Karlsruher Institut für Technologie

Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S. Gieseke – Exercises: Dr. Christoph Borschensky, Dr. Cody B Duncan

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Exercise Sheet 9

Hand-in Deadline: Mo 10.01.22, 12:00.
Discussion: Di 11.01.22, Mi 12.01.22.

Please note that due to the Christmas break, you have three weeks to solve this sheet.

Exercise 1 will introduce an important ingredient for calculating scattering amplitudes with scalar particles, and exercises 2 and 3 will focus on showing mathematical identities that will be extremely helpful when we get around to calculating scattering processes including fermions.

1. **[3 points] Real Scalar Field: Feynman Propagator**

   The propagator of a field describes its probability amplitude for a propagation from one place to another and is an important ingredient in calculating scattering amplitudes. With the quantization of the fields as discussed on the previous sheet and the field operators being expressed in terms of creation and annihilation operators, the so-called Feynman propagator of a real scalar field $\phi(x)$ is defined as the vacuum expectation value of the time-ordered product between the field at different space-time points x and x',

   $$i\Delta_F(x-x') \coloneqq \langle 0|T\phi(x)\phi(x')|0\rangle,$$

   where $T\phi(x)\phi(x') \coloneqq \Theta(t-t')\phi(x)\phi(x') + \Theta(t'-t)\phi(x')\phi(x)$ defines the time-ordering operator. Upon several manipulations (which we do not consider here), the above expression takes the following form in momentum space:

   $$\Delta_F(x) = \lim_{\epsilon\to 0^+} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}. \tag{1}$$

   Show that the Feynman propagator of Eq. (1) fulfills the inhomogeneous Klein-Gordon equation,

   $$\left(\Box + m^2\right)\Delta_F(x) = -\delta^{(4)}(x),$$

   where $\delta^{(4)}(x)$ denotes the 4-dimensional δ-distribution, being "defined" as

   $$\delta^{(4)}(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip\cdot x}.$$

2. **[6 points] Gamma Algebra (I)**

   The Dirac matrices, which appear in the Dirac equation and play an important role when calculating amplitudes with fermions, e.g. scattering or decay processes involving leptons and/or quarks, are defined by the anticommutation relation, namely the Clifford Algebra

   $$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\cdot \mathbb{1}_4 \tag{2}$$

with $\mathbb{1}_4$ the identity matrix in four-dimensional spinor space. This means that two Dirac matrices with different Lorentz indices anticommute:

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \qquad \text{for } \mu \neq \nu,$$

and that the square of γ matrices is proportional to the identity matrix multiplied by $+1$ or -1:

$$\gamma^\mu \gamma^\mu = (\gamma^\mu)^2 = g^{\mu\mu} \cdot \mathbb{1}_4 \qquad (\text{no sum over } \mu)$$

with the convention $g^{00} = -g^{11} = -g^{22} = -g^{33} = +1$. We furthermore define

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

as the commutator between two γ matrices.

We will make use of the Feynman slash notation, i.e. with an arbitrary 4-vector a_μ (e.g. a momentum), we define $\not{a} := a_\mu \gamma^\mu$. The scalar product between two 4-vectors a_μ and b_μ is simply given by the contraction of the Lorentz index, $a \cdot b = a_\mu b^\mu = a_\mu b_\nu g^{\mu\nu}$. Note that these 4-vectors will clearly commute with the γ matrices.

To solve this exercise, you do not need any explicit representation of the γ matrices. It is possible to prove the relations solely with the anticommutative property relation of Eq. (2).

Prove the following relations for any generic four-vectors a^μ and b^μ:

(a) $\not{a}\not{a} = a^2 \cdot \mathbb{1}_4$,

(b) $\not{a}\not{b} = (a \cdot b) \cdot \mathbb{1}_4 - i\sigma^{\mu\nu} a_\mu b_\nu$,

(c) $\gamma_\mu \gamma^\mu = 4 \cdot \mathbb{1}_4$,

(d) $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$,

(e) $\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu = 4 g^{\nu\rho} \cdot \mathbb{1}_4$,

(f) $\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$.

