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Dynamic vacuum energy: Quantum aspects of a three-form gauge field

Master's Thesis of

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I declare that I have developed and written the enclosed thesis completely by myself, and have not used sources or means without declaration in the text.

Mühlhausen, January 8, 2021

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(Max Haubenwallner)

Abstract

The three-form gauge field may provide a dynamical solution of the cosmological constant problem. In previous works, the nonkinematic three-form gauge field was used. Motivated by q-theory, the three-form gauge field theory was extended by the introduction of a kinematic term for the field. In this thesis, a Hamiltonian analysis is used to examine both variants of the three-form gauge field. Following from the gauge invariance of the three-form gauge field, the Hamiltonian analysis includes constraints. This analysis enables the calculation of the number of local degrees of freedom and the formulation of the path integral formalism of the nonkinematic and kinematic three-form gauge field. Finally, different interactions of the three-form gauge field are considered.

Zusammenfassung

Das drei-Form Eichfeld könnte eine dynamische Lösung des Problemes der kosmologischen Konstante bereitstellen. In früheren Arbeiten wurde hierzu das konstante drei-Form Eichfeld herangezogen, das keinerlei Kinematik besitzt. Jüngst hat sich allerdings, motiviert von der q-Theorie, eine Erweiterung des drei-Form Eichfeldes ergeben, welche einen kinematischen Term beinhaltet. In dieser Arbeit werden beide Varianten des drei-Form Eichfeldes im Lichte der Hamiltonischen Mechanik betrachtet, welche auf Grund der Eichinvarianz ebenjenes Feldes Zwangsbedingungen enthält. Neben der Berechnung der Anzahl globaler und lokaler Freiheitsgrade, ermöglicht die Hamiltonische Analyse die Formulierung des Pfadintegralformalismus des drei-Form Eichfeldes. Zuletzt werden mögliche Interaktionen des drei-Form Eichfeldes betrachtet und deren Implikationen für das Problem der kosmologischen Konstante diskutiert.

Notation

Let us review some notation and conventions used in this thesis.

- We use natural units with $\hbar = c = 1$.
- We use the metric signature $(-+++)$.
- The determinant of the metric tensor is
$$g \equiv \det(g_{\mu\nu}) \ .$$
- Latin indices, like i, j, k, run over spatial coordinate labels $1, 2, 3$ or x, y, z.
- Greek indices, like μ, ν, ρ, run over the four spacetime coordinate labels $0, 1, 2, 3$ or t, x, y, z.
- Throughout, unless specified otherwise, we use the Einstein sum convention.
- Square brackets around spacetime indices denote complete antisymmetrization.
- The covariant derivative is denoted by ∇_μ.
- The box operator is defined by
$$\Box \equiv \nabla_\alpha \nabla^\alpha .$$
- A prime on a function defines the total derivative of this function with respect to its argument
$$f'(x) \equiv \frac{df(x)}{dx} .$$
- A dot over a function denotes the time derivative of this function.

Contents

1. Introduction

In this introductory chapter, we, first, consider the Einstein field equation with a genuine cosmological constant and the effects of this cosmological constant on a Friedmann-Robertson-Walker universe. Second, we discuss contributions to vacuum energy density by quantum field theory and the cosmological constant problem. Last, we introduce the three-form gauge field and discuss connections to the aforementioned cosmological constant problem.

1.1. The Einstein field equation and a cosmological constant

The Einstein field equation describes the evolution of the metric tensor $g_{\mu\nu}$ depending on the energy-momentum tensor $T^{\mu\nu}$. Including a cosmological constant λ, which was first introduced to enable a static universe solution in reference [1], the Einstein field equation can be derived from the variation of the action

$$S[\psi, g_{\mu\nu}] = \int_{\mathbb{R}^4} d^4x\sqrt{-g}\left(-\frac{1}{16\pi G}(R-2\lambda) + \mathcal{L}(\phi, g_{\mu\nu})\right) \tag{1.1}$$

with respect to the metric $g_{\mu\nu}$. Defining the energy-momentum tensor as the variation of the matter action with respect to the metric tensor

$$\delta \int d^4x\sqrt{-g}\,\mathcal{L}(\phi, g_{\mu\nu}) \equiv \frac{1}{2}\int d^4x\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu}, \tag{1.2}$$

we get the Einstein field equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}. \tag{1.3}$$

To analyze the effects of the cosmological constant λ, we make a simple *Ansatz* to solve the Einstein field equation (1.3). We use the metric tensor of a Friedmann-Robertson-Walker universe, which describes a maximally symmetric, homogeneous and isometric universe, with the line element

$$ds^2 = -dt^2 + a(t)^2\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{1.4}$$

in the absence of matter, thereby setting the energy-momentum tensor to zero. This is a de Sitter universe. The dimensionless scale factor $a(t)$ characterizes the expansion or contraction of the universe. The Einstein field equation gives the Friedmann equations

$$3H(t)^2 = \lambda, \tag{1.5a}$$

$$3H(t)^2 + 2\dot{H}(t) = \lambda, \tag{1.5b}$$

where we defined the Hubble parameter $H(t)$ as

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{1.6}$$

The solution of equations (1.5) is a Hubble constant

$$H = \pm\sqrt{\frac{\lambda}{3}}. \tag{1.7}$$

Thereby, in a universe without matter, the cosmological constant λ gives rise to an expanding, or contracting, universe. In following sections, the nature of terms acting as a cosmological constant in the Friedmann equations (1.5), are elaborated.

1.2. Contributions to the vacuum energy density

In the last section, we considered a Friedmann-Robertson-Walker universe with a cosmological constant but without matter. Here, we introduce contributions to the energy-momentum tensor (1.2) that have the same effect on the evolution of the metric tensor $g_{\mu\nu}$ as the cosmological constant λ. Therefore, we consider a perfect fluid with energy-momentum tensor [2, Section 14.2]

$$T^{\mu\nu} = (\rho + P)\, U^\mu U^\nu + P g^{\mu\nu}, \tag{1.8}$$

where ρ is the energy density, P is the pressure of the perfect fluid and U^μ is the velocity four-vector with

$$U^0 = 1, \quad U^i = 0, \tag{1.9}$$

The equation of state parameter w of a perfect fluid is defined as

$$w \equiv \frac{P}{\rho}. \tag{1.10}$$

We are particularly interested in perfect fluids with equations of state parameter -1 since they have the energy-momentum tensor

$$T^{\mu\nu} = -\rho g^{\mu\nu}, \tag{1.11}$$

which has, comparing to the cosmological constant term in the Einstein field equation (1.3), the same structure as the cosmological constant term. Note here, that the energy density in equation (1.11) can be positive or negative. Of course, the energy-momentum tensor in the Einstein field equations is not restricted to a single contribution of a perfect fluid with equations of state parameter -1. Different contributions add to an overall vacuum energy density.

In the remainder of this section, we discuss different contribution to the vacuum energy density. First, as seen in the previous section, a raw cosmological constant contributes to the vacuum energy density by

$$\rho_\lambda = \frac{\lambda}{8\pi G}. \tag{1.12}$$

This contribution is of order of the Planck energy E_{Pl} since it is a pure gravitational effect.

For the next points, we consider a scalar field. It is described by the action

$$S[\phi, g^{\mu\nu}] = \int d^4x\sqrt{-g}\left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\right), \tag{1.13}$$

where $V(\phi)$ is the potential of the scalar field. The energy-momentum tensor is obtained by the variation with respect to the metric tensor $g^{\mu\nu}$

$$T^{\mu\nu} = -g^{\mu\nu}\left(\frac{1}{2}\nabla_\mu\phi\nabla^\nu\phi + V(\phi)\right) - \nabla^\mu\phi\nabla^\nu\phi. \tag{1.14}$$

Returning to the contributions to the vacuum energy density, the second contribution we discuss, is the vacuum expectation value of a spontaneously broken theory. The solution of the equation of motion of such a theory has not the same symmetry as the action of the theory itself. In particular, we consider a scalar field with

$$V(\phi_0) \neq 0, \tag{1.15}$$

where ϕ_0 is the minimum of the potential $V(\phi)$. Expanding the field around this minimum

$$\phi(x) = \phi_0 + \chi(x), \tag{1.16}$$

gives the energy-momentum tensor

$$T^{\mu\nu} = -g^{\mu\nu}V(\phi_0) - g^{\mu\nu}\left(\frac{1}{2}\nabla_\mu\chi\nabla^\mu\chi + \frac{1}{2}V''(\phi_0)\,\chi^2\right) - \nabla^\mu\chi\nabla^\nu\chi + O\left(\chi^3\right). \tag{1.17}$$

This tensor has a contribution of $V(\phi_0)$ to the vacuum energy density. The energy scale of the particular contribution is determined by the theory. A experimentally confirmed spontaneous broken theory is the Higgs theory.

Third, the zero-point fluctuation of a quantum field theory contribute a formally infinite amount to the vacuum energy density. The zero-point fluctuation of a scalar field is calculated by the vacuum expectation value of the energy-momentum tensor (1.14). The resulting energy density [3, 4] is

$$\rho = \frac{1}{4\pi^2}\int_0^\infty dp p^2\,\sqrt{p^2+m^2} = \infty, \tag{1.18}$$

which is formally infinite. A relativistically invariant regularization scheme [4] can be used to show that the energy density (1.18) behaves like a perfect fluid with equation of state parameter −1 and, thereby, contributes to the vacuum energy density. It is worth mentioning, that the contributions of fermions and bosons to vacuum energy density are of opposite sign and thus would completely cancel in a supersymmetric universe [5]. Accordingly, the contribution of the zero-point fluctuations of quantum fields would be replaced by the energy scale of supersymmetry due to its vacuum expectation value like above.

As seen in the last section, all these contributions behave like a cosmological constant term in the Einstein field equation (1.3). Accordingly, the vacuum energy density can be measured by cosmological observations and is of order $\left(10^{-3}eV\right)^4$ [6]. How to get from above contributions to the vacuum energy density to this comparatively small value is the so-called cosmological constant problem [7]. Another aspect of this problem is that, assuming only constant contributions to the vacuum energy density, the vacuum energy density was in the early universe enormous [7] since symmetries were potentially unbroken and thereby had no contributions. This suggest a dynamical way to cancel the vacuum energy density. An approach is presented in the next section and the next chapter.

1.3. The three-form-gauge-field approach

In this section, we consider a particular gauge field, that was used [8–14] to solve the cosmological constant problem from previous section. This gauge field is the antisymmetric three-form gauge field $A(x) = A_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$. For a short introduction to p-form gauge fields, cf. reference [15, Section 8.8]. In the further course of this section, we discuss some properties of this field.

As a consequence of gauge-invariance, physical quantities derived from the three-form gauge field $A_{\mu\nu\rho}$ do not change under the gauge transformation

$$A_{\mu\nu\rho}(x) \longrightarrow \tilde{A}_{\mu\nu\rho}(x) = A_{\mu\nu\rho}(x) + \nabla_{[\mu}\lambda_{\nu\rho]}(x)\,. \tag{1.19}$$

In particular, the field strength tensor $F_{\mu\nu\rho\lambda}$, defined by

$$F_{\mu\nu\rho\lambda} = \nabla_{[\mu}A_{\nu\rho\lambda]}, \tag{1.20}$$

does not change under above gauge transformation. This four-form has only one independent component since it is completely antisymmetric in four indices in four spacetime dimensions. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{48}F^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}, \tag{1.21}$$

where we follow the notation of reference [14], because coming results are compared to this reference. The dynamics of the three-form gauge field $A_{\mu\nu\rho}$ is described by the action

$$S\left[A_{\mu\nu\rho}, g_{\alpha\beta}\right] = -\frac{1}{48}\int_{\mathbb{R}^4} d^4x\sqrt{-g}\; F^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}. \tag{1.22}$$

Variation with respect to the three-form gauge field $A_{\mu\nu\rho}$

$$
\begin{aligned}
\delta S &= -\frac{1}{24}\int_{\mathbb{R}^4} d^4x\sqrt{-g}\, F^{\mu\nu\rho\lambda}\delta\big(\nabla_{[\mu}A_{\nu\rho\lambda]}\big) \\
&= -\int_{\mathbb{R}^4} d^4x\sqrt{-g}\, F^{\mu\nu\rho\lambda}\delta\big(\nabla_{\mu}A_{\nu\rho\lambda}\big) \\
&= \int_{\mathbb{R}^4} d^4x\sqrt{-g}\, \nabla_\mu F^{\mu\nu\rho\lambda}\delta A_{\nu\rho\lambda},
\end{aligned}
\tag{1.23}
$$

gives the field equation of the three-form gauge field

$$
\partial_\mu\left(\sqrt{-g}F^{\mu\nu\rho\lambda}\right) = 0, \tag{1.24}
$$

where we used the well-known formula, cf. [2, Section 4.7], to rewrite the covariant derivative of an antisymmetric tensor in terms of the four-gradient. Field equation (1.24) can be rewritten

$$
\partial_\mu\left(\sqrt{-g}\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\lambda}f\right) = \epsilon^{\mu\nu\rho\lambda}\partial_\mu f = 0 \Rightarrow \partial_\mu f = 0, \tag{1.25}
$$

where we defined f as

$$
\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\lambda}f \equiv F^{\mu\nu\rho\lambda}. \tag{1.26}
$$

According to equation (1.25), the three-form gauge field of theory (1.22) has no kinematics. Consequently, we refer to it as nonkinematic three-form gauge field to differentiate it from the kinematic three-form gauge field we introduce later.

The striking property of the nonkinematic three-form gauge field is the contribution to the energy-momentum tensor. We calculate it by the variation of action (1.22), where we used the definition (1.26), with respect to the metric tensor $g_{\mu\nu}$

$$
\begin{aligned}
\delta S &= -\frac{1}{2}\int d^4x\, f^2\delta\sqrt{-g} \\
&= -\frac{1}{4}\int d^4x\sqrt{-g}\, f^2 g^{\mu\nu}\delta g_{\mu\nu}.
\end{aligned}
\tag{1.27}
$$

Accordingly, the energy-momentum tensor

$$
T^{\mu\nu} = -\frac{1}{2}f^2 g^{\mu\nu} \tag{1.28}
$$

gives a term proportional to the metric tensor $g^{\mu\nu}$ and thus contributes to the vacuum energy density as discussed in previous sections. This is a direct consequence of the missing kinematics of the nonkinematic three-form gauge field, since a term proportional to the gradient of the field would not exclusively give a contribution to vacuum energy density.

