Karlsruher Institut für Technologie

Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borschensky, Dr. Cody B Duncan

Exercise Sheet 4

Hand-in Deadline: Mo 22.11.21, 12:00.
Discussion: Di 23.11.21, Mi 24.11.21.

1. **[6 points] Pion-Nucleon Scattering and Isospin Invariance**

   Consider pion-nucleon scattering processes of the form $\{\Pi + N \to \Pi' + N'\}$, with pions denoted $\Pi = \{\pi^+, \pi^0, \pi^-\}$ and nucleons $N = \{p, n\}$, mediated by the strong interaction.

   (a) **[1 points]** List down all of the possible processes (there are 10) and classify them according to the total isospin of the initial (final) states in the coupled basis.

   (b) **[1 point]** If we denote the 3rd component of the isospin of each initial (and final respectively) state particles as:

   $$I_3^{\Pi} = \mu, \;\; I_3^{N} = \nu \qquad \text{and} \;\; I_3^{\Pi'} = \mu', \;\; I_3^{N'} = \nu',$$

   show that the S-matrix elements $\langle \Pi' + N' | \hat{S} | \Pi + N \rangle$ in the isospin space can be written as:

   $$\langle 1, \mu', \frac{1}{2}, \nu' | \hat{S} | 1, \mu, \frac{1}{2}, \nu \rangle = \Big[C_{1\mu, \frac{1}{2}\nu; \frac{1}{2}, \mu+\nu} C_{1\mu', \frac{1}{2}\nu'; \frac{1}{2}, \mu'+\nu'} \mathcal{A}^{\frac{1}{2}}_{\Pi+N\to\Pi'+N'} + C_{1\mu, \frac{1}{2}\nu; \frac{3}{2}, \mu+\nu} C_{1\mu', \frac{1}{2}\nu'; \frac{3}{2}, \mu'+\nu'} \mathcal{A}^{\frac{3}{2}}_{\Pi+N\to\Pi'+N'} \Big] \delta_{\mu+\nu, \mu'+\nu'}$$

   in terms of the elements of the S-matrix $\mathcal{A}^{I}_{\Pi+N\to\Pi'+N'} = \langle I, \mu' + \nu' | \hat{S} | I, \mu + \nu \rangle$ in the coupled isospin basis, along with the corresponding Clebsch-Gordan coefficients. How does this result reflect the Wigner-Eckart theorem?

   (c) **[2 points]** Apply the above result to the specific channels $\pi^- + p \to \pi^- + p$ and $\pi^- + p \to \pi^0 + n$ to demonstrate that:

   $$\left\langle \hat{S} \right\rangle_{\pi^-+p\to\pi^-+p} = \frac{1}{3} \Big[\mathcal{A}^{\frac{3}{2}}_{\pi^-+p\to\pi^-+p} + 2\mathcal{A}^{\frac{1}{2}}_{\pi^-+p\to\pi^-+p} \Big]$$

   $$\left\langle \hat{S} \right\rangle_{\pi^-+p\to\pi^0+n} = \frac{\sqrt{2}}{3} \Big[\mathcal{A}^{\frac{3}{2}}_{\pi^-+p\to\pi^0+n} - \mathcal{A}^{\frac{1}{2}}_{\pi^-+p\to\pi^0+n} \Big]$$

   Hint: You will need the Clebsch-Gordan coefficient table from last week.

   (d) **[2 points]** Verify that the cross section ratio between the scattering processes with $\pi^+ + p$ and $\pi^- + p$ initial states is approximately:

   $$\frac{\sigma(\pi^+ + p)}{\sigma(\pi^- + p)} \approx 3,$$

   when measured in collision experiments with centre-of-mass energies $\sqrt{s} \approx m_{\Delta^0}$ (see last sheet for details about the neutral Delta baryon).

2. **[5 points] Relativistic Kinematics**

We consider the fixed-target experiment $\pi^+ + p \to \pi^+ + p$, where the proton is initially at rest. Find an expression for the lab-frame velocity of the outgoing pion as a function of an arbitrary centre-of-mass energy $\sqrt{s}$.

