Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borschensky, Dr. Cody B Duncan

Exercise Sheet 6

Hand-in Deadline: Mo 06.12.21, 12:00.
Discussion: Di 07.12.21, Mi 08.12.21.

1. **[6 points] Relativistic Wave Equation for A Scalar Particle**

   (a) **[1 point]** Explain why the Schrödinger Equation is not suitable for a relativistic formulation of quantum mechanics.

   Hinweis: Books like Peskin & Schroeder's Quantum Field Theory, or Srednicki's QFT (available for free online: `https://web.physics.ucsb.edu/~mark/qft.html`) provide helpful discussions on this topic.

   (b) **[1 point]** Identify, and *briefly* discuss **two** reasons why the Klein-Gordon Equation does not yet provide a fully satisfactory alternative.

   (c) Starting from the Klein-Gordon Equation for a free scalar field:

$$\Big(\Box + m^2\Big)\phi(\vec{x},t) := \Big(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\Big)\phi(\vec{x},t) = 0,$$

   where $\Box = \partial_\mu \partial^\mu$ is the 4-dimensional d'Alembertian operator.

   i. **[1 point]** Apply the *minimal coupling* prescription:

$$-i\vec{\nabla} \to -i\vec{\nabla} + q\vec{A}$$

   to derive the equation that describes a relativistic scalar particle ϕ coupled to a classical magnetic field A.

   ii. **[2 points]** Making (and briefly justifying) suitable approximations, study the equation obtained in part (i) above *in the non-relativistic limit*.
   Hinweis: It is useful to rewrite the scalar field as: $\phi(x) = \psi(\vec{x},t)\exp(-imt)$, where the ψ encodes the non-relativistic part. Why? In non-relativistic problems, do we consider long time scales or short times scales? What happens to $|\partial_t\psi|$ vs $|\partial_t^2\psi|$?

   iii. **[1 point]** Show that the result is equivalent to imposing the minimal coupling Ansatz to the Schrödinger equation directly.

2. **[4 points] Scalar Field Lagrangians: Noether theorem**

   (a) **[2 points]** Show that the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\Big((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\Big) - \frac{m^2}{2}(\phi_1^2+\phi_2^2) - \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2,$$

remains invariant under the transformation ($\theta \in \mathbb{R}$):

$$\begin{aligned}
\phi_1 &\to \phi_1' = \phi_1 \cos\theta - \phi_2 \sin\theta, \\
\phi_2 &\to \phi_2' = \phi_1 \sin\theta + \phi_2 \cos\theta, \\
x_\mu &\to x_\mu' = x_\mu
\end{aligned}$$

Note that in the Lagrangian, the derivatives are shorthand for:

$$(\partial_\mu \phi)^2 = (\partial_\mu \phi)(\partial^\mu \phi)$$

(b) **[2 points]** Compute the Noether current associated to the above transformation:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i - T^{\mu\nu} \delta x_\nu,$$

with:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} (\partial^\nu \phi_i) - \mathcal{L} g^{\mu\nu}$$

as well as its corresponding Noether charge:

$$Q = \int \mathrm{d}^3x J^0$$

3. **[8 points] Lagrangian Formulation of Classical Electrodynamics**

Classical electrodynamics is governed by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F^{\mu\nu} - J^\mu A_\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor and $J_\mu = (\rho, \vec{J})$ is a source term.

The field-strength tensor can be explicitly written as:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

where $\vec{E} = (E_x, E_y, E_z)$ and $\vec{B} = (B_x, B_y, B_z)$ are the electric and magnetic fields respectively.

The Euler-Lagrange equations take the Lagrangian density above, and allow us to generate the equations of motion for the fields involved (in this case A^μ):

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial\big(\partial_\mu A_\nu\big)} \Big) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

Note that in general for a Lagrangian involving n unique fields, you will have n corresponding Euler-Lagrange equations (one for each unique field).

(a) **[3 points]** Show that calculating the Euler-Lagrange equation yields:

$$\partial_\mu F^{\mu\nu} - J^\nu = 0$$

Consider now the gauge transformation:

$$A^\mu(x) \to A^{'\mu}(x) = A^\mu(x) + \partial^\mu f(x)$$

and show that the transformed Lagrangian $\mathcal{L}'$ leads to the same equations of motion for $A^\mu(x)$, i.e. that the Euler-Lagrange equations are the same.

(b) **[3 points]** Show how Maxwell's equations (in Heavyside-Lorentz units $c = \epsilon_0 = \mu_0 = 1$):

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \times \vec{B} = \vec{J} + \partial_t \vec{E}$$
$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \cdot \vec{E} = -\partial_t \vec{B},$$

can be obtained from:

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} := \partial_\mu \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0,$$

where $\tilde{F}^{\mu\nu}$ is known as the *dual* field-strength tensor (can you explicitly write it?).
Prove that, in the particular case where the source term is absent, the homogeneous Maxwell's Equations can be derived from the so-called Bianchi Identity:

$$\partial^{[\mu} F^{\nu\rho]} := \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0 \tag{1}$$

(c) **[2 points]** Fermi proposed an alternative formulation:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu) - J^\mu A_\mu$$

Obtain the equations of motion in this case, and determine the *necessary condition* to reproduce the standard form of Maxwell's equations.

Extra work (no points awarded)

For those interested in taking the above questions to the next step, here are two extra question that extend question 3's Gedankenexperiment.

(d) Let's now consider a speculative extension of classical electrodynamics. We add a new term to the field-strength tensor:

$$F'^{\mu\nu} = F^{\mu\nu} - \epsilon^{\mu\nu\rho\sigma} \partial_\rho A'_\sigma,$$

as a function of a *second* 4-potential: $A'_\mu = (\phi', \vec{A}')^T$.
Show that the inhomogeneous Maxwell's equations in this theory are equivalent to those of the standard electrodynamics.

(e) How would Maxwell's equations differ from their standard form if we *also* allowed an additional source term J'_μ such that the dual field-strength tensor is affected as follows:

$$\partial_\mu \tilde{F}'^{\mu\nu} = J'^\nu?$$

Can you attribute a physical interpretation to the new term in the modified equations?