In reference [14], the path integral formalism of the three-form gauge theory (1.22) is formulated. The wick-rotated partition function is found to be

$$Z = \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \exp\left\{ \int_V d^4x \left(-\frac{1}{2} f^2 \right) \right\}. \tag{1.29}$$

Some authors [10–12] argue, that the most probable field configuration of the three-form gauge field is, according to the path integral (1.29), the field configuration that gives a zero overall vacuum energy density since the vacuum energy contributions of previously discussed sources appear in the exponent of integral (1.29) together with the energy density of the nonkinematic three-form gauge field (1.28) and the most probable field configuration is found by setting this exponent to zero. Path integral (1.29) is derived in section 4.2. The vacuum energy cancellation by this path integral is discussed in section 5.4. The path integral for the, later introduced, kinematic three-form gauge field is obtained in section 4.4.

2. q-theory

In this chapter, we, first, introduce q-theory as a theory of thermodynamics and Lorentz invariance. Second, we discuss the three-form gauge field representation of q-theory including the so-called kinematic three-form gauge field, which is a main concern of this thesis. Third, we discuss the field equations and some quantities of the three-form gauge field representation.

2.1. The inputs of q-theory

Here, we introduce q-theory [16–21] as an effective theory of the quantum vacuum, following from thermodynamics and Lorentz invariance.

Henceforth, we assume the vacuum to be described by a self-sustained medium, i.e. a medium with a definite volume even in the absence of an environment. Self-sustained media are characterized by a conserved quantity, which determines the volume of the system. Thereby, we consider the conserved vacuum variable $q^{\mu\nu}$

$$\nabla_\mu q^{\mu\nu} = 0. \tag{2.1}$$

As a first model, we take a portion of vacuum isolated from its environment, so that there is no external pressure P. The quantity Q with conservation law

$$\frac{dQ}{dt} = 0, \tag{2.2}$$

determines the volume V of the vacuum by

$$V = \frac{Q}{q}, \tag{2.3}$$

where we assume q to be a genuine thermodynamic variable that is constant over space and time. Thereby, we describe a homogeneous vacuum. The thermodynamic potential to consider is the Gibbs free energy [22, Section 2.15]

$$W = E - TS + PV \tag{2.4}$$

at zero temperature. According to the assumption of zero pressure, the Gibbs free energy is given by

$$W = \int d^3r\ \epsilon(q, \psi) = \int d^3r\ \epsilon(Q/V, \psi)\,, \tag{2.5}$$

where ψ is a low-energy effective matter field. The equilibrium state of system (2.5) is obtained by

$$\frac{\delta W}{\delta \psi} = 0, \quad \frac{dW}{dV} = 0. \tag{2.6}$$

The second equation of (2.6) gives

$$-\epsilon(q, \psi_0) + q\frac{d\epsilon(q, \psi_0)}{dq} = 0, \tag{2.7}$$

where ψ_0 is the spacetime independent equilibrium value of the matter field. The solution of the above equation (2.7) determines the equilibrium value q_0 of the thermodynamic variable q and, thereby, gives, since the charge Q is constant, the equilibrium value of the volume V_0 by equation (2.3).

We continue our model by allowing for a vacuum under the external pressure P. The thermodynamic potential is still the Gibbs free energy and equivalent to equation (2.7), we get

$$P = -\epsilon(q, \psi_0) + q\frac{d\epsilon(q, \psi_0)}{dq}, \tag{2.8}$$

an integrated version of the Gibbs-Duhem equation

$$Nd\mu = -SdT + VdP, \tag{2.9}$$

which can be obtained [22, Section 2.24] by equation

$$W = N\mu, \tag{2.10}$$

and the differential of the Gibbs free energy. Here, the chemical potential μ is defined as

$$\mu \equiv \frac{dE}{dV} = \frac{d\epsilon(q, \psi_0)}{dq}, \tag{2.11}$$

so that we can rewrite equation (2.8) as

$$P = -\epsilon(q, \psi_0) + q\mu. \tag{2.12}$$

In this thermodynamic discussion, the conserved quantity Q plays the same role as the particle number N.

According to equation (2.8), the thermodynamic relevant energy density is

$$\tilde{\epsilon}(q, \psi)_{vac} = \epsilon(q, \psi) - q\frac{\epsilon(q, \psi)}{q}. \tag{2.13}$$

It is this cancellation of energy density, that allows the q-field to be a high energy field, that still provides a low energy vacuum energy density.

A stable vacuum requires the equilibrium value, specified by q_0 and ψ_0, to be in a minimum, i.e.

$$\frac{d^2W}{d^2V} = \left[\frac{1}{V}q^2\frac{d^2\epsilon(q, \psi)}{d^2q}\right]_{q=q_0} \geq 0, \tag{2.14}$$

which motivates the introduction of the isothermal compressibility χ

$$\chi^{-1} = \left[q^2 \frac{d^2\epsilon(q,\psi)}{d^2 q} \right]_{q=q_0} \geq 0. \tag{2.15}$$

Up to this point, we did not use the Lorentz invariance of the vacuum variable q. The importance of Lorentz invariance can be seen from the fact, that a potential Lorentz-violating energy scale exceeds the Planck scale [23]. To describe the thermodynamic energy density (2.13) as a perfect fluid with equation of state parameter w, cf. section 1.2 and [2], we assume the Lorentz invariance of the vacuum energy density, so that we get

$$P_{vac} = -\tilde{\epsilon}_{vac}. \tag{2.16}$$

Consequently, the thermodynamic energy density $\tilde{\epsilon}_{vac}$ is a perfect fluid with equation of state parameter -1.

The thermodynamic equilibrium condition (2.8) gives the energy density of the vacuum in equilibrium. In a perfect quantum vacuum, for example, there is no external pressure and, thereby, no vacuum energy density. If there is an external pressure, like the existence of thermal matter, though, the vacuum state is Lorentz noninvariant, making the vacuum variable q shift to a new equilibrium value that compensates the external pressure and gives a nonzero vacuum energy density.

2.2. The three-form gauge field representation of q-theory

In this chapter, we discuss the three-form gauge field representation of the previous introduced q-theory. The three-form gauge field was already considered in section 1.3 and is known for introducing a constant vacuum energy density to the energy-momentum tensor.

First, we give the definition of the q-field in terms of the field strength tensor

$$F_{\mu\nu\rho\lambda}(x) = \sqrt{-g}\epsilon_{\mu\nu\rho\lambda} q(x)\,. \tag{2.17}$$

The above equation was already used in the previous section 1.3 to rewrite the field strength tensor in terms of a single variable. Since the field strength tensor on the left-hand side of equation (2.17) is a fundamental quantity, we can make some remarks about the properties of the q-field. First, the q-field is a composite field of the metric and the field strength tensor, which has only one independent component due to its antisymmetry. Second, the q-field changes sign under parity transformations since the completely antisymmetric tensor density $\epsilon_{\mu\nu\rho\lambda}$ also changes sign. Third, the q-field is a (pseudo-)scalar field since the combination of $\sqrt{-g}$ and antisymmetric tensor density ϵ transforms like a tensor, just like the four-form field strength. Last, the q-field is of mass dimension -2.

The action of the three-form representation of q-theory is, in general, given by

$$S[q] = \int_{\mathbb{R}^4} dx^4 \sqrt{-g}\, f(q)\,, \tag{2.18}$$

where f is a function of the q-field. The field equations are obtained by the variation of above action with respect to the three-form gauge field $A_{\mu\nu\rho}$. The variation is

$$
\begin{aligned}
\delta S[q] &= \int_{\mathbb{R}^4} dx^4 \sqrt{-g} \left[f'(q)\, \frac{1}{2q} \delta\left(q^2\right) \right] \\
&= -\frac{1}{24} \int_{\mathbb{R}^4} dx^4 \sqrt{-g} \left[f'(q)\, \frac{1}{q} F^{\mu\nu\rho\lambda} \delta\left(\nabla_\mu A_{\nu\rho\lambda}\right) \right] \\
&= \frac{1}{24} \int_{\mathbb{R}^4} dx^4 \sqrt{-g} \left[\frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\lambda} \partial_\mu (f'(q))\, \delta A_{\nu\rho\lambda} \right],
\end{aligned}
\tag{2.19}
$$

where we have used

$$
q^2 = -\frac{1}{24} F_{\mu\nu\rho\lambda} F^{\mu\nu\rho\lambda}. \tag{2.20}
$$

According to the variation (2.19), the equation of motion of the q-field is

$$
\partial_\alpha \left(\frac{d}{dq} f(q) \right) = 0. \tag{2.21}
$$

In q-theory, we consider the Lagrangian density

$$
\mathcal{L}_q = \frac{1}{2} C(q)\, \nabla_\mu q \nabla^\mu q + \epsilon(q)\,, \tag{2.22}
$$

so that the action of q-theory and gravity [24] is

$$
S\left[q, g_{\mu\nu}\right] = -\int_{\mathbb{R}^4} dx^4 \sqrt{-g} \left(\frac{R}{16\pi G(q)} + \frac{1}{2} C(q)\, \nabla_\mu q \nabla^\mu q + \epsilon(q) \right), \tag{2.23}
$$

where R is the Ricci scalar, $G(q)$ is the gravitational coupling parameter, $C(q)$ is a prefactor of mass dimension -2 and $\epsilon(q)$ is the potential of q-theory. According to equation (2.21), the equation of motion, following from the action (2.23), is

$$
\partial_\alpha \left(\frac{d\epsilon(q)}{dq} - \frac{dG(q)}{dq} \frac{R}{16\pi G(q)^2} - \frac{1}{2} \frac{dC(q)}{dq} \nabla_\mu q \nabla^\mu q - C(q)\, \nabla_\mu \nabla^\mu q \right) = 0, \tag{2.24}
$$

which is solved by

$$
\frac{d\epsilon(q)}{dq} - \frac{dG(q)}{dq} \frac{R}{16\pi G(q)^2} - \frac{1}{2} \frac{dC(q)}{dq} \nabla_\mu q \nabla^\mu q - C(q)\, \nabla_\mu \nabla^\mu q = \mu, \tag{2.25}
$$

where μ is a constant over space and time. The identification of μ in equation (2.25) with the chemical potential is motivated by the energy-momentum tensor. As we see in chapter 3, this quantity plays a crucial role in the dynamics of the three-form gauge field. The energy-momentum tensor is defined by the variation of the action with respect to the metric tensor $g_{\mu\nu}$ as in equation (1.2). Therefore, it is convenient to calculate the variation of the generic function $f(q)$, as introduced earlier, with respect to the metric tensor $g_{\mu\nu}$

$$\delta f(q) = f'(q)\,\delta\left(\frac{F_{\mu\nu\rho\lambda}}{\epsilon_{\mu\nu\rho\lambda}\sqrt{-g}}\right) = -\frac{1}{2} q f'(q)\, g^{\mu\nu}\delta g_{\mu\nu}, \tag{2.26}$$

where the middle part of the equation only holds for mutually different indices and we do not use the Einstein sum convention. The energy-momentum tensor corresponding to the action (2.23) is, thereby,

$$T^{\mu\nu} = -g^{\mu\nu}\left(\epsilon(q) - \mu q + \frac{1}{2}C(q)\,\nabla_\alpha q \nabla^\alpha q\right) + C(q)\,\nabla^\mu q \nabla^\nu q. \tag{2.27}$$

Here, we replaced the derivative of the integrand in action (2.23) with respect to the q-field, as obtained in equation (2.26), by the solution of the equation of motion (2.25), which introduces the parameter μ into the energy-momentum tensor. By comparing with the previous result (2.13), we identify the constant μ with the chemical potential from previous section.

In the course of this thesis, we usually consider a simpler version of equation of motion (2.25). By removing the dependence of the gravitational coupling parameter on q and replacing the q dependent prefactor $C(q)$ by the constant $\bar{C}$, the simplified equation of motion reads

$$\frac{d\epsilon(q)}{dq} - \bar{C}\nabla_\alpha \nabla^\alpha q = \mu. \tag{2.28}$$

This equation can also be written as a scalar-like equation

$$\frac{d\rho_v(q)}{dq} - \bar{C}\nabla_\alpha \nabla^\alpha q = 0, \tag{2.29}$$

with potential

$$\rho_v(q) = \epsilon(q) - \mu q, \tag{2.30}$$

where ρ_v is the vacuum energy density of the q-field, as seen in the energy-momentum tensor (2.27).

Finally, we make a remark about the kinematic term for the q-field in equation of motion (2.25). This is a higher derivative term since the q-field is defined by the field-strength tensor $F_{\mu\nu\rho\lambda}$, which already describes the dynamics of the three-form gauge field. We discuss this in more detail in chapter 3. Henceforth, we refer to the three-form gauge field with this higher derivative term as the kinematic three-form gauge field.

2.3. Kinematics of the q-field

In this section, we examine the equation of motion of the q-field, make an *Ansatz* for the potential $\epsilon(q)$ and show some numerical solutions of the equation of motion of q-theory.

First, we consider the q-field close to equilibrium [25]. The chemical potential is chosen so that the equilibrium spacetime is Minkowski spacetime. We refer to this chemical potential as μ_0. It is defined by equation

$$\epsilon(q_0) - \mu_0 q_0 = 0, \tag{2.31}$$

where q_0 is the equilibrium value of the q-field. In equilibrium, the q-field is constant and satisfies

$$\left.\frac{d\epsilon(q)}{dq}\right|_{q=q_0} = \mu_0. \tag{2.32}$$

To describe the q-field close to equilibrium, we substitute for the q-field

$$q(x) = q_0 + \chi(x)\,, \quad \chi \ll q_0, \tag{2.33}$$

in equation of motion (2.28) and get

$$\left(\frac{1}{\bar{C}}\left.\frac{d^2\epsilon(q_0)}{d^2q}\right|_{q=q_0} - \nabla_\alpha\nabla^\alpha\right)\chi + O\left(\chi^2\right) = 0, \tag{2.34}$$

where we have used the definition of the equilibrium value (2.32).