You may find the Källén function useful:

$$\lambda^2(a^2, b^2, c^2) = (a^2 - b^2 - c^2)^2 - 4b^2c^2$$

Hinweis: You may find the hints attached at the end of the sheet to be a useful guide or set of benchmarks to solving the problem.

3. **[9 points]** Flavour-spin wave function of a hadron

To completely characterize the quantum state of hadrons, we need a product wavefunction which factorizes into several pieces, each of them corresponding to separate Hilbert spaces:

- a **spatial** part, describing the relative location and motion of the quarks,
- a **spin** part, representing the orientation of their spins,
- a **flavour** part, indicating the quark type (e.g. u, d),
- a **colour** part, specifying the individual quark colour charges.

$$|\Psi\rangle_{\text{hadron}} = |\Psi^{\text{space}}\rangle \otimes |\Psi^{\text{spin}}\rangle \otimes |\Psi^{\text{flavour}}\rangle \otimes |\Psi^{\text{colour}}\rangle\,.$$

From Pauli's Exclusion Principle, we know that the total wavefunction of a baryon must be antisymmetric under the permutation of two quarks.

For the spatial part, we will assume that the lowest-lying hadronic states are to be bound states of (anti)quarks with no relative angular momenta, $\vec{L} = 0$. The spatial wavefunction is therefore **symmetric** for this question. The spin state can be either completely symmetric ($j = s = 3/2$) or of mixed symmetry ($j = s = 1/2$). Finally, due to colour confinement, all hadron states are colour singlets, hence the colour wavefunction is completely anti-symmetric.

(a) **[0 points]** With all these ingredients in mind, convince yourself, that the spin-flavour (sf) wavefunction of the state Δ^{++} from the baryon decouplet is given by (quite trivially!):

$$|\Psi^{\text{sf}}\rangle_{\Delta^{++}} = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle = |u\uparrow u\uparrow u\uparrow\rangle$$

(b) **[2 points]** Using the same notation, write down the *normalized* spin-flavour wavefunction for the Δ^+ ($J = 3/2$, spin-half, uud bound state). If it were feasible to pull one quark out, what would the probability that it would be a d-quark with spin up?

Constructing $|\Psi^{\text{spin}}\rangle \otimes |\Psi^{\text{flavour}}\rangle$ for states of the baryon octet is a little trickier, as we must combine states of mixed symmetry to make a completely symmetric combination. The general recipe is:

$$|\Psi^{\text{spin}}\rangle \otimes |\Psi^{\text{flavour}}\rangle = N\Big\{\, |\Psi^{\text{spin}}_{12}\rangle \otimes |\Psi^{\text{flavour}}_{12}\rangle + |\Psi^{\text{spin}}_{13}\rangle \otimes |\Psi^{\text{flavour}}_{13}\rangle + |\Psi^{\text{spin}}_{23}\rangle \otimes |\Psi^{\text{flavour}}_{23}\rangle \,\Big\},$$

where Ψ_{ij} denotes a wavefunction with mixed symmetry, viz. antisymmetric under $i \leftrightarrow j$ quark-pair exchange. This way, the product wavefunction $\Psi^{\text{flav}}_{ij} \otimes \Psi^{\text{spin}}_{ij}$ is symmetric under such quark exchanges.

(c) **[3 points]** Write down the *six* mixed symmetry spin-half wavefunctions $|\Psi_{ij}^{\text{spin}}\rangle$ for $i, j = 1, 2, 3$. Notice that the exact same structure applies to the isospin-half wavefunction.

