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**Exercise 1: Saddle-point approximation for a path integral - continued**

Remember the second exercise on sheet 6, in which we showed in Euclidean space that in the classical limit with $\hbar \to 0$ the path integral was just given by the classical action

$$S_E[\phi_0, J] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_0)^2 + \frac{1}{2}m^2\phi_0^2 + V(\phi_0) - J(x)\phi_0\right] ,$$

where $\phi_0$ fulfills the relation

$$-\partial_\mu^2 \phi_0(x) + m^2\phi_0(x) + V'(\phi_0(x)) - J(x) = 0 \,. \tag{1}$$

We consider the $\phi^4$ theory with

$$V(\phi) = \frac{\lambda}{4!}\phi^4 \,.$$

In analogy to the lecture we want to derive the connected, but Euclidean Green's functions $\tau_c^E(x_1, x_2)$ and $\tau_c^E(x_1, x_2, x_3, x_4)$.

(a) Show that the classical action can be rewritten in the form

$$S_E[\phi_0, J] = -\frac{1}{2}\int d^4x \left(J\phi_0 + 2\frac{\lambda}{4!}\phi_0^4\right) . \tag{2}$$

(b) We expand $\phi_0 = \phi^{[0]} + \lambda\phi^{[1]} + \lambda^2\phi^{[2]} + \ldots$. In Eq. 1 we sort by orders in $\lambda$ and for the lowest order, $\lambda^0$, obtain $-(\partial^2 - m^2)_x \phi^{[0]}(x) = J(x) =: J_x$. The Euclidean two-point function $G^E(x, y)$ is defined through

$$(\partial^2 - m^2)_x G^E(x, y) = -\delta^{(4)}(x - y) \quad \leftrightarrow \quad G^E_{xy} := G^E(x, y) = \int \frac{d^4q_E}{(2\pi)^4} \frac{e^{iq_E(x-y)}}{q_E^2 + m^2} \,.$$

Show that it yields

$$\phi^{[0]}(x) = \int d^4a G^E_{xa} J_a \,.$$

(c) Consider the orders $\lambda^1$ and $\lambda^2$ in Eq. 1 to prove

$$\phi^{[1]}(x) = -\frac{1}{6}\int d^4a\, d^4b\, d^4c\, d^4d\, G^E_{xa}G^E_{ab}G^E_{ac}G^E_{ad}J_bJ_cJ_d$$
$$\phi^{[2]}(x) = \frac{1}{12}\int d^4a\, d^4b\, d^4c\, d^4d\, d^4e\, d^4f\, d^4y\, G^E_{xa}G^E_{ab}G^E_{ac}G^E_{ay}G^E_{yd}G^E_{ye}G^E_{yf}J_bJ_cJ_dJ_eJ_f \,.$$

(d) Finally expand also Eq. 2 in powers of $\lambda$ up to $\lambda^1$ and insert the results from the previous two subexercises. According to the lecture the generating functional of the connected Green's functions is just given by the classical action $W_E[J] = S_E[\phi_0, J]$, such that

$$\tau_c^E(x_1, x_2) = -\left.\frac{\delta^2 W_E[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0}, \quad \tau_c^E(x_1, x_2, x_3, x_4) = -\left.\frac{\delta^4 W_E[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}.$$

Derive the connected Green's functions starting from the expanded version of Eq. 2. Explain the physical meaning of the connected Green's functions by identifying the corresponding Feynman diagrams.

**Exercise 2: Propagator of the gauge field in the Stückelberg Lagrangian**

We again consider the Stückelberg Lagrangian of a single free massive gauge field given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \frac{m^2}{2}A^\mu A_\mu .$$

Therein we use the Abelian field strength tensor defined by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ as well as a covariant gauge fixing term employing the free parameter $\xi$ and a mass term with mass $m$. We motivated the Stückelberg Lagrangian already on sheet 2 and now add the gauge fixing term obtained by the Fadeev-Popov trick in the covariant gauge $\partial_\mu A^\mu = 0$. We will discuss other gauge choices later in the course.

(a) Derive the equation of motion for the gauge field, for which you should obtain

$$\left[(\Box + m^2)g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\nu = 0 .$$

*Hint:* You can use the functional derivative of the action $S = i\int d^4x\mathcal{L}$ with respect to $A_\rho$ or use the Euler-Lagrange equation for the pair $A_\rho$ and $\partial_\rho A_\sigma$. Be careful to perform the derivatives with respect to a new index $A_\rho$ and thus add $\delta^\rho_{\mu,\nu,\ldots}$ where appropriate.

(b) Fourier transformation ($\partial_\mu \to ik_\mu$) of this equation allows to determine the Green's function $\Delta_{\nu\rho}(k)$ in momentum space

$$\left[(-k^2 + m^2)g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu\right]\Delta_{\nu\rho}(k) = \delta^\mu_\rho .$$

Make the ansatz $\Delta_{\nu\rho}(k) = A(k^2)g_{\nu\rho} + B(k^2)k_\nu k_\rho$ and determine $A(k^2)$ and $B(k^2)$ by equating the coefficients. You should obtain

$$\Delta_{\mu\nu}(k) = \frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2} - \frac{\frac{k_\mu k_\nu}{m^2}}{k^2 - \xi m^2} .$$

(c) Discuss the cases $\xi \to 0$ (Landau gauge), $\xi \to 1$ (Feynman gauge) and $\xi \to \infty$ (unitary gauge) as well as $m \to 0$. Compare the latter result with the result of the photon/gluon propagator.