The equation of motion close to equilibrium (2.34) is, neglecting higher order term, a scalar field equation for the (pseudo-)scalar field χ with mass

$$m_\chi^2 \equiv \frac{1}{\bar{C}}\left.\frac{d^2\epsilon(q)}{d^2q}\right|_{q=q_0}. \tag{2.35}$$

The energy-momentum tensor close to equilibrium is

$$T^{\mu\nu} = \bar{C}\left[-g^{\mu\nu}\left(\frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}\nabla_\alpha\chi\nabla^\alpha\chi\right) + \nabla^\mu\chi\nabla^\nu\chi\right] + O\left(\chi^3\right) \tag{2.36}$$

and resembles, up to the prefactor $\bar{C}$, the energy-momentum tensor of a scalar field. These observations lead to the assumption that the kinematic three-form gauge field has only one local degree of freedom, as we discuss in chapter 3 in much more detail.

Second, we consider a particular potential $\epsilon(q)$, cf. [17]. The potential has to satisfy two requirements. First, the equilibrium value q_0 defined by equation (2.32) is nonzero. Second, the vacuum is stable by equation (2.15). Accordingly, we choose the potential

$$\epsilon(q) = \frac{1}{4}\alpha q^4 - \frac{1}{2}\beta q^2. \tag{2.37}$$

The positivity of the isothermal compressibility gives a condition on the parameters α and β

$$q_0 \geq \frac{\beta}{3\alpha}, \tag{2.38}$$

in dependence of the equilibrium value q_0, which, in turn, depends on the chemical potential μ.

To solve the field equations (2.28) with potential (2.37) numerically, we introduce dimensionless quantities f, a, b, c, r_v, τ, m, representing the quantities $q, \alpha, \beta, \bar{C}, \rho_v, t, \mu$, respectively, by

$$f \equiv \frac{q}{E_q^2}, \quad a \equiv \frac{\alpha}{E_q^4}, \quad b \equiv \beta, \quad c \equiv \bar{C}E_q^2, \quad r_v \equiv \frac{\rho_v}{E_q^4}, \quad \tau \equiv tE_q, \quad m \equiv \frac{\mu}{E_q^2}, \tag{2.39}$$

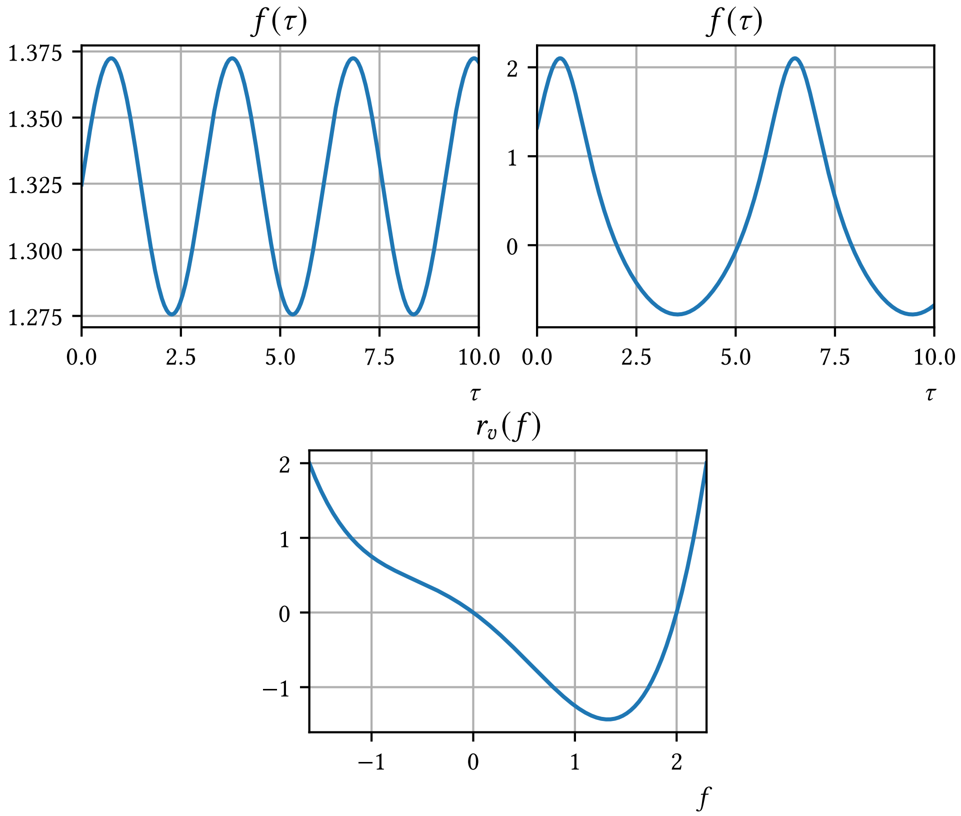

Figure 2.1.: On the top, numerical solutions of equation (2.40) are shown. The initial conditions are $\{f(\tau), f'(\tau)\} = \{f_0, 0.1\}$ and $\{f(\tau), f'(\tau)\} = \{f_0, 1\}$, respectively. Here, f_0 is the equilibrium value of the f-field. The left-hand side panel shows the, in this section discussed, f-field close to equilibrium. On the bottom panel, the potential (2.41) is shown. The parameters for all of the plots are $\{a, b, c, m\} = \{1, 1, 1, 1\}$.

where E_q is the energy scale of q-theory and t is the time. The equation of motion for the dimensionless field f is, assuming spatial homogeneity,

$$\frac{dr_v(f)}{df} + c\frac{d^2 f}{d^2\tau} = 0, \tag{2.40}$$

with dimensionless potential

$$r_v(f) = \frac{1}{4}af^4 - \frac{1}{2}bf^2 - mf. \tag{2.41}$$

In figure 2.1 the numerical solutions of equation (2.40) with two different initial conditions and the potential (2.41) are shown.

3. Theory of Hamiltonian mechanics

In this chapter, we obtain the Hamiltonian formulation of the nonkinematic and kinematic three-form gauge field theory. This enables the calculation of the equation of motion and, more importantly, of the number of local degrees of freedom and the transition to the path integral formulation of q-theory in the next chapter. Henceforth, unless specified differently, we only use Minkowski spacetime.

The chapter is structured in the following way. First, we give an introduction to the terminology and basics of constrained Hamiltonian mechanics. Second, we apply this analysis, after a short example of the Standard Maxwell gauge theory, to the nonkinematic three-form gauge field. Last, we examine the constrained Hamiltonian mechanics of a higher-order Lagrangian and thereby find a way to analyze the kinematic three-form gauge field theory.

3.1. Lagrangian mechanics

Since most theories are expressed by a Lagrangian density $\mathcal{L}$, we start by a short introduction to Lagrangian mechanics, cf. [26, 27]. For simplicity, we restrict the following to a system with a finite number of degrees of freedom. Then, the dynamics is described by the action

$$S = \int_{t_1}^{t_2} dt \; L(q_i, \dot{q}_i) \,, \tag{3.1}$$

where $L(q_i, \dot{q}_i)$, the Lagrangian, is a function of the generalized coordinates $q_i(t)\,, i = 1, ..., N$, describing N degrees of freedom, and the generalized velocities $\dot{q}_i(t)$ are defined by

$$\dot{q}_i(t) \equiv \frac{dq_i(t)}{dt}. \tag{3.2}$$

According to the principle of stationary action, we obtain the equations of motion of system (3.1) by varying the action with respect to the generalized coordinates q_i

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \left[\delta q_i \frac{\partial L}{\partial q_i} + \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right] \\ &= \frac{\partial L}{\partial q_i} \delta q_i \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \delta q_i, \end{aligned} \tag{3.3}$$

where we have used partial integration in the last line of the above computation. Assuming the variation of the start point $\delta q_i(t_1)$ and end point $\delta q_i(t_2)$ to be zero, we obtain the well-known Euler-Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{3.4}$$

This equation allows for a complete determination of the time evolution of the generalized coordinates q_i.

3.2. Construction of Hamiltonian mechanics from the Lagrangian

In the last section, we concluded how to determine the time evolution of a generalized coordinate q_i by varying the action containing the Lagrangian L. But, there is an equivalent way to obtain the equations of motion for q_i. This goes by the name of Hamiltonian dynamics, whose formulation of classical mechanics is closer to quantum theory since it allows quantization by replacing Poisson brackets by commutators and is later also useful for the path integral formulation.

To construct the Hamiltonian dynamics of a system, cf. [27, 28], we start with a Lagrangian L and define the canonical momenta $p_i(t)$ by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}. \tag{3.5}$$

Henceforth, we consider the quantity

$$H(q_i, p_i) \equiv p_i \dot{q}_i - L(q_i, \dot{q}_i)\,, \tag{3.6}$$

called Hamiltonian. The variational principle of the Hamiltonian (3.6) is

$$0 = \delta H = \dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \delta L = \dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i}\delta q_i. \tag{3.7}$$

Assuming the canonical momenta (3.5) to be independent functions of the velocities $\dot{q}_i$, we write the Hamiltonian as function of the canonical coordinates q_i and the canonical momenta p_i since the variation of the Hamiltonian (3.7) only contains variations of q_i and p_i, but not of the velocities $\dot{q}_i$. Due to the variations of q_i and p_i being independent of each other, we are able to write equation (3.7) as

$$\left[\frac{\partial H}{\partial q_i} + \frac{\partial L}{\partial q_i}\right]\delta q_i + \left[\frac{\partial H}{\partial p_i} - \dot{q}_i\right]\delta p_i = 0. \tag{3.8}$$

Consequently, the canonical equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{3.9a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{3.9b}$$

where we have used the definition of the canonical momenta (3.5) in the Euler-Lagrange equation (3.4) to obtain equation (3.9b). According to equations (3.9), we calculate the time evolution of a generic variable $g(q_i, p_i)$ to be

$$\dot{g} = \frac{\partial g}{\partial q_i}\dot{q}_i + \frac{\partial g}{\partial p_i}\dot{p}_i = \frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i}, \tag{3.10}$$

which motivates the definition of the Poisson brackets

$$[g, f] = \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}, \tag{3.11}$$

so that the time evolution of a variable can be expressed as

$$\dot{g} = [g, H]\,. \tag{3.12}$$

3.3. Construction of constrained Hamiltonian mechanics

In the last section, we have obtained the Hamiltonian mechanics from a given Lagrangian. In this section, we want to extend our analysis to a certain class of Lagrangians, called singular Lagrangians. This analysis originated from Dirac [29]. Calling these Lagrangians singular goes back to Bergmann [30] and is motivated by rewriting the Euler-Lagrange equation (3.4) to

$$\ddot{q}_j\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial L}{\partial q_i} - \dot{q}_j\frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}. \tag{3.13}$$

To solve above equation for the accelerations $\ddot{q}_i$, we have to invert the matrix

$$\Lambda_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \tag{3.14}$$

Consequently, we can only solve uniquely for the accelerations if the matrix Λ_{ij} is nonsingular. Accordingly, from the Lagrangian point of view, a singular Lagrangian does not give a unique solution of the Euler-Lagrange equation for the accelerations $\ddot{q}_i$.

Returning to Hamiltonian mechanics, singular Lagrangians lead to so-called constraints. The analysis of a constrained Hamiltonian system requires more care since one of our previous assumptions does not hold. We follow in this section the textbooks [26, 31], but there are also [32, 33] for reference.

In Hamiltonian mechanics constraints are realized by equations

$$\phi_m(q_i, p_i) = 0, \quad m = 1, ..., M. \tag{3.15}$$

These constraints follow directly from the definition of the canonical momenta (3.5) and are thus called primary constraints. We still define the Hamiltonian as in the last section. But, due to the constraints (3.15), we can not assume the variations of q_i and p_i to be independent of each other. As we show in Appendix A.2, the Hamiltonian equations of motion involve, as a consequence of constraints, arbitrary functions u_m

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + u_m \frac{\partial \phi_m}{\partial p_i}, \tag{3.16a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + u_m \frac{\partial \phi_m}{\partial q_i}. \tag{3.16b}$$

For simplicity, the time evolution of the variables (3.16) can also be obtained by the introduction of the extended Hamiltonian

$$H_E \equiv H + u_m \phi_m, \tag{3.17}$$

and defining

$$[u_m, H] \equiv 0. \tag{3.18}$$

The time evolution of a generic variable $g(q_i, p_i)$ is

$$\dot{g} = [g, H] + u_m [g, \phi_m] . \tag{3.19}$$

Although, up to this point, the functions u_m seem to be completely arbitrary there is a basic consistency requirement, concretely

$$\dot{\phi}_m = [\phi_m, H] + u_n [\phi_m, \phi_n] = 0, \tag{3.20}$$

which ensures the description of the dynamics to be valid at all times. Constraints, which are obtained by these consistency conditions, are called secondary constraints. Equations (3.20) are a system of inhomogeneous linear equations in the unknowns u_n. Assuming we find a special solution

$$u_m = U_m, \tag{3.21}$$

we can construct the general solution of system (3.20) by adding to (3.21) solutions of the homogeneous equation

$$V_m [\phi_n, \phi_m] = 0. \tag{3.22}$$

We find, thereby, the most general solution of (3.20) as

$$u_m = U_m + v_a V_{am}, \tag{3.23}$$

where v_a are completely arbitrary functions, which can not be determined by the consistency conditions (3.20). The total Hamiltonian is, consequently, defined as

$$H_T = H + U_m \phi_m + v_m \phi_m. \tag{3.24}$$

It gives the time evolution of the canonical variables in the most determined way possible.

Except of denoting the constraints as primary or secondary there is another difference between constraints. If the Poisson brackets of a constraint with all other constraints are zero, the constraint is called first-class, while constraints where this is not the case are called second-class constraints. These second-class constraints lead to the introduction of the Dirac bracket, which replaces the Poisson bracket. In this thesis however, we do not have to deal with second-class constraints, so that we do not extend this topic further.

But how is this formalism connected to the three-form gauge field that is the main issue of this thesis? The defining feature of a gauge theory is the invariance of physical (i.e. measurable) quantities, derived from the gauge field, under certain gauge transformations acting on the field, which is in our case the three-form gauge field. Since the field, and thereby the canonical variables representing the field in a Hamiltonian analysis, does change under a gauge transformation, there is a direction in phase space not representing any change in physical quantities, but still changing the canonical variables. This arbitrariness is represented by the arbitrary functions v_m in constrained Hamiltonian mechanics. It means that not all degrees of freedom of the theory represent physical degrees of freedom. This is the main focus in section 4.3.

The connection of gauge theory and constrained Hamiltonian mechanics is also represented by the fact that first-class constraints are generating gauge transformations of the canonical variables. Accordingly, the Poisson brackets of a gauge-invariant quantity with all first-class constraints are zero.

