Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borchensky, Dr. Cody B Duncan

Exercise Sheet 2

Hand-in Deadline: Mo 15.11.21, 12:00.
Discussion: Di 16.11.21, Mi 17.11.21.

Please note that you have two weeks to complete this sheet!

1. **[7 points]** A closer look at $SU(3)$

   A generic finite transformation $U \in SU(3)$ in the fundamental representation may be written in terms of 8 real parameters (*why?*) and corresponding generators of the Lie algebra as

$$U = \exp\left(-\frac{i}{2}\sum_{i=1}^{8}\alpha_i\lambda_i\right) \quad \text{with} \quad \mathrm{Tr}\left(\frac{\lambda_i}{2}\cdot\frac{\lambda_j}{2}\right) = \delta_{ij}/2$$

   where λ_i are the 3×3 Gell-Mann matrices. These implement the $SU(3)$ generators in the fundamental representation and, as such, are hermitian, traceless matrices satisfying:

$$\left[\frac{\lambda_a}{2},\frac{\lambda_b}{2}\right] = i f^{ab}_{c}\frac{\lambda_c}{2} \quad \text{or} \quad [t^a, T^b] = i f^{ab}_{c} T^c. \tag{1}$$

   (a) **[2 points]** Using Eq. (1), and recalling that the generators fulfill the Jacobi identity

$$[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0 \tag{2}$$

   Show that $\mathrm{Tr}(T^aT^b) = C_F\delta^{ab}$, where C_F is a real numerical factor. Evaluate it explicitly, using the first Gell-Mann matrix:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   Another important representation is the *adjoint* representation, for which the generators are given by the structure constants themselves:

$$[F_i(A)]_{jk} = -i f_{ijk}$$

   (b) **[2 points]** With the help of the identities in Eq. (1) and (2), show that the adjoint generators $F(A)$ satisfy:

$$(F_iF_j)_{mn} - (F_jF_i)_{mn} = i f_{ijk}(F_k)_{mn}$$

   and thus:

$$[F_i, F_j]_{mn} = i f_{ijk}(F_k)_{mn}$$

   Note that these are exactly the same relations satisfied by the $SU(3)$ generators in the fundamental representation T^a.

(c) **[1 point]** Check that $\mathrm{Tr}(F_a(A)F_b(A)) = f_{ade}f_{bde} = C_A$, where again C_A is a real number.

(d) **[1 points]** Verify that the quadratic Casimir of the group, $F^2 = F^a F_a$ commutes with every individual generator F_a.

2. **[9 points] Hadron Decays and Isospin Invariance**

Recall that the lightest quark flavours are members of an isospin $SU(2)$ doublet $I = \frac{1}{2}$:

$$|I, I_3\rangle_u = |u; \frac{1}{2}, +\frac{1}{2}\rangle \qquad |I, I_3\rangle_d = |d; \frac{1}{2}, -\frac{1}{2}\rangle\,.$$

Similarly, the nucleons also constitute an isospin doublet:

$$|I, I_3\rangle_p = |uud; \frac{1}{2}, +\frac{1}{2}\rangle \qquad |I, I_3\rangle_n = |udd; \frac{1}{2}, -\frac{1}{2}\rangle\,,$$

as do the nucleon resonances:

$$|I, I_3\rangle_{N^+} = |uud; \frac{1}{2}, +\frac{1}{2}\rangle \qquad |I, I_3\rangle_{N^0} = |udd; \frac{1}{2}, -\frac{1}{2}\rangle$$

The three pions form an isospin triplet $I = 1$:

$$|I, I_3\rangle_{\pi^+} = |u\bar{d}; 1, +1\rangle \quad |I, I_3\rangle_{\pi^0} = |\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}); 1, 0\rangle \quad |I, I_3\rangle_{\pi^-} = |d\bar{u}; 1, -1\rangle$$