3. **[9 points] Properties of Dirac Spinors**

The momentum-space Dirac spinors $u_s(\vec{p})$ and $v_s(\vec{p})$ (with mass m, spin s, and momentum $\vec{p}$) are defined via the ansatz for plane wave solutions of the Dirac equation:

- $\psi_s(x) = u_s(\vec{p})\, e^{-ip\cdot x}$ for positive energy solutions,
- $\psi_s(x) = v_s(\vec{p})\, e^{ip\cdot x}$ for negative energy solutions.

(a) **[1 point]** Show that:

(i) $\not{p} u_s(\vec{p}) = m u_s(\vec{p})$, $\bar{u}_s(\vec{p})\not{p} = m\bar{u}_s(\vec{p})$,

(ii) $\not{p} v_s(\vec{p}) = -m u_s(\vec{p})$, $\bar{v}_s(\vec{p})\not{p} = -m\bar{v}_s(\vec{p})$.

The $u_s(\vec{p})$ and $v_s(\vec{p})$ fulfill the following orthogonality constraints/conditions:

$$\bar{u}_s(\vec{p}) u_{s'}(\vec{p}) = -\bar{v}_s(\vec{p}) v_{s'}(\vec{p}) = 2m\delta_{ss'},$$
$$\bar{u}_s(\vec{p}) v_{s'}(\vec{p}) = -\bar{v}_s(\vec{p}) u_{s'}(\vec{p}) = 0,$$

and the completeness relations:

$$\sum_s \Big[u_s(\vec{p})\bar{u}_s(\vec{p}) - v_s(\vec{p})\bar{v}_s(\vec{p}) \Big] = 2m.$$

(b) **[1 point]** Check the completeness relation by showing that when it is applied to the basis states $u_{s'}(\vec{p}), v_{s'}(\vec{p}), \bar{u}_{s'}(\vec{p})$, and $\bar{v}_{s'}(\vec{p})$, you obtain the correct result.

(c) **[2 points]** Now we define the projection operators:

$$\Lambda^{\pm}(\vec{p}) = \frac{\pm \not{p} + m}{2m},$$

which project the states of positive and negative energy, respectively, out of an arbitrary state:

$$f(\vec{p}) = \sum_s \left[\alpha_s u_s(\vec{p}) + \beta_s v_s(\vec{p})\right], \quad \alpha, \beta \in \mathbb{C}.$$

Show that $\Lambda^{\pm}(\vec{p})$ are indeed projectors, namely:

$$\begin{aligned}\left(\Lambda^{\pm}(\vec{p})\right)^2 &= \Lambda^{\pm}(\vec{p}),\\ \Lambda^{+}(\vec{p}) f(\vec{p}) &= \sum_s \alpha_s u_s(\vec{p}),\\ \Lambda^{-}(\vec{p}) f(\vec{p}) &= \sum_s \beta_s v_s(\vec{p}),\\ \Lambda^{+}(\vec{p}) + \Lambda^{-}(\vec{p}) &= 1.\end{aligned}$$

(d) **[2 points]** Finally, using the previous results, show that:

$$\begin{aligned}\sum_s u_s(\vec{p}) \bar{u}_s(\vec{p}) &= \not{p} + m,\\ \sum_s v_s(\vec{p}) \bar{v}_s(\vec{p}) &= \not{p} - m.\end{aligned}$$

(e) **[3 points]** Prove the *Gordon identity*,

$$\bar{u}_s(\vec{p}\,')\gamma^\mu u_s(\vec{p}) = \bar{u}_s(\vec{p}\,') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m}\right] u_s(\vec{p}),$$

with $q = p' - p$ and $\sigma^{\mu\nu}$ as defined in exercise 2.

We wish you a Merry Christmas,
a relaxing time between the holidays,
and a Happy New Year 2022!