After the analysis of the constraints, there is a possibility to calculate the local degrees of freedom of the system. This was developed in [31] and is given [28] as

$$\#\mathrm{dof} = \frac{1}{2}\left[\# \begin{pmatrix}\text{canonical}\\ \text{variables}\end{pmatrix} - 2 \cdot \# \begin{pmatrix}\text{first-class}\\ \text{constraints}\end{pmatrix} - \# \begin{pmatrix}\text{second-class}\\ \text{constraints}\end{pmatrix}\right], \tag{3.25}$$

where the # symbol means "number of". In the above equation the counting of the first-class constraints is doubled, which is connected to the fact that these constraints are generating gauge transformations as mentioned before.

3.4. Example: Standard Maxwell gauge theory

Connecting last section's analysis with a well-known gauge theory, we give the example of the Standard Maxwell gauge theory in Minkowski spacetime. This analysis can also be found in textbooks [26, 28, 32]. As another example, the constrained Hamiltonian analysis of the two-form gauge field can be found in reference [34].

The Standard Maxwell theory is described by the one-form gauge field $A = A_\mu dx^\mu$. The theory is invariant under gauge transformations

$$A_\mu(x) \longrightarrow A_\mu(x) + \nabla_\mu \Lambda(x)\,. \tag{3.26}$$

The Lagrangian density is

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{3.27}$$

where F is the exterior derivative

$$F_{\mu\nu} \equiv \nabla_{[\mu} A_{\nu]} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{3.28}$$

of the one-form field A. In terms of the generalized coordinates A_0, A_1, A_2, A_3 the Lagrangian density is

$$\mathcal{L}_{EM} = \frac{1}{2}\dot{A}_i \dot{A}_i + \frac{1}{2}\nabla_i A_0 \nabla_i A_0 - \dot{A}_i \nabla_i A_0 - \frac{1}{2}\nabla_i A_j \left(\nabla_i A_j - \nabla_j A_i\right). \tag{3.29}$$

The canonical momenta π_0 and π_i corresponding to A_0 and A_i, respectively, are

$$\pi_0 = \frac{\partial \mathcal{L}}{\partial(\nabla_0 A_0)} = 0, \quad \pi_i = \frac{\partial \mathcal{L}}{\partial(\nabla_0 A_i)} = \nabla_0 A_i - \nabla_i A_0, \tag{3.30}$$

where we see that momentum π_0 represents the primary constraint

$$\phi^1 \equiv \pi_0. \tag{3.31}$$

The extended Hamiltonian density is

$$\mathcal{H}_E = \frac{1}{2}\pi_i\pi_i - \pi_i \nabla_i A_0 + \frac{1}{2}\nabla_i A_j \left(\nabla_i A_j - \nabla_j A_i\right) + \pi_0 \left(\dot{A}_0 + u^1\right), \tag{3.32}$$

where u^1 is an arbitrary function. For consistency, we check the time evolution of constraint ϕ_1 to be zero

$$\dot{\phi}^1 = [\pi_0, \mathcal{H}_E] = -\nabla_i \pi_i \equiv \phi^2, \tag{3.33}$$

which is actually Gauß's law. This does not fix the function u_1 and is thereby a new constraint. For consistency, we check the time evolution of the new constraint ϕ^2, which is in fact zero

$$\dot{\phi}^2 = [\nabla_i \pi_i, \mathcal{H}_E] = \nabla_i \nabla_j \nabla_j A_i - \nabla_i \nabla_i \nabla_j A_j = 0. \tag{3.34}$$

After checking the consistency of all constraints, we obtain the total Hamiltonian density as

$$\mathcal{H}_T = \frac{1}{2}\pi_i\pi_i - \pi_i \nabla_i A_0 + \frac{1}{2}\nabla_i A_j \left(\nabla_i A_j - \nabla_j A_i\right) + \pi_0 \left(\dot{A}_0 + u^1\right) + u^2 \nabla_i \pi_i. \tag{3.35}$$

Since the Poisson brackets of constraints ϕ_1 and ϕ_2 are zero, they are first-class constraints. The time evolution of the canonical variables is given by the Poisson brackets with the total Hamiltonian (3.35).

$$A_0 = \dot{A}_0 + u_1, \qquad \dot{\pi}_0 = 0, \tag{3.36a}$$

$$\dot{A}_i = \pi_i + \nabla_i A_0 - \nabla_i u_2, \qquad \dot{\pi}_i = \nabla_j \nabla_j A_i - \nabla_i \nabla_j A_j. \tag{3.36b}$$

In equations of motion (3.36), the arbitrary functions u_1 and u_2 represent the gauge freedom of electromagnetism. According to equation (3.25), we can give the number of local degrees of freedoms as

$$\#\mathrm{dof} = \frac{1}{2}(8 - 2 \cdot 2) = 2, \tag{3.37}$$

which is of course a well-known result and represented by the two polarizations of the photon.

We note here, that equations of motion (3.36) do not allow to connect these two local degrees of freedom to pairs of canonical variables, since all pairs of canonical variables involve some arbitrary function. We come back to this point later, when discussing the same point for the kinematic three-form gauge field. There is also a nice discussion about this topic in reference [35, Appendix E.2], where the author describes equivalence classes $\tilde{A}_\mu$ in terms of only one configuration for every physical equivalent configuration of the standard vector potential A_μ.

3.5. The nonkinematic three-form gauge field

After familiarizing with constrained Hamiltonian mechanics, we analyze the nonkinematic three-form gauge field of section 1.3. This theory was already reported to have no local degrees of freedom, cf. [9] and [14]. In reference [15] it is also stated that in four dimensions the three-form field "describes no degrees of freedom at all". As potential we use a more general form as in (1.21)

$$\epsilon\left(F^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}\right) = \epsilon\left(\dot{X}_0 + \nabla_i X_i\right), \tag{3.38}$$

which is assumed to be a polynomial of $F^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}$ to maintain parity invariance and where we have defined the variables X_0 and X_i in terms of the three-form gauge field $A_{\mu\nu\rho}$ as

$$X_0 \equiv -6A_{123}, \quad X_1 \equiv 6A_{023}, \quad X_2 \equiv -6A_{013}, \quad X_3 \equiv 6A_{012}. \tag{3.39}$$

In terms of these variables the q-field is given by

$$q^2 = \left(\dot{X}_0 + \nabla_i X_i\right)^2. \tag{3.40}$$

Considering Lagrangian density

$$\mathcal{L}(X_0, X_i) = -\epsilon\left(\dot{X}_0 + \nabla_i X_i\right), \tag{3.41}$$

we immediately expect constraints since there is only a time derivative of one coordinate. The canonical momenta P_0 and P_i to X_0 and X_i are, respectively,

$$P_0 = -\frac{\partial\epsilon\left(\dot{X}_0 + \nabla_i X_i\right)}{\partial\dot{X}_0}, \quad P_i = 0. \tag{3.42}$$

Therefore, we have three constraints

$$\phi_i^1 \equiv P_i. \tag{3.43}$$

Assuming we can solve $\dot{X}_0$ in terms of the momentum P_0 and $\nabla_i X_i$, the extended Hamiltonian density (3.17) is

$$\mathcal{H}_E = \dot{X}_0(P_0, \nabla_i X_i)\, P_0 + \dot{X}_i P_i + \epsilon\left(\dot{X}_0(P_0, \nabla_i X_i), \nabla_i X_i\right) + u_i^1 P_i. \tag{3.44}$$

According to consistency condition (3.20), the time evolution of the constraints ϕ_i^1 is set to zero

$$\dot{\phi}_i^1 = [P_i, \mathcal{H}_E] = \nabla_i \frac{\partial\epsilon\left(\dot{X}_0(P_0, \nabla_i X_i), \nabla_i X_i\right)}{\partial(\nabla_i X_i)} = \nabla_i P_0 \equiv \phi_i^2, \tag{3.45}$$

where we have used the definition of P_0 in equation (3.42). Not fixing an arbitrary function, there is a new constraint ϕ_i^2. Since there is no X_0 in Hamiltonian density (3.44), the time evolution of this constraint is zero. After obtaining all constraints and fixing a maximal amount of arbitrary functions, we can give the total Hamiltonian density

$$\mathcal{H}_T = \dot{X}_0(P_0, \nabla_i X_i)\, P_0 + \dot{X}_i P_i + \epsilon\left(\dot{X}_0(P_0, \nabla_i X_i), \nabla_i X_i\right) + u_i^1 P_i + u_i^2 \nabla_i P_0. \tag{3.46}$$

The constraints are

$$\phi_i^1 = P_i, \quad \phi_i^2 = \nabla_i P_0. \tag{3.47}$$

The equations of motion are obtained by the Poisson brackets of the variables with the total Hamiltonian density (3.46)

$$\dot{X}_0 = \dot{X}_0(P_0, \nabla_i X_i) + \frac{d\epsilon\big(\dot{X}_0(P_0, \nabla_i X_i), \nabla_i X_i\big)}{dP_0} - \nabla_i u_i^2, \qquad \dot{P}_0 = 0, \tag{3.48a}$$

$$\dot{X}_i = \dot{X}_i + u_i^1, \qquad \dot{P}_i = 0. \tag{3.48b}$$

From equations (3.48a), (3.45) and (3.40) follow the Lagrangian equations of motion (2.24)

$$\nabla_\nu \left(\frac{d\epsilon(q)}{dq} \right) = 0. \tag{3.49}$$

From here, the local degrees of freedom can be calculated by the introduced manner. But, let us first discuss the equations of motion (3.48) to gain some insight about the dynamic variables of this theory. The time evolution of the variables X_0 and X_i is completely arbitrary since their equations of motion (3.48a) and (3.48b) involve the arbitrary functions u_i^1 and u_i^2. Their respective canonical momenta, P_0 and P_i are also subject to the constraints (3.47). The pairs of canonical variables X_0, P_0 and X_i, P_i describe, consequently, no propagating degrees of freedom, rendering the theory nonkinematic.

Since there are eight canonical variables and four constraints, the local degrees of freedom are, according to equation (3.25),

$$\#\mathrm{dof} = \frac{1}{2}(8 - 2 \cdot 4) = 0, \tag{3.50}$$

which we expect given above discussion.

But there is a subtlety arising from the form of the constraints (3.47). The momentum P_0 is only constrained to a constant value in space and, by consistency, thereby also constrained to a constant value in time. In contrast to the other momenta P_i though, the momentum P_0 is not constrained strictly to zero. Accordingly, there is a global degree of freedom in the nonkinematic three-form gauge field theory. In a later section, we use exactly this global degree of freedom to obtain the path integral formalism of this theory like in the introduction 1.3.

3.6. Higher-order Lagrangians

Since the Lagrangian density of the kinematic three-form gauge field (2.22) contains time derivatives of higher than first-order, we have to extend the previous analysis of constrained Hamiltonian mechanics to these kinds of Lagrangians, called higher-order Lagrangians. This was done directly for singular Lagrangians in reference [36]. Here, we adopt a simpler way [37] by transforming the higher-order Lagrangian into a first-order Lagrangian at the cost of additional constraints. An example in literature of this procedure applied to a relativistic particle is given in reference [38]. After obtaining the first-order

Lagrangian, we can use the familiar formalism of section 3.2. Later, we, additionally, introduce the Ostrogradsky instability [39], which is also apparent in singular Lagrangians [40].

First, we follow reference [37] by transforming a higher-order singular Lagrangian to a first-order singular Lagrangian with additional constraints. A m-order Lagrangian is a Lagrangian of form

$$L\left(x, x^{(1)}, x^{(2)}, ..., x^{(m)}\right), \tag{3.51}$$

where we only use one variable x for simplicity. We want to show that the constrained first-order Lagrangian

$$L'(q_1, ..., q_m; \dot{q}_1, ..., \dot{q}_m; \lambda_1, ..., \lambda_{m-1}) = L(q_1, ..., q_m; \dot{q}_m) + \lambda_j \left(\dot{q}_j - q_{j+1}\right), \tag{3.52}$$

where q_i is defined by

$$q_i \equiv x^{(i-1)}, \quad i = 1, ..., m, \tag{3.53}$$

and the λ_j, $j = 1, ..., m-1$ are Lagrange multipliers enforcing equations (3.53), gives the same dynamics as Lagrangian (3.51). The variation of the action with respect to the generalized position x gives

$$\delta S = \int dt \, \sum_{i=0}^{m} (-1)^i \, \frac{d^i}{dt^i} \frac{\partial L}{\partial x^{(i)}} \delta x, \tag{3.54}$$

so that the equations of motion of Lagrangian (3.51) are

$$\sum_{i=0}^{m} (-1)^i \, \frac{d^i}{dt^i} \frac{\partial L}{\partial x^{(i)}} = 0. \tag{3.55}$$

The equations of motion of system (3.52) are obtained by the Euler-Lagrange equation (3.4) since it is only of first order. The equations of motion of the Lagrange multipliers λ_j are

$$\dot{q}_j - q_{j+1} = 0. \tag{3.56}$$

For the coordinates q_i we get

$$\frac{\partial L}{\partial q_i} - \lambda_{i-1} - \dot{\lambda}_i = 0. \tag{3.57}$$

By solving equations (3.57) for λ_{i-1}, we can eliminate all λ_i and use equation (3.56) to obtain equation of motion (3.55). Thereby, we have shown that both Lagrangian (3.51) and (3.52) give the same dynamics. Up to this point constraints have not placed a role, since we stayed with Lagrangian mechanics. In the next chapter, this changes.

Another feature of higher-order Lagrangians is the Ostrogradsky instability [39, 41]. To discuss this, we limit the examined system to a second-order Lagrangian with only one variable x and thereby two momenta p_1 and p_2. The Hamiltonian of such a system is

$$H = p_1 x + p_2 \dot{x} - L. \tag{3.58}$$

The Ostrogradsky instability lies in the appearance of the term linear in p_1. It allows for runaway solutions since parts of the Hamiltonian can grow indefinitely in time while the

Hamiltonian stays constant. We note here that even the minimal example of one variable x, leads to a situation of two degrees of freedom, as discussed in more detail in reference [41]. The kinematic three-form gauge field does not suffer from the Ostrogradsky instability, as was already analyzed in reference [42]. In the next section, we obtain the same result as a byproduct of the Hamiltonian analysis.

3.7. The kinematic three-form gauge field

After discussing constrained Hamiltonian mechanics and higher-order Lagrangians, we are finally ready to combine both methods to analyze the kinematic three-form gauge field. After obtaining the total Hamiltonian density, we calculate the number of local degrees of freedom and obtain the equations of motion for the canonical variables.

