Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borchensky, Dr. Cody B Duncan

Exercise Sheet 2

Hand-in Deadline: Mo 01.11.21, 12:00.
Discussion: Di 02.11.21, Mi 03.11.21.

1. **[7 points] Lie Algebra structure of $SU(N)$ and $SO(N)$**

   (a) **[1 point]** Show that real $N \times N$ unimodular (i.e. $\det(R) = +1$) and orthogonal ($R^{\mathrm{T}} R = \mathbb{1}_N$) matrices (i.e. the elements of $SO(N)$) have $N(N-1)/2$ independent parameters.

   (b) **[1 point]** Show that complex $N \times N$ unimodular (i.e. $\det(U) = +1$) and unitary ($R^{\dagger} R = \mathbb{1}_N$) matrices (i.e. the elements of $SU(N)$) have $N^2 - 1$ independent parameters.

   (c) **[2 points]** Prove that any finite transformation $U \in SU(N)$ may be written as a combination of infinitesimal transformations $U(\theta) = \exp(i\theta_a J^a)$ where $a = 1, 2, ..., N^2 - 1$, and the basis elements of the Lie algebra J_a are: (i) *hermitian*, and (ii) traceless matrices, $\mathrm{Tr}(J_a) = 0 \forall a$

   Hint: Take the limit $\theta^a \to 0 \forall a$. To check that $Tr(J_a) = 0$, recall that the identity $det(e^A) = e^{TrA}$.

   We will from now on focus only on the $SU(2)$ case.

   (d) **[1 points]** Convince yourself that a generic $SU(2)$ transformation can be parameterized as:

   $$A = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} = a_0 + i\vec{a} \cdot \vec{\sigma}$$

   where $(a_0, \vec{a}) = (a_0, a_1, a_2, a_3)$ are real-valued parameters satisfying $\sum_i a_i^2 = 1$, and the vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_2)$ contains the Pauli matrices:

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   (e) **[2 points]** Prove that the parameterization of A above is equivalent to:

   $$A = \exp\left[i\frac{\theta}{2}\left(\vec{\sigma} \cdot \vec{n}\right)\right],$$

   where the rotation angle θ and the unit vector $\vec{n}$ are related to $(a_0, \vec{a})$. Make these relations explicit.

2. **[6 points] A closer look at $SO(3)$**

   (a) **[1 point]** Given a generic $SO(3)$ transformation represented in the form $R(\theta) = \exp(\theta_a K_a)$, show that the generators K_a are real and antisymmetric matrices.

   (b) **[1 point]** If instead the transformation is represented as $R(\theta) = \exp(i\theta_a J_a)$, show that the generators J_a are imaginary and hermitian.

   The following matrices implement independent rotations of angles $\theta_a, a = 1, 2, 3$ around the x, y, z axis in 3-dimensional Euclidean space:

$$R_x(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

$$R_y(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

$$R_z(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   (c) **[2 points]** Taking the limit $R(\delta\theta_a) = 1 + i\delta\theta_a J^a + \mathcal{O}(\theta^2)$, obtain the matrix form of $J_a, a = 1, 2, 3$. Check that your result is equivalent to the systematic recipe for computing the group generators:

$$J_a = \frac{1}{i} \frac{\partial U(\theta_a)}{\partial \theta_a}\bigg|_{\theta_a = 0}.$$

   (d) **[2 points]** Using their explicit matrix form, check that J_a satisfy the commutation relation:

$$[J_a, J_b] = f^c_{ab} J_c.$$

   Determine the structure constants of the Lie algebra f^c_{ab}.

3. **[7 points] Tensors under Lorentz transformations**

   Taking $x \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ and the orthogonality condition, $\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma g_{\mu\rho} = g_{\nu\sigma}$,

   (a) **[1 point]** Show that $a^\mu b_\mu$ is invariant under Lorentz transformations.

   (b) **[2 points]** Let $(\Lambda^{-1})^\mu{}_\nu$ be the inverse of $\Lambda_\nu{}^\mu$, meaning that

$$(\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\rho = \delta^\mu_\rho\,.$$

   Show that $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$.

   (c) **[2 points]** Show that $\partial_\mu{}' = (\Lambda^{-1})^\nu{}_\mu \partial_\nu$ where $\partial'_\mu = \frac{\partial}{\partial x'^\mu}$.

   (d) **[2 points]** Show that for a general 2-index tensor $M_{\mu\nu}$, that $M^\mu{}_\nu$ and $M_\nu{}^\mu$ are not necessarily equal. On the other other hand, show that $\delta^\mu{}_\nu = \delta_\nu{}^\mu$. Why is the ordering not relevant in the latter case?