Karlsruher Institut für Technologie

Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borschensky, Dr. Cody B Duncan

Exercise Sheet 7

Hand-in Deadline: Mo 13.12.21, 12:00.
Discussion: Di 14.12.21, Mi 15.12.21.

1. **[7 points] Complex Scalar Fields (I)**

On the previous sheet, you discussed the Lagrangian for real scalar fields. Consider now a similar Lagrangian for a free theory with two complex-valued scalar fields $\phi_1(x)$ and $\phi_2(x)$:

$$\mathcal{L} = (\partial_\mu \phi_1^*)(\partial^\mu \phi_1) + (\partial_\mu \phi_2^*)(\partial^\mu \phi_2) - m^2(\phi_1^*\phi_1 + \phi_2^*\phi_2). \tag{1}$$

It is easiest to analyze this theory by considering $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$ as basic dynamical variables, rather than the real and imaginary parts of $\phi_{1,2}(x)$.

(a) **[1 point]** Derive the equations of motion for $\phi_{1,2}$ and $\phi_{1,2}^*$.

(b) **[1 point]** Determine the conjugate momenta $\pi_{1,2}$ and $\pi_{1,2}^*$ belonging to $\phi_{1,2}$ and $\phi_{1,2}^*$, respectively.

(c) **[2 points]** Show that the Hamiltonian H is given by:

$$H = \int \mathrm{d}^3\, \mathcal{H},$$

with the Hamiltonian density $\mathcal{H}$:

$$\mathcal{H} = \pi_1^*\pi_1 + \pi_2^*\pi_2 + (\vec{\nabla}\phi_1^*)\cdot(\vec{\nabla}\phi_1) + (\vec{\nabla}\phi_2^*)\cdot(\vec{\nabla}\phi_2) + m^2(\phi_1^*\phi_1 + \phi_2^*\phi_2).$$

Here, $\vec{\nabla}$ is the three-dimensional derivative operator, $\vec{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)^T$.

(d) **[3 points]** On the previous sheet, you have discussed global SO(2) transformations for real scalar fields $\phi_{1,2}$. Discuss now for the Lagrangian of Eq. (1) global SU(2) transformations,

$$\phi_i(x) \longrightarrow \phi_i'(x) = [\exp(i\alpha_k\sigma_k)]_{ij}\,\phi_j(x) \quad \text{for} \quad i,j = 1,2, \quad k = 1,2,3,$$

with parameters $\alpha_k \in \mathbb{R}$ independent of x and the three Pauli matrices σ_k. What are the transformation rules for the complex-conjugated variables $\phi_{1,2}^*(x)$? Show that the Lagrangian of Eq. (1) is invariant under such a transformation. Determine also the three conserved Noether currents and charges.

Hint: It might be useful to rewrite the fields $\phi_{1,2}$ as a two-component vector.

2. **[10 points] Lagrangian of a Massive Vector Field**

The dynamics of a real vector field B_μ of mass $m > 0$ is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}m^2 B_\mu B^\mu, \tag{2}$$

where the field strength tensor $G_{\mu\nu}$ is defined by $G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$ as usual.

(a) **[2 points]** Derive from the Lagrangian the Proca equation, i.e. the equation of motion for a massive vector field B^μ:

$$\partial_\nu G^{\mu\nu} = m^2 B^\mu.$$

(b) **[2 points]** Show with your result from **(a)** that the field B^μ also satisfies

$$(\Box + m^2)B^\mu = 0 \qquad \textbf{and} \qquad \partial_\mu B^\mu = 0,$$

with the d'Alembert operator $\Box = \partial_\mu \partial^\mu$. Are these two equations always valid? What is the difference to the massless case?

(c) **[2 points]** Consider the following local, i.e. x-dependent, transformation of the vector field B_μ:

$$B_\mu(x) \longrightarrow B'_\mu(x) = B_\mu(x) + \partial_\mu \Lambda(x),$$

with an x-dependent function $\Lambda(x)$. This is a gauge transformation that, for the massless case, leaves the physics of the system invariant, as discussed in the lecture. Check if the Lagrangian of Eq. (2) is also invariant under this transformation.

(d) **[1 point]** In Fourier space the vector field is written as (ignoring quantization for now)

$$B^\mu(x) = \int \frac{d^4k}{(2\pi)^4} \varepsilon^\mu(k) e^{ik\cdot x},$$

and $\varepsilon^\mu(k)$ is the polarization vector of the field. We choose the z-axis as the direction of propagation of the field B^μ. Thus we have the following 4-momentum $k^\mu = (E, 0, 0, p)^T$.
Using the results in **(b)**, show that $k_\mu \varepsilon^\mu = 0$.

(e) **[2 point]** The polarization vectors are conventionally normalized to $\varepsilon^*_\mu \varepsilon^\mu = -1$. What are two independent polarization vectors $\varepsilon^\mu_\pm$ transverse to the direction of propagation? With the normalization condition and the result of the previous question, deduce the form of the longitudinal polarization vector ε^μ_L.

(f) **[1 point]** Show that the $\{\varepsilon^\mu_\pm, \varepsilon^\mu_L\}$ satisfy the following completeness relation,

$$\sum_{\lambda=\pm,L} \varepsilon^{\mu *}_\lambda \varepsilon^\nu_\lambda = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}.$$