The Lagrangian density to consider is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{48}\nabla^\alpha F^{\mu\nu\rho\lambda}\nabla_\alpha F_{\mu\nu\rho\lambda} - \epsilon\left(F^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}\right) \\ &= \frac{1}{2}\bar{C}\left(\ddot{X}_0 + \nabla_i\dot{X}_i\right) - \frac{1}{2}f\left(\dot{X}_0 + \nabla_i X_i\right),\end{aligned} \tag{3.59}$$

where $f(x_0 + \nabla_i X_i)$ is defined as

$$f\left(\dot{X}_0 + \nabla_i X_i\right) \equiv \frac{1}{2}\bar{C}\nabla_j\left(\dot{X}_0 + \nabla_i X_i\right)\nabla_j\left(\dot{X}_0 + \nabla_i X_i\right) + \epsilon\left(\dot{X}_0 + \nabla_i X_i\right) \tag{3.60}$$

with the potential ϵ like in the nonkinematic theory. By section 3.6, we can rewrite this second-order Lagrangian density into a first-order Lagrangian density

$$\tilde{\mathcal{L}} = \frac{1}{2}\bar{C}(\dot{x}_0 + \nabla_i x_i)^2 - \frac{1}{2}f(x_0 + \nabla_i X_i) + \lambda_0\left(x_0 - \dot{X}_0\right) + \lambda_i\left(x_i - \dot{X}_i\right), \tag{3.61}$$

where we introduced the new canonical variables

$$x_0 \equiv \dot{X}_0, \quad x_i \equiv \dot{X}_i, \tag{3.62}$$

and the canonical variables X_0, X_i are the same as in section 3.5

$$X_0 \equiv -6A_{123}, \quad X_1 \equiv 6A_{023}, \quad X_2 \equiv -6A_{013}, \quad X_3 \equiv 6A_{012}. \tag{3.63}$$

By using Lagrangian density (3.61), the momenta associated with the variables $x_0, x_i, X_0, X_i, \lambda_0$ and λ_i are, respectively,

$$p_0 = \bar{C}(\dot{x}_0 + \nabla_i x_i), \quad p_i = 0, \quad P_0 = \lambda_0, \quad P_i = \lambda_i, \quad \Gamma_0 = 0, \quad \Gamma_i = 0, \tag{3.64}$$

where we conclude eleven constrains

$$\phi_i^1 \equiv p_i, \quad \phi^2 \equiv P_0 - \lambda_0, \quad \phi_i^3 \equiv P_i - \lambda_i, \quad \phi^4 \equiv \Gamma_0, \quad \phi_i^5 \equiv \Gamma_i. \tag{3.65}$$

The extended Hamiltonian density is

$$\begin{aligned}\mathcal{H}_E =& \frac{p_0^2}{2\bar{C}} + x_i \nabla_i p_0 + p_i \dot{x}_i + P_0 \dot{X}_0 + P_i \dot{X}_i + f(x_0 + \nabla_i X_i) \\ & + \Gamma_0 \dot{\lambda}_0 + \Gamma_i \dot{\lambda}_i + \lambda_0 \left(x_0 - \dot{X}_0\right) + \lambda_i \left(x_i - \dot{X}_i\right) + \\ & u_i^1 p_i + u^2 (P_0 - \lambda_0) + u_i^3 (P_i - \lambda_i) + u^4 \Gamma_0 + u_i^5 \Gamma_i, \end{aligned} \tag{3.66}$$

where the u^i, $i = 1, \ldots, 5$ are arbitrary functions. For consistency, we set the time evolution, obtained by the Poisson bracket with the extended Hamiltonian density, of the constraints to zero

$$\begin{aligned}
&\dot{\phi}_i^1 = \nabla_i p_0 + \lambda_i && \Rightarrow \quad \phi_i^6 \equiv -\nabla_i p_0 + \lambda_i, && (3.67a)\\
&\dot{\phi}^2 = u^4 + \dot{\lambda}_0 && \Rightarrow \quad u^4 = -\dot{\lambda}_0, && (3.67b)\\
&\dot{\phi}_i^3 = -\frac{\partial f(x_0 + \nabla_k X_k)}{\partial X_i} + \dot{\lambda}_i + u_i^5 && \Rightarrow \quad u_i^5 = -\nabla_i f'(x_0 + \nabla_k X_k) - \dot{\lambda}_i, && (3.67c)\\
&\dot{\phi}^4 = x_0 - \dot{X}_0 - u^2 && \Rightarrow \quad u^2 = x_0 - \dot{X}_0, && (3.67d)\\
&\dot{\phi}_i^5 = x_i - \dot{X}_i - u_i^3 && \Rightarrow \quad u_i^3 = x_i - \dot{X}_i. && (3.67e)
\end{aligned}$$

By (3.67), we have determined eight functions u^i and got three new constraints ϕ_i^6 by equation (3.67a). Using equations (3.67b–e), we get the extended Hamiltonian density

$$\begin{aligned}\mathcal{H}_E =& \frac{p_0^2}{2\bar{C}} + P_0 x_0 + P_i x_i + f(x_0 + \nabla_i X_i) + (\nabla_i p_0)\, x_i \\ & + p_i \left(\dot{x}_i + u_i^1\right) + \nabla_j f'(x_0 + \nabla_i X_i)\, \Gamma_j + u_i^6 (\lambda_i - \nabla_i p_0)\,. \end{aligned} \tag{3.68}$$

Since we have obtained new constraints, we check for consistency again:

$$\begin{aligned}
&\dot{\phi}_i^1 = \nabla_i p_0 + P_i && \Rightarrow \quad \phi_i^7 \equiv \nabla_i p_0 + P_i, && (3.69a)\\
&\dot{\phi}^2 = 0, && && (3.69b)\\
&\dot{\phi}_i^3 = \nabla_i \left(\nabla_j f''(x_0 + \nabla_k X_k)\, \Gamma_j\right), && && (3.69c)\\
&\dot{\phi}^4 = 0, && && (3.69d)\\
&\dot{\phi}_i^5 = -u_i^6 && \Rightarrow \quad u_i^6 = 0, && (3.69e)\\
&\dot{\phi}_i^6 = \nabla_i P_0 + \nabla_i \left(\nabla_j f''(x_0 + \nabla_k X_k)\, \Gamma_j\right) && \Rightarrow \quad \phi_i^8 \equiv \nabla_i P_0 - \nabla_i \left(\nabla_j f''(x_0 + \nabla_k X_k)\, \Gamma_j\right). && (3.69f)
\end{aligned}$$

Equation (3.69e) fixed the arbitrary function u_i^6 to zero and renders the extended Hamiltonian thereby independent of λ_0 and λ_i. Accordingly, the equations of motion for the variables x_0, x_i, X_0 and X_i do not include these variables and we can immediately impose the constraints ϕ^4 and ϕ_i^5. This procedure corresponds to the introduction of the Dirac brackets [37]. Since we have imposed the constraints ϕ_i^4 and ϕ^5, constraints ϕ^2 and ϕ_i^3 are rendered unnecessary and we get from (3.69) six new constraints ϕ_i^7 and ϕ_i^8. For bookkeeping, our remaining constraints are

$$\phi_i^1 = p_i, \quad \phi_i^7 = \nabla_i p_0 + P_i, \quad \phi_i^8 = \nabla_i P_0. \tag{3.70}$$

The time evolution of the new constraint ϕ_i^7 is the other new constraint ϕ_i^8. The time evolution of ϕ_i^8 is zero since the extended Hamiltonian is independent of X_0. Thereby, we have completely analyzed the constrained Hamiltonian mechanics and obtain the total Hamiltonian density

$$\mathcal{H}_T = \frac{p_0^2}{2\bar{C}} + f(x_0 + \nabla_i X_i) + P_0 x_0 + (P_i + \nabla_i p_0)\left(x_i + u_i^7\right) + p_i\left(\dot{x}_i + u_i^1\right) + (\nabla_i P_0)\, u_i^8. \quad (3.71)$$

Just like in the nonkinematic theory, we find the momentum P_0 to be constant by constraints and thereby resulting in one global degree of freedom. Thereby, the appearance of a global degree of freedom is a general feature of a three-form gauge field. It is exactly this constant momentum, that stabilizes the theory in respect to the Ostrogradsky, as discussed in the last section. We already know from section 2.2 that the chemical potential μ is also a constant of the theory. We find a connection to the constant momentum P_0 below. Before, let us calculated the local degrees of freedom. Constraints (3.70) are seven first-class constraints and we have sixteen canonical variables. Thereby, we calculate the number of local degrees of freedom

$$\#\text{dof} = \frac{1}{2}(16 - 2 \cdot 7) = 1. \quad (3.72)$$

This result was already suspected in reference [42] and is hereby confirmed. It makes the kinematic three-form gauge field effectively a (pseudo-)scalar field. We conclude this section by giving the equation of motion and finding the connection between the constant canonical variable P_0 and the chemical potential μ.

The equations of motion are obtained by the Poisson bracket of the variable with the total Hamiltonian. We get

$$\dot{X}_0 = x_0, \qquad \dot{P}_0 = 0, \quad (3.73a)$$

$$\dot{X}_i = x_i + u_i^7, \qquad \dot{P}_i = \nabla_i f'(x_0 + \nabla_i X_i), \quad (3.73b)$$

$$\dot{x}_0 = \frac{p_0}{\bar{C}} - \nabla_i x_i - \nabla_i u_i^7, \qquad \dot{p}_0 = -f'(x_0 + \nabla_i X_i) - P_0, \quad (3.73c)$$

$$\dot{x}_i = \dot{x}_i + u_i^1, \qquad \dot{p}_i = -P_i - \nabla_i p_0. \quad (3.73d)$$

To recover the original equations of motion from section 2.2, we use the time evolution of the canonical momentum p_0 (3.73c), the definition of the momentum p_0 (3.64) and the definition of $f(x_0 + \nabla_i X_i)$ (3.60)

$$-P_0 = \dot{p}_0 + f' = \bar{C}[(\ddot{x}_0 + \nabla_i \dot{x}_i) + \nabla_i \nabla_i (x_0 + \nabla_i X_i)] + \frac{d\epsilon(x_0 + \nabla_i X_i)}{d(x_0 + \nabla_i X_i)}$$

$$\Rightarrow \mu = \frac{d\epsilon(q)}{dq} - \bar{C}\nabla_\nu \nabla^\nu q, \quad (3.74)$$

where we have used (3.40) to identify the q-field. In equation (3.74), we have identified the constant canonical variable P_0 with the chemical potential μ by the relation

$$\mu \equiv -P_0. \quad (3.75)$$

Above identification is one of the main results of this thesis. The connection of a canonical variable with the chemical potential allows for dynamical vacuum energy in the path integral formalism of the kinematic three-form gauge field, as we see in section 5.4.

Our last remark concerns the gauge-invariant local degree of freedom we expect by result (3.72). There is no particular variable in the equation of motion (3.73) not involving arbitrary functions. So, similar to the Standard Maxwell gauge theory 3.4, we can not identify immediately a pair of canonical variables that correspond to the physical local degree of freedom. This is resolved in section 4.3, where a canonical transformation is used to find the physical direction in phase space.

4. Path integral for a three-form gauge field

This chapter is dedicated to obtain the path integral formalism for the nonkinematic and kinematic three-form gauge field. We heavily use the results of the previous chapter.

The chapter is structured in the following way. First, we give an introduction to the path integral formalism incorporating first-class constraints. Second, we formulate the path integral for the nonkinematic three-form gauge field and compare our result to the results in literature. Third, we use a canonical transformation to identify the gauge-invariant degrees of freedom of the kinematic three-form gauge field. Last, we give the path integral formulation of the kinematic three-form gauge field.

4.1. The path integral of a constrained system

In this first section of the chapter, we introduce the path integral formalism and examine the consequences of constraints. The path integral formalism goes back to Feynman [43]. The influence of first-class constraints was first analyzed by Faddeev [44] from where we draw a lot. The influence of second-class constraints was later considered by Senjanovic [45].

We follow the notation of reference [44]. The path integral of an unconstrained system is given by [15, 46, 47]

$$\langle out|\, S\, |in\rangle = \int\limits_{\text{asymp}} \exp\left\{ i \int\limits_{-\infty}^{\infty} dt\ (p_i \dot{q}_i - H) \right\} \prod_i \frac{dp_i dq_i}{(2\pi)^n}, \tag{4.1}$$

where H is the Hamiltonian of the system, q_i and p_i are the canonical variables and n is the number of degrees of freedom. Here, the states $\langle out|$ and $|in\rangle$ are defined by the solutions of the equations of motion in the asymptotic regime $t \to \pm\infty$, these conditions are indicated by the subscript "asymp".

Note here that we are not using the action in the path integral directly. The process to identify the term in the path integral with the Lagrangian and calculating the integral over the momenta is described in [15, Section 9.2].

The measure of integral (4.1) describes an integration over the complete phase space. Thereby, it is clear that a gauge theory leads to difficulties since not all of phase space represents different physical configurations. This leads to an infinite overcounting of every physical field configuration since there are infinitely many equivalent configuration in phase space for the same physical configuration. This overcounting is addressed by Faddeev's method.

The main point of this method is to prove the following equality

$$\int_{\text{asymp}} \exp\left\{ i \int_{-\infty}^{\infty} dt \, (p_i \dot{q}_i - H - \lambda_a \varphi_a) \right\} \prod_a \delta(\chi_a) \det \| \{\chi_a, \varphi_b\} \| \prod_i \frac{dp_i dq_i}{(2\pi)^n} \prod_b d\lambda_b \tag{4.2a}$$

$$= \int_{\text{asymp}} \prod_i \frac{dp_i^*(t)\, dq_i^*(t)}{(2\pi)^{n-m}} \exp\left[i \int_{-\infty}^{\infty} dt \, p_i^* \dot{q}_i^* - H^* \right], \tag{4.2b}$$

where in the first line of the equation a runs from 1 to m, λ_a are the Lagrange multipliers enforcing the constraints φ_a and χ_a, $a = 1, \ldots, m$ are the gauge conditions. In the second line, the variables denoted with an asterisk are obtained by a canonical transformation, where the canonical momenta of the new system of variables p_a are the gauge conditions

$$\chi_a = p_a. \tag{4.3}$$

These momenta p_a and their corresponding variables q_a are dropping out of the formalism, so that only the pairs q_i^* and p_i^* remain. This method reduces the phase space of the original problem and thereby describes the path integral with fewer canonical variables. More information on that calculation is in reference [44] and in textbook [28, Section 16].

