

Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)  
Martin Gabelmann (martin.gabelmann@kit.edu) (Office 12/17 - Build. 30.23)  
Jonas Müller (jonas.mueller@kit.edu) (Office 12/17 - Build. 30.23)

### Exercise 1: Vertex of non-abelian gauge fields

The normalized generating functional of a pure non-abelian free gauge theory with a source  $J_{a\mu}$  for the gauge field takes the form

$$\mathcal{Z}_0[J] = \exp \left[ i \int d^4x d^4y \left( \frac{1}{2} J_{a\mu}(x) D_{ab}^{\mu\nu}(x-y) J_{b\nu}(y) \right) \right].$$

Therein  $D_{ab}^{\mu\nu}$  is the causal Green function following the notation of the lecture. Calculate the Feynman rule for the vertex of three gauge fields starting from the interacting Lagrangian  $\mathcal{L}_I = g f_{abc} A_b^\mu(x) A_c^\nu(x) \partial_\mu A_{a\nu}(x)$ .

*Hint:* As the source terms for the ghost fields are not of relevance here, we removed the terms from  $\mathcal{Z}_0$ . Use the lecture to argue that the starting point is

$$\begin{aligned} & \langle 0 | T(A_\mu^a(x_1) A_\nu^b(x_2) A_\rho^c(x_3)) | 0 \rangle \\ &= \left( \frac{1}{i^3} \frac{\delta}{\delta J_{a\mu}(x_1)} \frac{\delta}{\delta J_{b\nu}(x_2)} \frac{\delta}{\delta J_{c\rho}(x_3)} \int d^4z g f^{klm} \frac{\delta}{\delta J_{k\alpha}(z)} \frac{\delta}{\delta J_{l\beta}(z)} \partial_\alpha^z \frac{\delta}{\delta J_m^\beta(z)} \mathcal{Z}_0[J] \right) \Big|_{J=0}. \end{aligned}$$

### Exercise 2: Ward identities in QED

We consider quantum electrodynamics (QED) with a single photon field and one fermion field. The generating functional takes the form

$$\mathcal{Z}[J, \eta, \bar{\eta}] = \frac{1}{\mathcal{N}} \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( i \int d^4x \mathcal{L}_{\text{tot}} \right),$$

where the Lagrangian contains a source  $J_\mu$  for the gauge boson and (Grassmann-type) sources  $\eta$  and  $\bar{\eta}$  for the fermion field

$$\mathcal{L}_{\text{tot}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \gamma^\mu (\partial_\mu - ie A_\mu) \psi - m \bar{\psi} \psi - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + J^\mu A_\mu + \bar{\eta} \psi + \bar{\psi} \eta.$$

*Add-on:* Separating interaction terms in  $\mathcal{L}_{\text{tot}}$  would again allow to define the generating functional of the free theory  $\mathcal{Z}_0$ , which we however don't need in this exercise.

- (a) Show that if  $\mathcal{Z}[J, \eta, \bar{\eta}]$  is gauge invariant under (infinitesimal) gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad \psi \rightarrow (1 + ie\Lambda)\psi, \quad \bar{\psi} \rightarrow \bar{\psi}(1 - ie\Lambda),$$

the following identity holds

$$\left[ -\frac{1}{\xi} \square \partial_\mu \frac{1}{i} \frac{\delta}{\delta J_\mu(x)} - \partial_\mu J^\mu + ie \left( \bar{\eta} \frac{\delta}{\delta \bar{\eta}(x)} - \eta \frac{\delta}{\delta \eta(x)} \right) \right] \mathcal{Z}[J, \eta, \bar{\eta}] = 0.$$

*Hint:* Replace fields with functional derivatives with respect to the external sources and thereby pay attention to signs for the fermion fields.

- (b) Remember that  $W$  obtained from  $\mathcal{Z} = \exp(iW)$  is the generating functional for connected diagrams. The generating functional for the vertex is then given by the Legendre transformation

$$\Gamma[A_\mu, \bar{\psi}, \psi] = W[J, \eta, \bar{\eta}] - \int d^4x (\bar{\psi}\eta + \bar{\eta}\psi + J_\mu A^\mu).$$

The classical fields and the inverse relations are given by

$$\begin{aligned} A_\mu &= \frac{\delta W}{\delta J_\mu}, & \bar{\psi} &= -\frac{\delta W}{\delta \eta}, & \psi &= \frac{\delta W}{\delta \bar{\eta}} \\ \frac{\delta \Gamma}{\delta A_\mu} &= -J^\mu, & \frac{\delta \Gamma}{\delta \psi} &= \bar{\eta}, & \frac{\delta \Gamma}{\delta \bar{\psi}} &= -\eta. \end{aligned}$$

Use the previous subexercise and these relations to show the generalized Ward identity

$$-\frac{1}{\xi} \square \partial_\mu A^\mu(x) + \partial_\mu \frac{\delta \Gamma}{\delta A_\mu(x)} + ie \left( \frac{\delta \Gamma}{\delta \psi(x)} \psi(x) + \bar{\psi}(x) \frac{\delta \Gamma}{\delta \bar{\psi}(x)} \right) = 0.$$

We can consider this generalized Ward identity for any combination of fermions and gauge bosons in the vertex. It holds to all orders in perturbation theory!

- (c) We can write the vertex function in the form

$$\begin{aligned} \Gamma[A_\mu, \psi, \bar{\psi}] &= \int d^4x_1 d^4x_2 \left( \bar{\psi}(x_1) S_F^{-1}(x_1 - x_2) \psi(x_2) + \frac{1}{2} A^\mu(x_1) D_{\mu\nu}^{-1}(x_1 - x_2) A^\nu(x_2) \right) \\ &+ \int d^4x_1 d^4x_2 d^4x_3 \bar{\psi}(x_1) e \Gamma^\mu(x_1, x_2, x_3) \psi(x_2) A^\mu(x_3) + \dots, \end{aligned}$$

where we added the inverse propagators  $S_F^{-1}$  and  $D_{\mu\nu}^{-1}$  of the fermion and gauge field, respectively, and the vertex function  $\Gamma^\mu$  coupling a gauge field to a pair of fermions. We first consider the Ward identity of the previous subexercise for two gauge bosons. Show that for the full inverse propagator in momentum space

$$D_{\mu\nu}^{-1}(k) = \left( -g_{\mu\nu} + \left( 1 - \frac{1}{\xi} \right) k_\mu k_\nu + \omega_{\mu\nu}(k) \right),$$

where  $\omega_{\mu\nu}(k)$  includes higher-order corrections, we obtain  $k^\mu \omega_{\mu\nu}(k) = 0$ , i.e. corrections are transversal to all orders in perturbation theory. *Add-on:* (You will understand the following sentence in a few weeks:) As a direct consequence of this the photon propagator is only logarithmically divergent, which is why QED has no hierarchy problem.

- (d) Consider the vertex of two fermions and one gauge field to show the most familiar form of the Ward identity being

$$\partial^\mu e \Gamma_\mu(x_1, x_2, x) = ie \left[ S_F^{-1}(x_1 - x_2) \delta^{(4)}(x_2 - x) - S_F^{-1}(x_1 - x_2) \delta^{(4)}(x - x_1) \right].$$

*Add-on:* In momentum space this relation reads  $q^\mu \Gamma_\mu(p, p+q, q) = S_F^{-1}(p+q) - S_F^{-1}(p)$ .