Finally, the isospin quadruplet states $I = \frac{3}{2}$:

$$|I, I_3\rangle_{\Delta^{++}} = |uuu; \frac{3}{2}, +\frac{3}{2}\rangle \qquad |I, I_3\rangle_{\Delta^+} = |uud; \frac{3}{2}, +\frac{1}{2}\rangle$$
$$|I, I_3\rangle_{\Delta^0} = |udd; \frac{3}{2}, -\frac{1}{2}\rangle \qquad |I, I_3\rangle_{\Delta^-} = |ddd; \frac{3}{2}, -\frac{3}{2}\rangle$$

Notice that the states N^+, Δ^+ have the same quark content and thus the same isospin quantum numbers as the proton. In this sense, they can be considered excited states (resonances) of the proton. Both resonances decay primarily to protons through the following channels:

$$\Delta^+ \to p + \pi^0, \text{or } n + \pi^+ \qquad N^+ \to p + \pi^0, \text{or } n + \pi^+$$

(a) **[2 points]** Represent each decay mode using a quark diagram.

(b) **[2 points]** Verify that quark flavour, electric charge, and total isospin are conserved. Which of the fundamental interactions (strong, weak, or electromagnetic) is therefore governing these decays?

(c) **[3 points]** Evaluate the decay width ratios:

$$R_{N^+} = \frac{\Gamma(N^+ \to n + \pi^+)}{\Gamma(N^+ \to p + \pi^0)} \qquad R_{\Delta^+} = \frac{\Gamma(\Delta^+ \to n + \pi^+)}{\Gamma(\Delta^+ \to p + \pi^0)}$$

Hint: Given the fact that total isospin is conserved, and making use of the Clebsch-Gordan coefficients (provide at the bottom of this question) to rewrite the isospin of the individual final states into the coupled basis. **To avoid being overwhelmed: you should only need the $1 \times 1/2$ table! (i.e. top-left, second down)**

(d) **[2 points]** To cross-check your results: verify that the corresponding branching fractions yield:

$$\mathrm{BR}_{N^+\to n+\pi^+} = \frac{\Gamma(N^+ \to n + \pi^+)}{\Gamma(N^+ \to n + \pi^+) + \Gamma(N^+ \to p + \pi^0)} \approx 66\%$$
$$\mathrm{BR}_{\Delta^+\to n+\pi^+} = \frac{\Gamma(\Delta^+ \to n + \pi^+)}{\Gamma(\Delta^+ \to n + \pi^+) + \Gamma(\Delta^+ \to p + \pi^0)} \approx 33\%$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

1/2×1/2

m1	m2	J=1, M=+1
+1/2	+1/2	1

m1	m2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m1	m2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m1	m2	J=3/2, M=+3/2
+1	+1/2	1

m1	m2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m1	m2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m1	m2	J=3/2, M=−3/2
−1	−1/2	1

2×1

m1	m2	J=3, M=+3
+2	+1	1

m1	m2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m1	m2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m1	m2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m1	m2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m1	m2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m1	m2	J=3, M=−3
−2	−1	1

1×1

m1	m2	J=2, M=+2
+1	+1	1

m1	m2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m1	m2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m1	m2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m1	m2	J=2, M=−2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

m1	m2	J=5/2, M=+5/2
+2	+1/2	1

m1	m2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m1	m2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m1	m2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m1	m2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m1	m2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m1	m2	J=2, M=+2
+3/2	+1/2	1

m1	m2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m1	m2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m1	m2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m1	m2	J=2, M=−2
−3/2	−1/2	1

3/2×1

m1	m2	J=5/2, M=+5/2
+3/2	+1	1

m1	m2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m1	m2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m1	m2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m1	m2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m1	m2	J=5/2, M=−5/2
−3/2	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

2×3/2

m1	m2	J=7/2, M=+7/2
+2	+3/2	1

m1	m2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m1	m2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m1	m2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m1	m2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m1	m2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m1	m2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m1	m2	J=7/2, M=−7/2
−2	−3/2	1

3/2×3/2

m1	m2	J=3, M=+3
+3/2	+3/2	1

m1	m2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m1	m2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m1	m2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m1	m2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m1	m2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m1	m2	J=3, M=−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

m1	m2	J=4, M=+4
+2	+2	1

m1	m2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m1	m2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m1	m2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m1	m2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m1	m2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m1	m2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m1	m2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m1	m2	J=4, M=−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.