1. [10 points] Poincaré Group: the Pauli-Lubanski Operator

The Pauli-Lubanski pseudovector describes the spin state of a moving particle:

\[ W_\mu = \frac{1}{2} \bar{M}_{\mu\sigma} P^{\sigma} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^{\sigma}, \]

where \( M^{\mu\nu} = i \left( x^\mu \partial^\nu - x^\nu \partial^\mu \right) \) denotes the relativistic angular momentum tensor operator, and \( P^\mu = i \partial^\mu \) is the 4-momentum. Its commutation relation is given as:

\[ [W_\mu, W_\nu] = -i \varepsilon_{\mu\nu\rho\sigma} W^{\rho} P^{\sigma}. \]

The simultaneous eigenvalues of \( P^2 \) and \( W^2 \) can be used to classify particles according to their mass and spin as irreducible representations of the Poincaré algebra.

We define the generalized Levi-Civita symbol in four dimensions as:

\[ \varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 
1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an odd permutation of } \{0, 1, 2, 3\} \\
-1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an even permutation of } \{0, 1, 2, 3\} \\
0 & \text{otherwise}
\end{cases}, \]

with \( \varepsilon_{0123} = g^{\mu_0} g^{\nu_1} g^{\rho_2} g^{\sigma_3} \varepsilon_{\mu_\nu\rho\sigma} = -\varepsilon_{0123}. \)

(a) [2 points] Show that the components of \( W_\mu \) for a particle at rest are \((0, -m\vec{J})^T\), where \( \vec{J} = \vec{x} \times \vec{P} \) is the total angular momentum operator in three dimensions.

(b) [3 points] Prove the following identities:

(i) \([M_{\mu\nu}, P_\rho] = i (g_{\rho\rho} P_\mu - g_{\mu\rho} P_\nu),\]
(ii) \(W_\mu P^\mu = 0,\]
(iii) \([W_\mu, P_\nu] = 0.\)

(c) [3 points] Show that \( P^2 \) and \( W^2 \) are the Casimir operators of the Poincaré group, i.e. that they commute with all its generators,

\[ [P^2, P_\mu] = [P^2, M_{\mu\nu}] = 0 \quad \text{and} \quad [W^2, P_\mu] = [W^2, M_{\mu\nu}] = 0. \]

You do not need to prove the last identity, \([W^2, M_{\mu\nu}] = 0, \) as the calculation is really tedious.
(d) [2 points] With
\[ W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} P^2 + M_{\mu\rho} M^{\nu\rho} P^\mu P^\nu, \]
show that
\[ W^2 |\vec{p} = \vec{0}, m, s\rangle = -m^2 s(s + 1) |\vec{p} = \vec{0}, m, s\rangle, \]
where \(|\vec{p} = \vec{0}, m, s\rangle\) is an eigenvector for a particle of mass \(m\), spin \(s\), and (vanishing) 3-momentum \(\vec{p} = \vec{0}\), and \(-m^2 s(s + 1)\) is the corresponding eigenvalue.

2. [5 points] Dimensional Analysis of the Lagrangian
Recall from Lagrangian mechanics that the action \(S\) is defined as the integral over time of the Lagrangian function \(L\):
\[ S = \int dt \, L. \]
The Lagrangian has dimension of energy. It can be written as an integral over space, \(L = \int d^3 x \, L\), of a quantity \(L\) that is known as the Lagrangian density. This way, the action can be written as
\[ S = \int d^4 x \, L. \]
Consider now the Klein-Gordon Lagrangian
\[ L_{KG} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2. \]
Here, \(\phi(x)\) is a real scalar field that depends on the space-time four vector \(x\), \(m\) is a mass, \(\partial_\mu\) and \(\partial^\mu\) are the derivatives with respect to the components of the four-vector \(x\) in the covariant and contravariant forms, respectively.

Make use of natural units throughout the exercise, i.e. when we say that something is dimensionless or of dimension \(x\), we always refer to the dimension of energy.

(a) [2 point] Determine the dimension of the action \(S\), of the integration element \(d^4 x\) and of the Lagrangian density \(L_{KG}\). Derive also the dimension of \(\phi(x)\). What is the dimension of the derivative \(\partial_\mu\)?

(b) [3 point] Argue if the following terms in a Lagrangian density are allowed, and, if they are not, point out all the reasons why they are not allowed:
(i) \( L_1 = g A(x) \bar{C}(x) C(x), \)
(ii) \( L_2 = m A(x) \bar{C}(x) C(x), \)
(iii) \( L_3 = i A(x) B^\mu(x) B^\nu(x), \)
(iv) \( L_4 = \frac{1}{m} B_\mu(x) B^\mu(x) \bar{C}(x) C(x), \)
(v) \( L_5 = \frac{g^2}{m} \partial_\mu B^\mu(x) A(x) \frac{\partial A(x)}{\partial t}, \)
(vi) \( L_6 = \frac{1}{4} g^4 m^4. \)
Assume that the coupling \(g\) is dimensionless, the coupling \(m\) is of dimension 1, the scalar and vector fields \(A(x)\) and \(B_\mu(x)\) are each of dimension 1, and the spinor fields \(C(x)\) and \(\bar{C}(x)\) are of dimension \(\frac{3}{2}\).

3. [3 points] Invariance of the Lagrangian Density
Consider the following transformation to the Lagrangian density \(L = L(\phi_i(x), \partial_\mu \phi_i(x))\) of scalar fields \(\phi_i(x)\):
\[ L' = L + \partial_\mu \Lambda^\mu, \]
where \( \Lambda^\mu = \Lambda^\mu(\phi_i(x)) \) denotes a generic function of the set of scalar fields \( \{\phi_i(x)\} \). Show that this transformation leaves the equations of motion for the \( \phi_i \) unchanged.

*Hint:* Assume that \( \Lambda^\mu \) is a smooth function that vanishes for field configurations at infinity.