Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borschensky, Dr. Cody B. Duncan

Exercise Sheet 11

Hand-in Deadline: Mo 24.01.22, 12:00.
Discussion: Di 25.01.22, Mi 26.01.22.

1. **[10 points] Gamma Algebra (II)**

   In this continuation of exercise 2 on sheet 9, we look once more at properties of the γ matrices. Again, only the Clifford algebra,
   $$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1}_4,$$
   is necessary, and you should not use any explicit form of the γ matrices.

   (a) **[1 point]** Besides the four standard γ matrices, a particular combination called γ^5 is usually defined as
   $$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3.$$
   Show that this is equivalent to
   $$\gamma^5 := -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,$$
   where $\varepsilon_{\mu\nu\rho\sigma}$ is the (totally antisymmetric) Levi-Civita symbol in four dimensions with $\varepsilon_{0123} = -1$.

   (b) **[2 points]** Show the following relations for γ^5:
   $$(\gamma^5)^\dagger = \gamma^5, \qquad (\gamma^5)^2 = \mathbb{1}_4, \qquad \{\gamma^\mu, \gamma^5\} = 0.$$
   Note: You may use the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ without deriving it.

   (c) **[4 points]** Show the following identities for traces over γ matrices:

   (i) $\mathrm{Tr}[\gamma^\mu] = 0$,
   (ii) $\mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] = 0$, if n is odd,
   (iii) $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$,
   (iv) $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$.

   (d) **[3 points]** Show the following identities for traces involving γ^5:

   (i) $\mathrm{Tr}[\gamma^5] = 0$,
   (ii) $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^5] = 0$,
   (iii) $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] = -4i\varepsilon^{\mu\nu\rho\sigma}$.

2. **[9 points] Scattering Processes in QED**

   In this exercise, you will discuss two important processes in QED: eletron-positron annihilation into photons, which will be discussed in the lecture, as well as electron-photon scattering, also known as Compton scattering.

(a) **[1 point]** Consider first the process of electron-positron annihilation into two photons,

$$e^-(p_1) + e^+(p_2) \to \gamma(k_1) + \gamma(k_2).$$

Draw all Feynman diagrams contributing at leading order. Label them with the momenta of the internal propagators and the wave function factors for the external particles (including their momenta).

(b) **[2 points]** Use the Feynman rules of QED to write down the contribution of the diagrams of **(a)** to the scattering amplitude. You should obtain the following result:

$$\begin{aligned} i\mathcal{M}_{e^-e^+} &= i\mathcal{M}_{e^-e^+,t} + i\mathcal{M}_{e^-e^+,u} \\ &= ie^2 \varepsilon_\mu^{(\lambda_1)*}(k_1)\, \varepsilon_\nu^{(\lambda_2)*}(k_2)\, \bar{v}_{s_2}(p_2) \left[\frac{2p_1^\mu \gamma^\nu - \gamma^\nu \not{k}_1 \gamma^\mu}{2p_1 \cdot k_1} + \frac{2p_1^\nu \gamma^\mu - \gamma^\mu \not{k}_2 \gamma^\nu}{2p_1 \cdot k_2} \right] u_{s_1}(p_1), \end{aligned}$$

where s_1, s_2 (λ_1, λ_2) are the spins (polarizations) of the electrons (photons).

Note: You may use any of the results that are derived in the lecture. Furthermore, the on-shell relations $p_1^2 = p_2^2 = m^2$ (with m the mass of the electron and positron) and $k_1^2 = k_2^2 = 0$, the anticommutation relation for γ matrices as well as the Dirac equations for the spinors might be helpful for simplifying the amplitude.

(c) **[2 points]** A correction to the process of **(a)** is given by an additional emission of a photon into the final state, leading to the process

$$e^- e^+ \to \gamma\gamma\gamma.$$

Draw the corresponding Feynman diagrams for this process (you do not need to add labels for the momenta). Can you guess how many diagrams there are for n photons in the final state?

(d) **[1 point]** A related process is Compton scattering, which is the dominant interaction between electromagnetic radiation and matter for photon energies of roughly 100 keV to 10 MeV. It is described by the inelastic scattering process:

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k').$$

Draw the leading-order Feynman diagrams for this process. How do they differ to the ones from part **(a)**?

(e) **[3 point]** Write down the scattering amplitude for the diagrams of **(d)**. You should obtain:

$$\begin{aligned} i\mathcal{M}_{e^-\gamma} &= i\mathcal{M}_{e^-\gamma,s} + i\mathcal{M}_{e^-\gamma,u} \\ &= -ie^2 \varepsilon_\mu^{(\lambda)}(k)\, \varepsilon_\nu^{(\lambda')*}(k')\, \bar{u}_{s'}(p') \left[\frac{2p^\mu \gamma^\nu + \gamma^\nu \not{k} \gamma^\mu}{2p \cdot k} - \frac{2p^\nu \gamma^\mu - \gamma^\mu \not{k}' \gamma^\nu}{2p \cdot k'} \right] u_s(p), \end{aligned}$$

where again s, s' (λ, λ') are the spins (polarizations) of the electrons (photons). Can you think of a set of rules (for the momenta, spinors, ...) to relate this scattering amplitude to the result of part **(b)**?