Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borschensky, Dr. Cody B Duncan

Exercise Sheet 5

Hand-in Deadline: Mo 29.11.21, 12:00.
Discussion: Di 30.11.21, Mi 01.12.21.

1. **[10 points] Poincaré Group: the Pauli-Lubanski Operator**

The *Pauli-Lubanski pseudovector* describes the spin state of a moving particle:

$$W_\mu = \frac{1}{2}\tilde{M}_{\mu\sigma}P^\sigma = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma,$$

where $M^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)$ denotes the relativistic angular momentum tensor operator, and $P^\mu = i\partial^\mu$ is the 4-momentum. Its commutation relation is given as:

$$[W_\mu, W_\nu] = -i\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma.$$

The simultaneous eigenvalues of P^2 and W^2 can be used to classify particles according to their mass and spin as irreducible representations of the Poincaré algebra.

We define the generalized Levi-Civita symbol in four dimensions as:

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an odd permutation of } \{0,1,2,3\} \\ -1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ 0 \text{ otherwise} \end{cases},$$

with $\varepsilon^{0123} = g^{\mu 0}g^{\nu 1}g^{\rho 2}g^{\sigma 3}\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon_{0123}$.

(a) **[2 points]** Show that the components of W_μ for a particle at rest are $(0, -m\vec{J})^T$, where $\vec{J} = \vec{x}\times\vec{P}$ is the total angular momentum operator in three dimensions.

(b) **[3 points]** Prove the following identities:

(i) $[M_{\mu\nu}, P_\rho] = i\left(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu\right)$,

(ii) $W_\mu P^\mu = 0$,

(iii) $[W_\mu, P_\nu] = 0$.

(c) **[3 points]** Show that P^2 and W^2 are the Casimir operators of the Poincaré group, *i.e.* that they commute with all its generators,

$$[P^2, P_\mu] = [P^2, M_{\mu\nu}] = 0 \qquad \text{and} \qquad [W^2, P_\mu] = [W^2, M_{\mu\nu}] = 0$$

You do not need to prove the last identity, $[W^2, M_{\mu\nu}] = 0$, as the calculation is really tedious.

(d) **[2 points]** With

$$W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} P^2 + M_{\mu\rho} M^{\nu\rho} P^\mu P_\nu,$$

show that

$$W^2 |\vec{p} = \vec{0}, m, s\rangle = -m^2 s(s+1) |\vec{p} = \vec{0}, m, s\rangle,$$

where $|\vec{p} = \vec{0}, m, s\rangle$ is an eigenvector for a particle of mass m, spin s, and (vanishing) 3-momentum $\vec{p} = \vec{0}$, and $-m^2 s(s+1)$ is the corresponding eigenvalue.

2. **[5 points] Dimensional Analysis of the Lagrangian**

Recall from Lagrangian mechanics that the action S is defined as the integral over time of the Lagrangian *function* L:

$$S = \int \mathrm{d}t \, L.$$

The Lagrangian has dimension of energy. It can be written as an integral over space, $L = \int \mathrm{d}^3 x \, \mathcal{L}$, of a quantity $\mathcal{L}$ that is known as the Lagrangian *density*. This way, the action can be written as

$$S = \int \mathrm{d}^4 x \, \mathcal{L}.$$

Consider now the Klein-Gordon Lagrangian

$$\mathcal{L}_{\mathrm{KG}} = \frac{1}{2} \partial_\mu \phi(x) \, \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2.$$

Here, $\phi(x)$ is a real scalar field that depends on the space-time four vector x, m is a mass, ∂_μ and ∂^μ are the derivatives with respect to the components of the four-vector x in the covariant and contravariant forms, respectively.

Make use of natural units throughout the exercise, *i.e.* when we say that something is *dimensionless* or *of dimension* x, we always refer to the dimension of energy.

(a) **[2 point]** Determine the dimension of the action S, of the integration element $\mathrm{d}^4 x$ and of the Lagrangian density $\mathcal{L}_{\mathrm{KG}}$. Derive also the dimension of $\phi(x)$. What is the dimension of the derivative ∂_μ?

(b) **[3 point]** Argue if the following terms in a Lagrangian density are allowed, and, if they are not, point out all the reasons why they are not allowed:

(i) $\mathcal{L}_1 = g A(x) \, \bar{C}(x) \, C(x)$,

(ii) $\mathcal{L}_2 = m A(x) \, \bar{C}(x) \, C(x)$,

(iii) $\mathcal{L}_3 = i A(x) \, B^\mu(x) \, B^\nu(x)$,

(iv) $\mathcal{L}_4 = \frac{1}{m} B_\mu(x) \, B^\mu(x) \, \bar{C}(x) \, C(x)$,

(v) $\mathcal{L}_5 = \frac{g^2}{m} \partial_\mu B^\mu(x) \, A(x) \, \frac{\partial A(x)}{\partial t}$,

(vi) $\mathcal{L}_6 = \frac{1}{4} g^4 m^4$.

Assume that the coupling g is dimensionless, the coupling m is of dimension 1, the scalar and vector fields $A(x)$ and $B_\mu(x)$ are each of dimension 1, and the spinor fields $C(x)$ and $\bar{C}(x)$ are of dimension $\frac{3}{2}$.

3. **[3 points] Invariance of the Lagrangian Density**

Consider the following transformation to the Lagrangian density $\mathcal{L} = \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$ of scalar fields $\phi_i(x)$:

$$\mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu,$$

where $\Lambda^\mu = \Lambda^\mu(\phi_i(x))$ denotes a generic function of the set of scalar fields $\{\phi_i(x)\}$. Show that this transformation leaves the equations of motion for the ϕ_i unchanged.

Hint: Assume that Λ^μ is a smooth function that vanishes for field configurations at infinity.