Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borschensky, Dr. Cody B Duncan

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Exercise Sheet 10

Hand-in Deadline: Mo 17.01.22, 12:00.
Discussion: Di 18.01.22, Mi 19.01.22.

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1. **[8 points] Bilinear Forms of Dirac Spinors**

   The basis of a relativistic field theory is the Lorentz invariance of the Lagrangian density (i.e. the Lagrangian transforms like a Lorentz scalar). It is therefore useful to understand the transformation properties of terms involving Dirac spinors under a Lorentz transformation (LT). The latter can be split up into two categories: the proper orthochonous LTs Λ_L and the discrete transformations such as parity transformations (Λ_P) and time reversal (Λ_T). In the following, we only discuss orthochronous transformations (i.e. no time reversal).

   Under a LT Λ,

   $$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu,$$

   a Dirac spinor $\psi(x)$ transforms as:

   $$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x),$$

   where $S(\Lambda)$ is a (4×4)-matrix which depends on the Lorentz transformations. For the proper LTs and parity transformations, it can be shown that:

   $$S^{-1} = \gamma^0 S^\dagger \gamma^0.$$

   For the Dirac equation to be covariant under LTs, the following relation for S and the γ matrices has to hold:

   $$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \qquad (1)$$

   For the proper LTs $S_L \equiv S(\Lambda_L)$ it can be shown that:

   $$[S_L, \gamma^5] = 0.$$

   For parity transformations $S_P \equiv S(\Lambda_P)$ with $\Lambda_P = \mathrm{diag}(1, -1, -1, -1)$, Eq. (1) becomes:

   $$[S_P, \gamma^0] = 0,$$
   $$\{S_P, \gamma^k\} = 0 \quad \text{for } k = 1, 2, 3,$$

   and the following applies:

   $$\{S_P, \gamma^5\} = 0.$$

   (a) **[7 points]** Show that the following bilinear forms exhibit the given properties:

(i) [1 point] $\bar{\psi}\psi$: scalar,
(ii) [1 point] $\bar{\psi}\gamma^5\psi$: pseudoscalar,
(iii) [1 point] $\bar{\psi}\gamma^\mu\psi$: vector,
(iv) [2 point] $\bar{\psi}\gamma^5\gamma^\mu\psi$: axialvector,
(v) [2 point] $\bar{\psi}\sigma^{\mu\nu}\psi$: (antisymmetric) tensor.

(b) **[1 points]** What about $\psi^\dagger\psi$? Does it transform like a Lorentz scalar?

2. **[8 points] Energy and Momentum of the Dirac Propagator**

In analogy to exercise 1 of sheet 8, where we considered and calculated the energy and momentum of the real Klein-Gordon field, we'll now consider the Dirac field. The corresponding Lagrangian is given by:

$$\mathcal{L} = \bar{\psi}(x)\big(i\partial\!\!\!/ - m\big)\psi(x) = \bar{\psi}(x)\big(i\gamma_\mu\partial^\mu - m\big)\psi(x)$$

where the fields ψ and $\bar{\psi}$ are considered as independent variables. For the Dirac field we use the Ansatz:

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} \sum_\lambda \left[a_\lambda(p)u_\lambda(p)e^{-ipx} + b_\lambda^\dagger(p)v_\lambda(p)e^{ipx}\right] \tag{2}$$

for $\lambda = \pm 1$.

(a) **[2 points]** What is the corresponding energy-momentum tensor $T^{\mu\nu}$? Why does the term proportional to $g^{\mu\nu}$ vanish?

(b) **[2 points]** Use this and the results from exercise 1 on sheet 8 to calculate the 4-momentum vector:

$$P^\mu = \int \mathrm{d}^3x T^{0\mu}$$

and show that this leads to the *normal-ordered* form:

$$: P^\mu := \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^\mu \sum_\lambda \left(\tilde{N}_\lambda^a(p) + \tilde{N}_\lambda^b(p)\right)$$

(c) **[2 points]** Show that the current:

$$j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$$

is conserved. *Hint:* This can be done without using the explicit form of ψ.

(d) **[2 points]** The corresponding charge is given by:

$$Q = \int \mathrm{d}^3x j^0(x)$$

Write the normal-ordered charge explicitly in terms of the operators $a, a^\dagger, b, b^\dagger$.

3. **[4 points] Two Particle Phase Space**

To calculate decay rates and cross sections we need an integration over the phase space of the particles in the final state. For a general process with two particles (momenta p_1, p_2, masses m_1, m_2) in the final state, this phase space integral is given by:

$$\int \mathrm{d}\Phi_2 = \int \frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(4)}(q - p_1 - p_2),$$

where q is the four-momentum of the incoming particles. This integral acts on the squared matrix element and a step function which represents the cuts on the final-state particles. Show that one can rewrite the integral as:

$$\int \mathrm{d}\Phi_2 = \int \mathrm{d}\Omega \frac{1}{32\pi^2 q^2} \lambda(q^2, m_1^2, m_2^2) \Theta(q_0) \Theta(q^2 - (m_1 + m_2)^2)$$

where we have used the Källén function:

$$\lambda(a^2, b^2, c^2) = \sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2} = \sqrt{(a^2 - b^2 - c^2)^2 - 4b^2c^2},$$

and the Heavyside step function Θ, and $\mathrm{d}\Omega = \mathrm{d}(\cos\theta_1)\mathrm{d}\phi_1$ is the integration over the solid angle of particle 1 in the centre-of-mass frame of the two-particle system. The function λ describes the momentum of both particles in the centre-of-mass frame:

$$|\vec{p}_1|^2 = |\vec{p}_2|^2 = \frac{\lambda(q^2, m_1^2, m_2^2)}{2\sqrt{q^2}}$$

Hints:

- Use the relation:

$$\frac{\mathrm{d}^3 p}{2E} = \mathrm{d}^4 p \Theta(p_0) \delta(p^2 - m^2)$$

- Work in the centre-of-mass frame of the two final-state particles. Justify this!