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Exercise 1: Vertex of non-abelian gauge fields

The normalized generating functional of a pure non-abelian free gauge theory with a source $J_{a\mu}$ for the gauge field takes the form

$$\mathcal{Z}_0[J] = \exp\left[i\int d^4x d^4y \left(\frac{1}{2}J_{a\mu}(x)D^{\mu\nu}_{ab}(x-y)J_{b\nu}(y)\right)\right].$$

Therein $D^{\mu\nu}_{ab}$ is the causal Green function following the notation of the lecture. Calculate the Feynman rule for the vertex of three gauge fields starting from the interacting Lagrangian $\mathcal{L}_I = gf_{abc}A^\mu_b(x)A^\nu_c(x)\partial_\mu A_{a\nu}(x)$.

Hint: As the source terms for the ghost fields are not of relevance here, we removed the terms from $\mathcal{Z}_0$. Use the lecture to argue that the starting point is

$$\begin{aligned}&\left\langle 0|T(A^a_\mu(x_1)A^b_\nu(x_2)A^c_\rho(x_3))|0\right\rangle\\ &= \left(\frac{1}{i^3}\frac{\delta}{\delta J_{a\mu}(x_1)}\frac{\delta}{\delta J_{b\nu}(x_2)}\frac{\delta}{\delta J_{c\rho}(x_3)}\int d^4z g f^{klm}\frac{\delta}{\delta J_{k\alpha}(z)}\frac{\delta}{\delta J_{l\beta}(z)}\partial^\alpha_z\frac{\delta}{\delta J^\beta_m(z)}\mathcal{Z}_0[J]\right)\Bigg|_{J=0}.\end{aligned}$$

Exercise 2: Ward identities in QED

We consider quantum electrodynamics (QED) with a single photon field and one fermion field. The generating functional takes the form

$$\mathcal{Z}[J,\eta,\overline{\eta}] = \frac{1}{\mathcal{N}}\int \mathcal{D}A_\mu\mathcal{D}\overline{\psi}\mathcal{D}\psi\exp\left(i\int d^4x\mathcal{L}_{\text{tot}}\right),$$

where the Lagrangian contains a source J_μ for the gauge boson and (Grassmann-type) sources η and $\overline{\eta}$ for the fermion field

$$\mathcal{L}_{\text{tot}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\overline{\psi}\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta.$$

Add-on: Separating interaction terms in $\mathcal{L}_{\text{tot}}$ would again allow to define the generating functional of the free theory $\mathcal{Z}_0$, which we however don't need in this exercise.

(a) Show that if $\mathcal{Z}[J,\eta,\overline{\eta}]$ is gauge invariant under (infinitesimal) gauge transformations

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \qquad \psi \to (1+ie\Lambda)\psi, \qquad \overline{\psi}\to\overline{\psi}(1-ie\Lambda),$$

the following identity holds

$$\left[-\frac{1}{\xi}\Box\partial_\mu\frac{1}{i}\frac{\delta}{\delta J_\mu(x)} - \partial_\mu J^\mu + ie\left(\overline{\eta}\frac{\delta}{\delta\overline{\eta}(x)} - \eta\frac{\delta}{\delta\eta(x)}\right)\right]\mathcal{Z}[J,\eta,\overline{\eta}] = 0.$$

Hint: Replace fields with functional derivatives with respect to the external sources and thereby pay attention to signs for the fermion fields.

(b) Remember that W obtained from $\mathcal{Z} = \exp(iW)$ is the generating functional for connected diagrams. The generating functional for the vertex is then given by the Legendre transformation

$$\Gamma[A_\mu, \overline{\psi}, \psi] = W[J, \eta, \overline{\eta}] - \int d^4x (\overline{\psi}\eta + \overline{\eta}\psi + J_\mu A^\mu) \,.$$

The classical fields and the inverse relations are given by

$$A_\mu = \frac{\delta W}{\delta J_\mu}, \qquad \overline{\psi} = -\frac{\delta W}{\delta \eta}, \qquad \psi = \frac{\delta W}{\delta \overline{\eta}}$$
$$\frac{\delta \Gamma}{\delta A_\mu} = -J^\mu, \qquad \frac{\delta \Gamma}{\delta \psi} = \overline{\eta}, \qquad \frac{\delta \Gamma}{\delta \psi} = -\eta \,.$$

Use the previous subexercise and these relations to show the generalized Ward identity

$$-\frac{1}{\xi}\Box \partial_\mu A^\mu(x) + \partial_\mu \frac{\delta \Gamma}{\delta A_\mu(x)} + ie\left(\frac{\delta \Gamma}{\delta \psi(x)}\psi(x) + \overline{\psi}(x)\frac{\delta \Gamma}{\delta \overline{\psi}(x)}\right) = 0 \,.$$

We can consider this generalized Ward identity for any combination of fermions and gauge bosons in the vertex. It holds to all orders in perturbation theory!

(c) We can write the vertex function in the form

$$\Gamma[A_\mu, \psi, \overline{\psi}] = \int d^4x_1 d^4x_2 \left(\overline{\psi}(x_1) S_F^{-1}(x_1 - x_2)\psi(x_2) + \frac{1}{2} A^\mu(x_1) D_{\mu\nu}^{-1}(x_1 - x_2) A^\nu(x_2)\right)$$
$$+ \int d^4x_1 d^4x_2 d^4x_3 \overline{\psi}(x_1) e \Gamma^\mu(x_1, x_2, x_3)\psi(x_2) A^\mu(x_3) + \dots \,,$$

where we added the inverse propagators S_F^{-1} and $D_{\mu\nu}^{-1}$ of the fermion and gauge field, respectively, and the vertex function Γ^μ coupling a gauge field to a pair of fermions. We first consider the Ward identity of the previous subexercise for two gauge bosons. Show that for the full inverse propagator in momentum space

$$D_{\mu\nu}^{-1}(k) = \left(-g_{\mu\nu}k^2 + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu + \omega_{\mu\nu}(k)\right) \,,$$

where $\omega_{\mu\nu}(k)$ includes higher-order corrections, we obtain $k^\mu \omega_{\mu\nu}(k) = 0$, i.e. corrections are transversal to all orders in perturbation theory. *Add-on:* (You will understand the following sentence in a few weeks:) As a direct consequence of this the photon propagator is only logarithmically divergent, which is why QED has no hierachy problem.

(d) Consider the vertex of two fermions and one gauge field to show the most familiar form of the Ward identity being

$$\partial^\mu e \Gamma_\mu(x_1, x_2, x) = ie\left[S_F^{-1}(x_1 - x_2)\delta^{(4)}(x_2 - x) - S_F^{-1}(x_1 - x_2)\delta^{(4)}(x - x_1)\right] \,.$$

Add-on: In momentum space this relation reads $q^\mu \Gamma_\mu(p, p+q, q) = S_F^{-1}(p+q) - S_F^{-1}(p)$.