In practice, it can be hard to find a fitting canonical transformation and one rather starts from the first line of equation (4.2) since it gives an analytic expression for the path integral of a constrained system. The vector potential A_μ of Standard Maxwell theory from section 3.4, for example, can not be easily reexpressed in a purely physical direction in phase space, so that the second line of equation (4.2) may not give a description of the path integral of Standard Maxwell gauge theory. The formulation of equivalence classes discussed earlier and in [35, Appendix 2.E] would of course solve this problem, but lacks a precise description in variables. The nonkinematic and kinematic three-form gauge field theories are in this aspect, as we see, different from the Standard Maxwell gauge theory since they have a description in terms of gauge invariant variables.

The nonkinematic three-form gauge field has, as seen in section 3.5, only one global degree of freedom expressed in the variable P_0, so that there is already a reduced phasespace description. The kinematic three-form gauge field, on the other hand, has no definite variables associated with the single propagating degree of freedom, so that we use a canonical transformation in section 4.3 to get to a reduced phasespace description.

4.2. The path integral of the nonkinematic three-form gauge field

In this section, we give the path integral formulation of the nonkinematic three-form gauge field and compare it to literature. As reminder, the path integral formulation of this theory was already found [14] and the wick-rotated partition functional $Z(V)$ is

$$Z(V; f) = \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \exp\left\{ -\frac{1}{2} \int_V d^4x \, f^2 \right\}, \tag{4.4}$$

where μ_0 is a fixed constant and f is defined by

$$F_{\mu\nu\rho\lambda} = f\epsilon_{\mu\nu\rho\lambda}. \tag{4.5}$$

To compare our result to (4.4), we use the same potential, that we left unspecified in the previous section 3.5, as in reference [14]. The potential is

$$\epsilon\left(F^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}\right) = -\frac{1}{48}F^{\mu\nu\rho\lambda}F_{\mu\nu\rho\lambda}, \tag{4.6}$$

which leads to the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\dot{X}_0 + \nabla_i X_i\right)^2, \tag{4.7}$$

in the variables of section 3.5. The momentum corresponding to X_0 is obtained from the Lagrangian density (4.7) as

$$P_0 = \dot{X}_0 + \nabla_i X_i. \tag{4.8}$$

According to section's 3.5 results, the total Hamiltonian density is

$$\mathcal{H}_T = \frac{1}{2}P_0^2 + \nabla_i P_0\left(X_i + u_i^2\right) + P_i\left(\dot{X}_i + u_i^1\right), \tag{4.9}$$

the constraints are

$$\phi_i^1 = P_i, \quad \phi_i^2 = \nabla_i P_0, \tag{4.10}$$

and the equations of motion are

$$\dot{X}_0 = P_0 - \nabla_i u_i^2, \qquad \dot{P}_0 = 0, \tag{4.11a}$$

$$\dot{X}_i = \dot{X}_i + u_i^1, \qquad \dot{P}_i = 0. \tag{4.11b}$$

According to the discussion at the end of section 3.5, the nonkinematic three-form gauge field has no local degrees of freedom, but has one global degree of freedom, since the canonical momentum P_0 is constrained to be any constant, in contrast to the other momenta P_i that are strictly zero. Thereby, we identify P_0 to be the only variable we integrate over in the path integral. The Poisson brackets of the momentum P_0 with the constraints are zero, consequently it is gauge-invariant. The Hamiltonian density of this reduced phase space is

$$\mathcal{H}^* = \frac{1}{2}\bar{P}_0^2, \tag{4.12}$$

where the bar above P_0 indicates that it is constant.

At this point an important remark is due. The field configurations we integrate over within the path integral formalism do not, in general, satisfy the equation of motion of the field. Accordingly, one might expect, that the momentum P_0 is not constant in time, since this condition is listed as equation of motion in (4.11). But, the constant time evolution of P_0 is also required as constraint since this is the consistency condition for the constraint ϕ^2 in equation (4.10). We do not include this constraint in the formalism, since the time variable t is singled out in Hamiltonian mechanics, which is of course in contrast to special relativity. A covariant formulation of constraint Hamiltonian mechanics can be achieved by parameterizing the theory by including the time t as a canonical variable [28, Chapter 4]. Thereby, P_0 is constant for every field configuration we integrate over.

Since P_0 is constant in space and time, the sum over all field configurations reduces to the ordinary integral

$$\int \mathcal{D}\bar{P}_0 \; = \int_{-\infty}^{\infty} d\bar{P}_0 \, . \tag{4.13}$$

Above equation resolves the last constraint of the formalism. Consequently, we can give the path integral (4.2) as

$$\int \mathcal{D}\bar{P}_0 \exp\left\{ i \int d^4x \, \left(\bar{P}_0 \dot{X}_0 - \mathcal{H}^*\right) \right\} = \int_{-\infty}^{\infty} d\bar{P}_0 \exp\left\{ i \int d^4x \left(-\frac{1}{2}\bar{P}_0^2\right) \right\}, \tag{4.14}$$

where we have used that P_0 is constant in time due to constraints, so that the term $P_0\dot{X}_0$ in the exponent vanishes at the boundary. Otherwise, we would need a gauge condition for X_0. This result is exactly the wick-rotated integral (4.4) up to a constant. In the next sections, we apply the formalism to the kinematic three-form gauge field. Since there is also a constant canonical momentum in the kinematic three-form gauge field, we may expect a similar result.

4.3. Separation of gauge-invariant degrees of freedom

In this section, we apply a canonical transformation to the kinematic three-form gauge field from section 3.7 to separate the pure gauge degrees of freedom from the pure physical degrees of freedom. Previous discussion has shown that the kinematic three-form gauge field has one global and one local degree of freedom. The global degree of freedom is connected to the canonical momentum P_0, just like in the nonkinematic theory, as we recall from section 3.5 and discussion in section 4.2. The local degree of freedom, on the other hand, is not directly connected to a single pair of canonical variables. This, we want to address in the following.

To clarify the procedure, let us give a little analogy. Consider a two-dimensional plane where the distance of a point to the origin is considered the physical quantity. If we describe this point by Cartesian coordinates, we need two coordinates to obtain the distance. These coordinates, of course, include more information than the distance, namely the angle, but this is considered gauge. We can not divide between the physical degree of freedom and gauge degree of freedom in Cartesian coordinates. Now, we consider a coordinate transformation into polar coordinates. Accordingly, one coordinate (the radius) gives directly the physical quantity, that is the distance of the point to the origin. The other coordinate (the angle) represents a gauge degree of freedom. Thereby, we separated gauge degrees of freedom and physical degrees of freedom from each other. In the remainder of this section we do a similar procedure to obtain this separation.

At this point, we can also make a short remark on the usage of the term "canonical". In textbooks [27, 28] the term is used like in this thesis, i.e. calling the momentum variable of

Hamiltonian theory or the equations of motion derived from the Hamiltonian "canonical". But there is also found another definition in literature. In reference [48], for example, the authors write (about a Hamiltonian formulation of gravity): "A precise determination of the independent dynamical modes of the gravitational field is arrived at when the theory has been cast into canonical form and consequently involves the minimal number of variables specifying the state of the system." In the sense of this reference, we are therefore casting the three-form gauge theory into its canonical form in this chapter.

Returning to the task of finding a canonical transformation, separating the gauge degrees of freedom from the physical ones, we already know a minimal variable representation of the kinematic three-form gauge field from section 2.2, namely the q-field, defined in terms of the canonical variables by equation (3.40). Therefore, we use a canonical transformation to get the q-field as one of the canonical variables. For the rest of the variables, we draw from reference [44] and choose the momenta so that they represent the constraints ϕ^1 and ϕ^7 of equation 3.70. Considering the above, we use the generating function [27, Section 9.1]

$$F_3\left(p_0, p_i, P_0, P_i; x_0^*, x_i^*, X_0^*, X_i^*; t\right) = -p_0 x_0^* - p_i x_i^* - X_0^* P_0 - \left(\nabla_i p_0 + P_i\right) X_i^* + P_0\left(\nabla_i X_i^*\right) t, \quad (4.15)$$

where the variables with the asterisk denote the new canonical variables. The relations between the new and old canonical variables are

$$x_l = -\frac{\partial F_3}{\partial k_l}, \quad k_l^* = -\frac{\partial F_3}{\partial x_l^*}, \quad H^* = H + \frac{\partial F_3}{\partial t}, \quad (4.16)$$

where x_l and k_l are, respectively, the canonical positions and momenta. Using transformation (4.15) and (4.16), we find the new variables to be

$$x_0^* = x_0 + \nabla_i X_i, \qquad p_0^* = p_0, \quad (4.17a)$$
$$x_i^* = x_i, \qquad p_i^* = p_i, \quad (4.17b)$$
$$X_0^* = X_0, \qquad P_0^* = P_0, \quad (4.17c)$$
$$X_i^* = X_i, \qquad P_i^* = \nabla_i p_0 + P_i. \quad (4.17d)$$

The new total Hamiltonian density is given by

$$\mathcal{H}_T^* = \frac{\left(p_0^*\right)^2}{2\bar{C}} + f\left(x_0^*\right) + P_0 x_0^* + \left(u_i^6 + x_i^*\right) P_i^* + \left(u_i^1 + \dot{x}_i^*\right) p_i^* + \left(\nabla_i P_0\right) u_i^8, \quad (4.18)$$

where the original total Hamiltonian density (3.71) was used. The constraints are

$$p_i^* = 0, \quad P_i^* = 0, \quad \nabla_i P_0^* = 0. \quad (4.19)$$

At this point, we still use eight canonical variables to describe one local degree of freedom. The canonical equations of motion for the variables x_i^*, X_i^*, p_i^* and P_i^* are

$$\dot{x}_0^* = \frac{p_0^*}{\bar{C}}, \qquad \dot{p}_0^* = -\frac{df\left(x_0^*\right)}{dx_0^*} - P_0^*, \quad (4.20a)$$
$$\dot{X}_0^* = x_0^* - \nabla_i u_i^8, \qquad \dot{P}_0^* = 0, \quad (4.20b)$$
$$\dot{x}_i^* = u_i^1 + \dot{x}_i^*, \qquad \dot{p}_i^* = 0, \quad (4.20c)$$
$$\dot{X}_i^* = x_i^* + u_i^6, \qquad \dot{P}_i^* = 0. \quad (4.20d)$$

With equation (4.20a), we finally found a gauge-invariant pair of canonical variables representing the one local degree of freedom of the kinematic three-form gauge field theory. The variables x_i^* and X_i^* represent gauge degrees of freedom since they are completely arbitrary. Additionally, they do not influence the equations of motion of the variables x_0^* and X_0^* and thereby can be completely removed from the formalism. Finally, the canonical variable X_0^* and, more importantly, the conjugated momentum P_0^* have the same structure as in the nonkinematic three-form gauge field theory and represent a global degree of freedom. Since the total Hamiltonian density (4.18) does not include the variable X_0^*, which is arbitrary since equation (4.20b) includes the arbitrary function u^8, we find a completely gauge-invariant formulation of the three-form gauge field in terms of the three variables x_0^*, p_0^* and P_0^*. In the next section, we use this formulation to obtain the path integral of the kinematic three-form field.

4.4. The path integral of the kinematic three-form gauge field

In the last section, we performed a canonical transformation to identify the relevant canonical variables with physical degrees of freedom of the kinematic three-form gauge field. This enables us, similar to the nonkinematic three-form gauge field, to obtain the path integral formalism of the kinematic three-form gauge field. To start with a familiar notation, we define

$$q \equiv x_0^*, \quad p_q \equiv p_0^*, \quad P_0^* \equiv -\mu, \tag{4.21}$$

so that the new total Hamiltonian density is

$$\mathcal{H}_q = \frac{p_q^2}{2\bar{C}} + \frac{1}{2}\bar{C}(\nabla_i q)^2 + \epsilon(q) - \mu q, \tag{4.22}$$

where we reinstated the function f according to equation (3.60). The canonical equation of motion is

$$\dot{p}_q = C\nabla_i\nabla_i q - \frac{d\epsilon(q)}{dq} + \mu, \quad \dot{q} = \frac{p_q}{\bar{C}} \tag{4.23a}$$

$$\Rightarrow \mu = \frac{d\epsilon(q)}{dq} - \bar{C}\Box q, \tag{4.23b}$$

which is in Minkowski spacetime equivalent to (2.28). According to section 4.1, the path integral is

$$Z_0 = \int \mathcal{D}\mu\mathcal{D}q\mathcal{D}p_q \exp\left\{i\int d^4x\,\left(\mu\dot{X}_0^* + \dot{q}p_q - \mathcal{H}_q\right)\right\} \tag{4.24a}$$

$$= \int_{-\infty}^{\infty} d\mu \int \mathcal{D}q\mathcal{D}p_q \exp\left\{i\int_{-\infty}^{\infty} d^4x\,\left(\frac{1}{2\bar{C}}p_q^2 - \mathcal{H}_q\right)\right\} \tag{4.24b}$$

$$= \mathcal{N}\int_{-\infty}^{\infty} d\mu \int \mathcal{D}q \exp\left\{i\int_{-\infty}^{\infty} d^4x\,\left(\frac{1}{2}\bar{C}\nabla_\nu q\nabla^\nu q + \epsilon(q) - \mu q\right)\right\}, \tag{4.24c}$$

where we have used that the term $\mu \dot{X}_0^*$ vanishes on the boundary since μ is constant and equation (4.13) to get from the first line (4.24a) to the second line (4.24b) and we integrate over the only quadratically appearing canonical momentum p_q, cf. [46, Chapter 5], to get from (4.24b) to (4.24c), where we included the normalization factor $\mathcal{N}$.

We note here that the path integral involves, just like in the nonkinematic theory, an integration over a constant quantity, here the chemical potential μ, and thereby also provides the dynamical cancellation process of the nonkinematic three-form gauge theory as discussed in references [10–12] and in section 5.4.