(d) **[3 points]** From the above result, show that the spin-flavour wavefunction of the proton with spin-up can be written as:

$$|\Psi\rangle_p^{\text{sf}} = \frac{1}{3\sqrt{2}}\Big[2\,|u\uparrow u\uparrow d\downarrow\rangle + 2\,|u\uparrow d\downarrow u\uparrow\rangle + 2\,|d\downarrow u\uparrow u\uparrow\rangle - |u\uparrow u\downarrow d\uparrow\rangle$$
$$- |u\uparrow d\uparrow u\downarrow\rangle - |d\uparrow u\uparrow u\downarrow\rangle - |u\downarrow u\uparrow d\uparrow\rangle - |u\downarrow d\uparrow u\uparrow\rangle - |d\uparrow u\downarrow u\uparrow\rangle\Big].$$

(e) **[1 point]** The interaction of a spin-half particle with a classical magnetic field $\vec{B}$ is governed by $\hat{\mathcal{H}}_{\text{Pauli}} = -\vec{\mu}\cdot\vec{B}$, where the magnetic moment operator is given by the 3rd component projection, $\hat{\mu}_z = \frac{q}{2m}\hat{S}_z$. Show that the magnetic moment of the proton can be written in terms of the up and down-quark magnetic moments as $\mu_p = \frac{1}{3}(4\mu_u - \mu_d)$, where:

$$\mu_u = \frac{2}{3}\frac{e}{2m_u}, \qquad \text{and} \qquad \mu_d = -\frac{1}{3}\frac{e}{2m_d}.$$

By direct analogy, evaluate the neutron magnetic moment μ_n and compare the ratio of μ_n/μ_p calculated here to the experimental measurement:

$$\left.\frac{\mu_n}{\mu_p}\right|_{\text{exp.}} = -0.68497945(58).$$

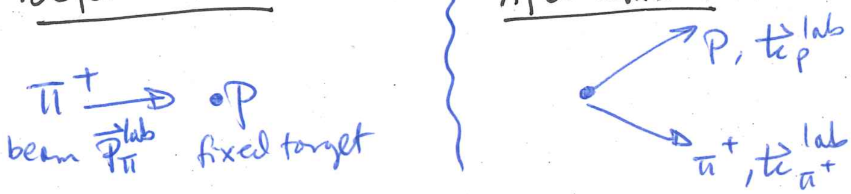
Hint: For the numerical estimate, recall that $m_u = m_d$ under the assumption of isospin invariance.

[Hints to solve Ex. 1e of sheet 4
ie. Useful tricks for relativistic kinematics]

Key Idea: Use Lorentz transformations to switch between the lab & CoM frames.
(CoM → centre-of-momentum / mass)

Lab Frame S

Before collision | After collision:

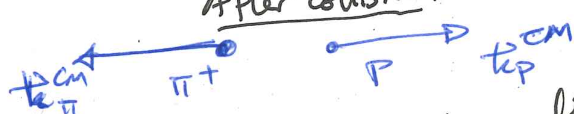


①. Use total 4-mom. before collision $P_\mu^{lab} = P_\mu^{lab,\pi^+} + P_\mu^{lab,p}$ to express E_π^{lab} in terms of $s = P_\mu^{lab} P^{lab\,\mu}$

Show:
$$E_\pi^{lab} = \frac{s - m_p^2 - m_\pi^2}{2m_p}.$$

Centre of mass frame

After collision:



②. Use total 4-mom. after collision $P_\mu^{cm} = P_\mu^{cm,\pi^+} + P_\mu^{cm,p}$ to calculate $s = P_\mu^{cm} P^{cm\,\mu}$ (How are $\vec{k}_\pi^{\,cm}$ & $\vec{k}_p^{\,cm}$ related?)

③. Find relative velocity between S & S':
$|\vec{u}|$
Calculate $|\vec{u}|$ by considering how $\vec{p}_\pi^{\,lab}$ & $\vec{p}_p^{\,lab}$ transform into the CoM frame $\vec{p}_\pi^{\,cm}$ & $\vec{p}_p^{\,cm}$.

④. Use the result of ③ to transform results in ② back to the lab frame, ie find $\vec{k}_\pi^{\,lab}$ in terms of E_π^{cm} & $\vec{k}_\pi^{\,cm}$.

Lab Frame S

⑤. With $\vec{k}_\pi^{\,lab}$ back in the lab frame, how are $|\vec{k}_\pi^{\,lab}|$ and v_π^{lab} related?