{D}\phi = \mathcal{D}\delta\phi$. Lastly define $\delta\phi' = \frac{1}{\sqrt{\hbar}}\delta\phi$ and count the orders in $\hbar$. Use $\int \prod_i dx_i e^{-x^T A x} = \int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_n e^{-x_i A_{ij} x_j} = \pi^{n/2} \det(A)^{-1/2}$ to replace $\int \mathcal{D}\delta\phi' \exp[-\int d^4x \delta\phi' [-\partial_\mu^2 + m^2 + V''(\phi_0)]\delta\phi']$.

Exercise 3: Grassmann variables

In the formulation of the path integral formalism for fermions we come across Grassmann variables. In this exercise we want to derive a few useful relations for them.

(a) Consider the Gaussian integral in the space of N real Grassmann variables θ_i with $i \in \{1, \dots, N\}$, which is of the form

$$I_N(M, \chi) = \int d\theta_1 \dots d\theta_N \exp\left[-\frac{1}{2}\theta^T M \theta + \chi^T \theta\right] ,$$

where M is an arbitrary antisymmetric matrix and $\chi = (\chi_1, \dots, \chi_N)^T$ is a vector of N independent Grassmann variables. Show that for $N = 4$ the identity

$$I_4(M, \chi = 0) = \sqrt{\det M}$$

holds by explicitly evaluating the integral.
Hint: Expand the exponential and argue why only one term of the expansion survives. The determinant is given by $\det M = (M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23})^2$ with the matrix elements M_{ij} of M.

(b) Show for non-vanishing χ and $N = 4$ the identity $I_4(M, \chi) = \sqrt{\det M} \exp\left[c\chi^T M^{-1} \chi\right]$. Determine the real constant c.
Hint: The inverse of the matrix M is given by

$$M^{-1} = \frac{1}{\sqrt{\det M}} \begin{pmatrix} 0 & -M_{34} & M_{24} & -M_{23} \\ M_{34} & 0 & -M_{14} & M_{13} \\ -M_{24} & M_{14} & 0 & -M_{12} \\ M_{23} & -M_{13} & M_{12} & 0 \end{pmatrix} .$$

(c) We consider integrals in the space of N Grassmann variables $\eta_1, \dots, \eta_N$. How does the integration measure transform under a linear transformation of the form $\eta_i' = B_{ij}\eta_j$?
Hint: Consider

$$\int d\eta_2 d\eta_1 \eta_1 \eta_2 = \int d\eta_2' d\eta_1' \eta_1' \eta_2' = 1$$

and replace $\eta_1'\eta_2'$ with $\eta_1\eta_2$. Try to generalize your findings for $N > 2$.

(d) Use the result of the previous subexercise to determine the complex, N-dimensional Gaussian integral

$$\int d\overline{\eta}_N \dots d\overline{\eta}_1 d\eta_N \dots d\eta_1 \exp\left[-\overline{\eta} B \eta\right]$$

for a hermitian matrix B.