Returning to the kinematic three-form gauge field, we can calculate the propagator of this theory since it has one propagating degree of freedom according to our previous calculations. Therefore, we introduce a current to the exponent of the path integral. More examination of this current is done in section 5.2. We follow in this calculation directly textbook [46, Section 6.1]. Before introducing the current, we shift the q-field to the minimum q_0 of its potential

$$\frac{d}{dq}(\epsilon(q) - \mu q)\bigg|_{q=q_0} = 0, \tag{4.25}$$

so that

$$q \to q_0(\mu) + q, \tag{4.26}$$

where we keep in mind that the equilibrium value q_0 depends on the chemical potential μ, which is also a variable integrate over. We obtain the path integral

$$\begin{aligned} Z[j] =& \mathcal{N} \int_{-\infty}^{\infty} d\mu \exp\left\{ i \int_{\mathbb{R}^4} d^4x \, \left(\epsilon(q_0) - \mu q_0\right) \right\} \times \\ & \int \mathcal{D}q \exp\left\{ -i \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \bar{C} q \left(\Box - m_q^2 \right) q - jq \right) \right\}, \end{aligned} \tag{4.27}$$

where we neglected the interaction terms of the q-field and we used the mass term m_q^2

$$m_q^2 \equiv \frac{1}{\bar{C}} \frac{d^2\epsilon(q)}{d^2 q}\bigg|_{q=q_0} . \tag{4.28}$$

We introduce the Feynman propagator by shifting the field q by

$$q \to q + \tilde{q}, \tag{4.29}$$

where $\tilde{q}$ solves the equation of motion

$$\left(\Box - m_q^2\right) \tilde{q} = j \quad \Rightarrow \tilde{q}(x) = \int_{\mathbb{R}^4} d^4y \, G(x-y,\mu) \, j(y) \, , \tag{4.30}$$

and the Feynman propagator $G(x-y,\mu)$ is defined by

$$\left(m_q^2 - \Box\right) G(x-y,\mu) = -\delta(x) \, . \tag{4.31}$$

Accordingly, the path integral is

$$Z[j] = \mathcal{N} \int_{-\infty}^{\infty} d\mu \exp\left\{ i \int_{\mathbb{R}^4} d^4x \, (\epsilon(q_0) - \mu q_0) \right\} \int \mathcal{D}q \exp\left\{ -i \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \bar{C} q \left(\Box - m_q^2 \right) q \right) \right\} \times$$
$$\exp\left\{ -\frac{1}{2} i \int_{\mathbb{R}^4} d^4x \; j(x) \, G(x-y,\mu) \, j(y) \right\}. \tag{4.32}$$

In the usual calculation for a fundamental scalar field we would conclude that the integration over the field q only contributes a normalization factor, so that we can get the propagator by functional differentiation with respect to the current. Here, this is not possible, since we additionally integrate over the chemical potential and the lowest energy state of the q-field, namely the mass term, depends on this chemical potential.

Let us rewrite this path integral before continuing the discussion. We define the quantities

$$\rho_v(\mu) \equiv \epsilon(q_0(\mu)) - \mu q_0(\mu) \,, \quad A(\mu) \equiv \int \mathcal{D}q \exp\left\{ -i \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \bar{C} q \left(\Box - m_q^2 \right) q \right) \right\}, \tag{4.33}$$

where $\rho_v(\mu)$ is the vacuum energy density from q-theory. We write the partition function as

$$Z[j] = \mathcal{N} \int_{-\infty}^{\infty} d\mu \, A(\mu) \exp\left\{ i \int_{\mathbb{R}^4} d^4x \; \rho_v(\mu) \right\} \exp\left\{ -\frac{1}{2} i \int_{\mathbb{R}^4} d^4x \; j(x) \, G(x-y,\mu) \, j(y) \right\}. \tag{4.34}$$

From here, we would have to examine the influence of the integration over the chemical potential μ to carry on the calculation. However, we can note that, if the chemical potential is fixed, the propagator of the kinematic three-form gauge field corresponds to the propagator of a scalar field. More discussion about the influence of the integration over the chemical potential μ is in section 5.4.

In this section, we found the path integral formulation of the kinematic three-form gauge field. It features the same integration over a constant field that led to the idea that the most probable field configuration of the three-form gauge field cancels the vacuum energy density, cf. [10–12]. Although the kinematic three-form gauge field has only one propagating degree of freedom, the integration over the global degree of freedom, represented by the chemical potential μ, makes the evaluation of the path integral more intricate, so that further analysis is needed.

5. Interactions of a three-form gauge field

In this chapter, we examine some possible interactions of the kinematic three-form gauge field. First, we use an analogy of the q-field with the neutral pion to motivate an interaction term with Standard model components. Second, we introduce a gauge-invariant current to our previous model. Third, we discuss the vacuum energy density cancellation induced by a gauge-invariant current. Last, we discuss possible vacuum energy density cancellation by the path integral formalism.

5.1. Decay of the q-field into photons

In this section, we draw from the Standard Model of particle physics an interaction term of the q-field with Standard Model matter and thereby obtain a decay rate in Minkowski spacetime. We use this decay rate to calculate the evolution of a universe with two perfect fluid components.

According to previous chapters the q-field is, in Minkowski spacetime, a pseudoscalar field with a mass dependent on the chemical potential μ of q-theory. Since we use the Standard Model to motivate an interaction term, we immediately arrive at the neutral pion π^0, which is a pseudoscalar nonfundamental particle [49]. We are interested in its decay into photons, which can be described by the effective interaction Lagrangian density [50, Section 22.1]

$$\mathcal{L}_{\pi\gamma\gamma} = g\pi^0 \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}, \tag{5.1}$$

where g is a constant of mass dimension -1, π^0 is the pion field and $F_{\mu\nu}$ is the field strength tensor of the vector potential A_μ of Standard Maxwell gauge theory. The interaction Lagrangian density (5.1) leads to the decay rate

$$\Gamma_{\pi^0\gamma\gamma} = \frac{m_\pi^3 g^2}{\pi}, \tag{5.2}$$

where m_π is the mass of the neutral pion π^0.

According to above equations, we introduce the interaction term of the q-field decaying into photons as [49]

$$\mathcal{L}_{q\gamma\gamma} = f q \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}, \tag{5.3}$$

where the constant f is of mass dimension -2, because of the dimensionality of the q-field.

Before giving the decay rate of the q-field into photons, we make a remark about an appealing feature of interaction Lagrangian density (5.3). For that, we need the equation of motion of the vector potential A_μ of the combined system of q-field and photons. The system is described by action

$$S = \int d^4x \sqrt{-g} \left(\mathcal{L}_q + \mathcal{L}_{EM} + \mathcal{L}_{q\gamma\gamma} \right), \tag{5.4}$$

where $\mathcal{L}_q$ is given by equation (2.22) with a constant $C(q) = \bar{C}$ and the Maxwell Lagrangian density has the standard form (3.27). The equation of motion of the vector potential A_μ is obtained by the variation of the action (5.4) with respect to the vector potential A_μ and is

$$\nabla_\nu F^{\alpha\nu} + f \frac{\epsilon^{\alpha\mu\nu\lambda}}{\sqrt{-g}} F_{\mu\nu} \nabla_\lambda q = 0. \tag{5.5}$$

This equation shows that the q-field does not effect the Standard Maxwell dynamics, if it is in the equilibrium state, where the four-gradient of it is zero. Thereby, Standard Maxwell theory is unaffected in Minkowski spacetime.

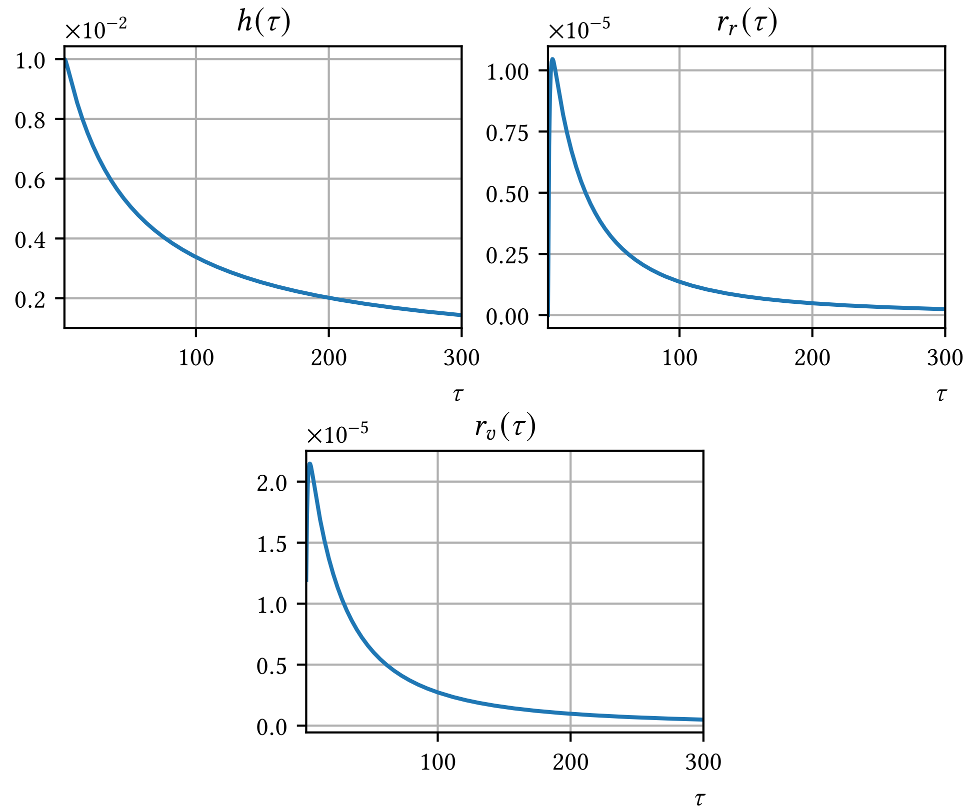

Figure 5.1.: In the upper left the solution to equation (A.4) is plotted with parameter $\gamma_{q\gamma\gamma} = 1$ and initial condition $h(1) = 5 \cdot 10^{-2}$ and $h'(1) = 0$. r_v is the dimensionless vacuum energy density and r_r the dimensionless radiation energy density obtained by equations (A.5a) and (A.5b).

Returning to the decay of the q-field, we find the decay rate in analogy to (5.2) to be

$$\Gamma_{q\gamma\gamma} = \frac{m_q^3 f^2}{\pi \bar{C}}, \tag{5.6}$$

where the additional $\bar{C}$ in the denominator is caused by the different mass dimensions of the scalar field and the q-field. In Appendix A.1, the evolution of a universe with two interacting perfect fluids is examined. Here, we only show the results of this calculation in figure 5.1. We can calculate the age of the universe $t_{now}$to be approximately

$$t_{\text{now}} \sim \left(\frac{10^{20}\ \text{eV}}{E_q}\right)^5 \cdot 1\ \text{s}, \tag{5.7}$$

where we used a starting vacuum energy density of Planck scale and the current vacuum energy density of $\rho_{\text{now}} \sim \left(10^{-3}\ \text{eV}\right)^4$.

5.2. Introduction of a gauge-invariant current

In this section, we introduce a gauge-invariant current to the Lagrangian density of the kinematic three-form gauge field to examine the influence of interactions in this theory. Therefore, we use the Lagrangian density

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_q, \tag{5.8}$$

where $\mathcal{L}_M$ is a generic matter Lagrangian density not further specified. Requiring gauge-invariance under the transformation

$$A_{\mu\nu\rho}(x) \longrightarrow \tilde{A}_{\mu\nu\rho}(x) = A_{\mu\nu\rho}(x) + \nabla_{[\mu}\lambda_{\nu\rho]}(x)\,, \tag{5.9}$$

introduces a conserved current to the Lagrangian density (5.8)

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_q + A_{\mu\nu\rho}J^{\mu\nu\rho}, \tag{5.10}$$

where the conserved current $J^{\mu\nu\rho}$ satisfies the conservation equation

$$\nabla_\mu J^{\mu\nu\rho} = 0. \tag{5.11}$$

This conservation condition gives strong constraints on the form of the current, as we see in the following analysis.

First, let us rewrite the interaction term in Lagrangian density (5.8) in a way that is more fitting for the Hamiltonian approach

$$A_{\mu\nu\rho}J^{\mu\nu\rho} = X_0 J_0 - X_i J_i, \tag{5.12}$$

where we have introduced the current variables

$$J_0 = -4J_{123}, \quad J_1 = 4J_{023}, \quad J_2 = -4J_{013}, \quad J_3 = 4J_{012}. \tag{5.13}$$

Consequently, the Hamiltonian density gets modified by

$$\tilde{\mathcal{H}} = \mathcal{H} - X_0 J_0 + X_i J_i, \tag{5.14}$$

where the tilde indicates quantities that are changed because of the conserved current.

The introduction of this conserved current changes the analysis of the Hamiltonian constraints from section 3.7. We do not repeat the complete analysis, but give an overview of the changes. Repeating the analysis gives the constraints

$$\phi_i^1 = p_i, \quad \phi_i^7 = \nabla_i p_0 + P_i, \quad \tilde{\phi}_i^8 = \nabla_i P_0 + J_i, \tag{5.15}$$

with the new total Hamiltonian density

$$\begin{aligned}\mathcal{H}_T =& \frac{p_0^2}{2\bar{C}} + f(x_0 + \nabla_i X_i) + P_0 x_0 - X_0 J_0 + X_i J_i \\ & (P_i + \nabla_i p_0)\left(x_i + u_i^7\right) + p_i\left(\dot{x}_i + u_i^1\right) + (\nabla_i P_0)\, u_i^8.\end{aligned} \tag{5.16}$$

The consistency condition of constraint $\tilde{\phi}^8$ gives the equation

$$\nabla_i J_0 + \dot{J}_i = 0, \tag{5.17}$$

which is only composed of the current variables J_0 and J_i and thereby no additional constraint, since these variables are external and not canonical variables. The above equation (5.17) is exactly conservation equation (5.11) with the variables (5.13). Accordingly, we could, instead of introducing a conserved current, consider a general addition to the theory and then conclude by consistency of the constraint $\tilde{\phi}^8$ that this addition has to fulfill the conservation equation (5.11). In principle, the equation of motion has the same form as in section 3.7

$$\tilde{\mu} = \frac{d\epsilon(q)}{dq} - \bar{C}\nabla_\nu \nabla^\nu q, \tag{5.18}$$

but, as a consequence of the conserved current, the chemical potential $\tilde{\mu}$ is no longer constant. Specifically, the constraint $\tilde{\phi}^8$ in (5.15) and the conservation equation (5.17) gives the change of the chemical potential in time and space as

$$\dot{\tilde{\mu}} = -J_0, \tag{5.19a}$$

$$\nabla_i \tilde{\mu} = J_i. \tag{5.19b}$$

Equations (5.19) seem to contradict what we found earlier, namely, that the constancy of canonical variable P_0, which the chemical potential represents, stabilizes the system so that the Ostrogradsky instability is circumvented. If we choose, for example, a constant value for the current variables J_0 and J_i, they satisfy the conservation equation (5.17), but lead to an infinite chemical potential on the spacetime boundary $\left(x_\mu \longrightarrow \pm\infty\right)$ of Minkowski spacetime. A solution to this problem is, of course, the introduction of physical boundary conditions for the current variables. Consequently, the chemical potential $\tilde{\mu}$ does only change by a constant amount. This is actually connected to the conserved quantities of the Noether theorem, as we see by integrating the conservation equation (5.17)

$$\int dx_i\, \nabla_i J_0 = J_0\big|_{\partial M} = \partial_t \int dx_i\, J_i = 0, \tag{5.20}$$

where ∂M is the boundary. Integrating equation (5.19b) in the same fashion, we find the chemical potential to change by these conserved quantities. A constant change of the chemical potential changes the q-field by a constant amount. The constant change of the field by the introduction of an antisymmetric current was also found in reference [14].

