Karlsruher Institut für Technologie

Einführung in Theoretische Teilchenphysik

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Exercise Sheet 8

Hand-in Deadline: Mo 20.12.21, 12:00.
Discussion: Di 21.12.21, Mi 22.12.21.

This problem set will look at the Klein-Gordon equation in much detail, first with a real and then a complex scalar field. We will discuss canonical quantization and how we can perform field decomposition, a skill we will in later sheets extend to the Dirac equation.

1. **[10 points] Real Scalar Field: Canonical Quantization**

   A real scalar field is governed by the Klein-Gordon Lagrangian:

   $$\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}\big(\partial_\mu \phi(x)\big)^2 - \frac{m^2}{2}\phi(x)^2.$$

   The Fourier Transform of the quantized field can be written as:

   $$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p}\Big[a_p e^{-ipx} + a_p^\dagger e^{ipx}\Big]$$

   where the plane wave solutions obey the orthogonality condition:

   $$\int \mathrm{d}^3 x e^{-ipx} e^{ip'x} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,').$$

   Note that, due to the energy-momentum relation $E_p = \sqrt{\vec{p}^2 + m^2}$, the above condition imples that $p_0' = p_0$. Here we have also used a shorthand for the annihilation and creation operators $a_p = a(\vec{p})$, $a_p^\dagger = a^\dagger(\vec{p})$, i.e. they depend on the 3-momentum $\vec{p}$.

   (a) **[2 points]** Show that the annihilation operator takes the form:

   $$a_p = \int \mathrm{d}^3 x e^{ipx}\big(i\dot{\phi} + E_p \phi(x)\big).$$

   What is the form for the creation operator $a_p^\dagger$?

   (b) **[2 points]** Given in terms of the fields, the canonical commutation relations are:

   $$\big[\phi(\vec{x},t), \phi(\vec{x}\,',t)\big] = \big[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{x}\,',t)\big] = 0$$
   $$\big[\phi(\vec{x},t), \dot{\phi}(\vec{x}\,',t)\big] = i\delta^{(3)}(\vec{x}-\vec{x}\,')$$

   Obtain the corresponding relations for the operator $a_p, a_p^\dagger$:

   $$\big[a_p, a_{p'}\big] = \big[a_p^\dagger, a_{p'}^\dagger\big] = 0$$
   $$\big[a_p, a_{p'}^\dagger\big] = (2\pi)^3 2E_p \delta^{(3)}(\vec{p}-\vec{p}\,')$$

(c) **[1 point]** Evaluate the stress-energy-momentum tensor $T^\mu{}_\nu$ for the scalar field $\phi(\vec{x}, t)$, where the tensor is given by:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi) - \delta^\mu_\nu \mathcal{L}$$

(d) **[3 points]** The energy spectrum of the scalar field is given by the eigenvalues of the Hamilton operator:

$$\hat{H} = \int \mathrm{d}^3x \mathcal{H} = \int \mathrm{d}^3x T^0_0$$

Using the Fourier Transform of the field, along with commutation relations for $a_p, a_p^\dagger$ in part (b), prove that $\hat{H}$ is diagonalized and can be written as:

$$\hat{H} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \Big[2\tilde{N}_p + C\Big],$$

where we have introduced the particle number operator:

$$N = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_p} a_p^\dagger a_p = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \tilde{N}_p,$$

and some constant term C. What is its physical interpretation?

(e) **[2 points]** Similarly, show that the 3-momentum of the field $P_i = \int \mathrm{d}^3x T^0{}_i$ yields:

$$\vec{P} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} \vec{p}\Big[2\tilde{N}_p + C\Big]$$

2. **[10 points] Complex Scalar Field (II)**

We continue the ideas we began studying in Question 1 of Sheet 7, though you do not necessarily need to have complete that problem to do this question.

The Hamiltonian of a complex-valued scalar field obeying the Klein-Gordon equation is given at the classical level by:

$$H = \int \mathrm{d}^3x \Big[\Pi^*\Pi + \big(\vec{\nabla}\phi^*\big) \cdot \big(\vec{\nabla}\phi\big) + m^2\phi^*\phi\Big].$$

The field variables ϕ, ϕ^* are treated as independent quantities, with respective canonically conjugate momenta:

$$\Pi := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi))}$$
$$\Pi^* := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*))}.$$

The theory is symmetric under real phase transformations $\phi \to \phi' = e^{i\alpha}\phi$, and $\phi^* \to \phi'^* = e^{-i\alpha}\phi^*$, $(\alpha \in \mathbb{R})$. This global $U(1)$ symmetry leads at the classical level to the Noether current j^0 and the corresponding Noether charge:

$$Q = \int \mathrm{d}^3x j^0 = i\int \mathrm{d}^3x \big(\Pi^*\phi^* - \Pi\phi\big) \tag{1}$$

In agreement with canonical quantization, the field operators are required to fulfill the following commutation relations at equal time:

$$\big[\hat{\phi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)\big] = \big[\hat{\phi}^\dagger(\vec{x}, t), \hat{\Pi}^\dagger(\vec{x}', t)\big] = i\delta^{(3)}(\vec{x} - \vec{x}'),$$

while the commutators involving the other combinations are required to vanish.

(a) **[2 points]** Prove that the Heisenberg equations:

$$
\begin{aligned}
i\frac{\partial \phi}{\partial t} &= [\phi, \hat{H}] \\
i\frac{\partial \hat{\Pi}}{\partial t} &= [\hat{\Pi}, \hat{H}],
\end{aligned}
$$

imply that ϕ satisfies the Klein-Gordon equation.

(b) **[3 points]** Field decomposition for the complex scalar field works in much the same as for the real scalar field we considered in question 1. Introducing the field decomposition into the quantized normal modes:

$$
\begin{aligned}
\phi(\vec{x}, t) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \left[a_p e^{-ipx} + b_p^\dagger e^{ipx} \right] \\
\phi^\dagger(\vec{x}, t) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \left[a_p^\dagger e^{ipx} + b_p e^{-ipx} \right],
\end{aligned}
$$

show that the classical Noether charge Q in Eq. (1) leads at the quantum level to the charge operator:

$$
\hat{Q} = i \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \left[\hat{N}_p^{(a)} - \hat{N}_p^{(b)} \right],
$$

with:

$$
\hat{N}_p^{(a)} = a_p^\dagger a_p, \qquad \text{and} \quad \hat{N}_p^{(b)} = b_p^\dagger b_p
$$

and $E_p = \sqrt{\vec{p}^2 + m^2}$

(c) **[3 points]** Check that $[\hat{Q}, \hat{H}] = 0$, namely that the charge is a conserved quantity in the quantized theory as well. To do this, recall that the expression of the quantized Hamiltonian (up to the zero-point energy term) is:

$$
\hat{H} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} E_p \left[\hat{N}_p^{(a)} + \hat{N}_p^{(b)} \right].
$$

To investigate the action of $\hat{Q}$ on the field states, let us assume $|\alpha\rangle$ represents an eigenstate of $\hat{Q}$ with eigenvalue q, i.e. $\hat{Q}\,|\alpha\rangle = q\,|\alpha\rangle$.

(d) **[2 point]** Show that:

$$
\begin{aligned}
\hat{Q}\phi^\dagger &= \phi^\dagger (\hat{Q} + 1) \\
\text{and thus} \quad \hat{Q}\phi^\dagger\,|\alpha\rangle &= (q+1)\phi^\dagger\,|\alpha\rangle\,,
\end{aligned}
$$

meaning that $\hat{Q}$, the charge operator, increases the charge of a state by one unit.