There is another way to introduce a conserved current to the kinematic theory. Instead of coupling the current to the three-form gauge field $A_{\mu\nu\rho}$, we can couple a scalar current to the variable representing the local degree of freedom, namely the q-field. The Lagrangian density (5.10) becomes

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_q + jq, \tag{5.21}$$

where j is the scalar current. By the same argument as above, we find this current to be constant. The introduction of this current is equivalent by shifting the chemical potential

$$\mu \to \mu + j. \tag{5.22}$$

Concluding, the introduction of an antisymmetric current $J^{\mu\nu\rho}$ leads to an effective change of the chemical potential, which in turn leads to a constant change in the q-field. This is true for all gauge-invariant interactions of the three-form gauge field, which leads to the following three remarks. First, the analysis of the three-form gauge field including interactions boils down to the analysis of the free three-form gauge field since all interaction can be absorbed in the chemical potential μ. Second, the path integral is not affected by a shift in the chemical potential μ, because the integration over μ has boundaries from $-\infty$ to ∞. Third, the inclusion of a gauge-invariant interaction is another contribution to the vacuum energy density. In the next section, we examine the effects of this contribution.

5.3. Vacuum energy density cancellation by a conserved current

In this section, we examine the cancellation of vacuum energy density by a conserved current, as introduced in the last section. This is not the cancellation mechanism according to the path integral of the nonkinematic three-form gauge field, like in section 1.3, but a cancellation process by interactions with other fields. We artificially introduce a current j_ρ, that leads to the cancellation of the energy density ρ of scale E^4.

In the last section, we concluded that the current j_ρ shifts the effective chemical potential. Accordingly, the change in the vacuum energy density ρ_v of the q-field due to the current j_ρ is

$$\Delta\rho_v = \epsilon(q) - \left(\mu + j_\rho\right) q = -j_\rho q, \tag{5.23}$$

where we assumed, that the vacuum energy density $\rho_v = \epsilon(q) - \mu q$ before the interaction was zero. To cancel the contribution ρ to the vacuum energy density, it has to hold

$$\rho + j_\rho q = 0, \tag{5.24}$$

which makes the the current j_ρ of scale E^4/E_q^2, where E_q is the energy scale of q-theory.

After the interaction, the equilibrium value q_0 is shifted due to the shift in the chemical potential. Assuming the energy scale of the contribution ρ to be much smaller than of

q-theory $E \ll E_q$, the shift in the equilibrium value δq_0 is

$$\epsilon'(q_0 + \delta q_0) = \mu + j \quad \Rightarrow \quad \delta q = \frac{j}{\epsilon''(q_0)} \tag{5.25}$$

The change in the mass term m_q is

$$\delta m_q = m_q(\mu) - m_q(\mu + j_\rho) = -\frac{1}{2\sqrt{\bar{C}}} \frac{\epsilon'''(q_0)}{\epsilon''(q_0)^{3/2}} j_\rho + O(j^2) \sim \frac{E^4}{E_q^3}. \tag{5.26}$$

Thereby, every gauge-invariant interaction changes the effective potential of the q-field and, in that way, every gauge-invariant interaction changes the mode solutions of the q-field.

In this section, we have examined how the q-field changes by the introduction of a gauge-invariant current that cancels a generic distribution to the vacuum energy density. This cancellation process, however, requires the current to be of a particular form and is no dynamical process of vacuum energy cancellation. Such a dynamical process we want to examine in the next section.

5.4. Vacuum energy cancellation by the path integral

In this section, we discuss the vacuum energy cancellation, which may occur due to the integration over the chemical potential μ in the path integral formulation of the kinematic three-form gauge field. This cancellation process involves the calculation of the most probable field configuration of the path integral and is, thereby, a full quantum calculation. In section 4.4, we already obtained the path integral of the kinematic three-form gauge field. But, first, let us discuss the main idea of references [10–12]. The path integral of the nonkinematic three-form gauge field with a cosmological constant term is

$$Z = \int_{-\infty}^{\infty} df \, \exp\left\{ \int_{\mathbb{R}^4} d^4x_E \left(V(f) + \frac{\lambda}{8\pi G} \right) \right\} = \int_{-\infty}^{\infty} df \, \exp\left\{ \int_{\mathbb{R}^4} d^4x_E \, \rho_{vac}(f) \right\}, \tag{5.27}$$

where we wick-rotated the integral into euclidean spacetime, indicated by x_E, and ρ_{vac} is the overall vacuum energy density. The most probable configuration of the field f, which is P_0 in the Hamiltonian formulation of the nonkinematic three-form gauge field, occurs, according to equation (5.27), when the exponent is zero and, thereby, provides Minkowski spacetime.

Returning to the kinematic three-form gauge field, it also features the integration over the constant chemical potential μ, but, in contrast, it additionally features a propagating mode, which is influenced by the value of the chemical potential μ.

The path integral for the kinematic theory with a cosmological constant term is, according to section 4.4,

$$Z_0 = \mathcal{N} \int_{-\infty}^{\infty} d\mu \, A(\mu) \exp\left\{ i \int_{\mathbb{R}^4} d^4x \, \rho_{vac}(\mu) \right\}. \tag{5.28}$$

Here, the integration over the constant chemical potential μ provides the same vacuum energy cancellation process as in the nonkinematic theory, neglecting the influence of the dynamical part of the path integral $A(\mu)$. Including $A(\mu)$, the final vacuum energy density, provided by μ, might be shifted due to the dynamics of the three-form gauge field and, thereby, might give a small vacuum energy density.

An interesting feature of this cancellation process is that the chemical potential, and thereby the mass term of the q-field, saves the contributions to the vacuum energy density it cancels. In that way, the kinematics of the three-form gauge field includes a history of the contributions to vacuum energy density.

Concluding, the kinematic three-form gauge field includes, in principle, the same cancellation process as the nonkinematic three-form gauge field like suggested in literature [10–12]. The propagating mode of the kinematic three-form gauge field although, makes the analysis more complicated and needs further analysis.

6. Conclusion

In this master's thesis, some aspects of the three-form gauge field are clarified and some new results are obtained. The first concern was to calculate the number of local degrees of freedom of the kinematic three-form gauge field. It was already suspected that the kinematic three-form gauge field only has one propagating degree of freedom, which was confirmed in this work. The extensive analysis of the Hamiltonian systems of the nonkinematic and kinematic three-form gauge fields, necessary for the calculation of the number of local degrees of freedom, showed a subtle feature of both three-form gauge fields. There is a global degree of freedom in the form of a constant canonical variable. It is exactly this constant canonical variable, which makes the three-form gauge field provide a possible solution to the cosmological constant problem.

After the Hamiltonian analysis, we obtained the path integral formulations of the kinematic and nonkinematic three-form gauge fields. We were able to reproduce the result for the nonkinematic three-form gauge field as found in literature. It was found that the path integral formulation of the kinematic three-form gauge field has also a constant canonical variable and, thereby, may provide a solution to the cosmological constant problem. We identified the chemical potential μ of q-theory with the constant canonical variable. Thereby, the chemical potential μ is included in the quantum theory of the three-form gauge field representation of q-theory and allows for a dynamical description of vacuum energy density.

The introduction of a gauge-invariant current was used to discuss the effects of interaction of the three-form gauge field. We have found that the current shifts the chemical potential at most by a constant amount. This gives the opportunity to describe a vacuum energy density cancellation process by this current.

Concluding, we have found a quantum theory for the kinematic three-form gauge field. The interpretation of this theory is more difficult than one might expect, although the kinematic three-form gauge field only has one propagating degree of freedom since there is a global degree of freedom included in this quantum theory. The path integral formulation allows, in principle, for full quantum calculations of the kinematic three-form gauge field.

A. Appendix

A.1. The evolution of the universe due to interacting perfect fluids

A simple model [51] to describe the exchange of energy between the q-field and the photon field is obtained by modifying the energy-momentum conservation in a Friedmann-Robertson-Walker universe, so that the combined energy density of two perfect fluids $\rho(t) = \rho_1(t) + \rho_2(t)$ is conserved and the perfect fluids can decay into each other with rates α and β. The expansion of the universe, described by the Hubble parameter $H(t)$, is, in this case, determined by the equations

$$3H^2 = 8\pi G_N(\rho_1 + \rho_2)\,, \tag{A.1a}$$

$$\dot{\rho_1} + 3H\Gamma\rho_1 = -\beta H\rho_1 + \alpha H\rho_2, \tag{A.1b}$$

$$\dot{\rho_2} + 3H\gamma\rho_2 = -\alpha H\rho_2 + \beta H\rho_1, \tag{A.1c}$$

where the perfect fluids ρ_1 and ρ_2 have the equation of state parameter $w_1 = 1 - \Gamma$ and $w_2 = 1 - \gamma$ respectively.
Combining equations (A.1a), (A.1b) and (A.1c), the Hubble parameter H is determined by a single equation

$$\ddot{H} + H\dot{H}(3\gamma + 3\Gamma + \alpha + \beta) + \frac{3}{2}H^3(3\Gamma\gamma + \beta\gamma + \alpha\Gamma) = 0. \tag{A.2}$$

In our case, we want to describe the decay of vacuum energy density into photons, so that equation (A.2) is with $\rho_1 = \rho_V$, $\rho_2 = \rho_R$, $\gamma = 4/3$, $\Gamma = 0$, $\alpha = 0$ and $\beta = \Gamma_{q\gamma\gamma}/H$

$$\ddot{H} + 4H\dot{H} + \dot{H}\Gamma_{q\gamma\gamma} + 2H^2\Gamma_{q\gamma\gamma} = 0. \tag{A.3}$$

By construction the resulting spacetime of equation (A.2) is Minkowski spacetime since vacuum energy $\neq 0$ is radiating. To obtain numerical solutions to equation (A.3), we rescale the parameters of the theory with respect to the energy scale of q-theory E_q. Parameters h, $\gamma_{q\gamma\gamma}$ and τ are dimensionless equivalents of H, $\Gamma_{q\gamma\gamma}$ and t, rescaled by appropriate powers of E_q. The prime stands for differentiation with respect to τ. The equation to solve numerically is

$$h''(\tau) + 4h(\tau)\,h'(\tau) + h'(\tau)\,\gamma_{q\gamma\gamma} + 2h(\tau)^2\,\gamma_{q\gamma\gamma} = 0. \tag{A.4}$$

To obtain the energy densities, the Friedmann equations are used. They are rescaled with respect to Planck energy E_{Pl}. The dimensionless densities r_v and r_r are given by

$$r_v = \left(\frac{E_q}{E_{\mathrm{Pl}}}\right)^2\left(\frac{3h^2}{8\pi} + \frac{2h'}{8\pi\gamma}\right), \tag{A.5a}$$

$$r_r = -\left(\frac{E_q}{E_{\mathrm{Pl}}}\right)^2 \frac{2h'}{8\pi\gamma}. \tag{A.5b}$$

The solution of equation (A.4) can be seen in figure 5.1.

A.2. Variations on a constrained phasespace

Here, we want to show that the Hamiltonian equations of motion in constrained phasespace are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + u_m \frac{\partial \phi_m}{\partial p_i}, \tag{A.6a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + u_m \frac{\partial \phi_m}{\partial q_i}. \tag{A.6b}$$

We follow in this calculation textbook [28]. As in the unconstrained case in section 3.2, we define the Hamiltonian by

$$H \equiv p_i \dot{q}_i - L(q_i, \dot{q}_i)\,. \tag{A.7}$$

Setting the variation of the Hamiltonian to zero we get

$$\left[\frac{\partial H}{\partial q_i} + \frac{\partial L}{\partial q_i}\right] \delta q_i + \left[\frac{\partial H}{\partial p_i} - \dot{q}_i\right] \delta p_i = 0. \tag{A.8}$$

In the unconstrained case, the variations δp_i and δq_i are independent of each other and thereby we find the familiar Hamiltonian equations of motion (3.9). In the constrained case, there are the constraints

$$\phi_m(q_i, p_i) = 0, \quad m = 1, ..., M, \tag{A.9}$$

which define a constrained surface Γ of dimension $2N - M$ on the phasespace, where N is the number of generalized coordinates. On this surface, we consider the equation

$$\lambda_n \delta q_n + \mu_n p_n = 0. \tag{A.10}$$

The solution of the above equation is given by

$$\lambda_n = u_m \frac{\partial \phi_m}{\partial q_n}, \tag{A.11a}$$

$$\mu_n = u_m \frac{\partial \phi_m}{\partial p_n}, \tag{A.11b}$$

if the gradients of the constraints are linearly independent. The linear independence of the gradients of the constraints is a regularity condition, which stems from the fact that the surface Γ, defined by the constraints, can be defined in different ways. For more information, see reference [28, Section 1.1.2.].

From equations (A.11) and equation (A.8), the Hamiltonian equations of motion (A.6) are obtained. Equations (A.6) can also be seen as the most general equations of motion in a constrained phasespace [32] since the Hamiltonian on the surface Γ is invariant under adding constraints ϕ_m

$$H \overset{\Gamma}{=} H + u_m \phi_m, \tag{A.12}$$

where the equality only holds on the surface Γ defined by the constraints. From this more general Hamiltonian the equations of motion (A.6) are obtained.

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