Karlsruhe Institute of Technology

Skyrmion spacetime defect, degenerate metric, and negative gravitational mass

Master's Thesis of

Martin Günther

at the Department of Physics
Institute for Theoretical Physics

Reviewer: Prof. Dr. F. R. Klinkhamer
Second reviewer: Prof. Dr. C. Adam

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I declare that I have developed and written the enclosed thesis completely by myself, and have not used sources or means without declaration in the text.

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(Martin Günther)

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(Prof. Dr. F. R. Klinkhamer)

Contents

1 Introduction

At scales currently accessible to experiments, i.e. from the scale of the visible universe to the TeV-scale probed at the LHC, spacetime seems to have a simply connected topology and a rather weak curvature. But in a (hypothetical) theory of quantum gravitation, one would expect large quantum fluctuations of the metric on smaller scales, e.g. the Planck scale. According to Wheeler [1, 2], the topology of space could also fluctuate wildly, resulting in a foam-like structure. It is an open question whether or not this is really the case. In order to get quantitative bounds, a simple classical model of such a spacetime foam was developed in [3]. This model consists of many localized topological defects embedded in Minkowski spacetime.

Since the topology of one type of topological defects is compatible with the matrix Lie group SO(3), it was suggested [4] that it could be stabilized by an SO(3) Skyrme field [5]. The spacetime obtained in Ref. [4] has a delta-function type curvature singularity; the existence of such a curvature singularity is implied by several topological censorship theorems [6, 7]. In Ref. [8], it was noticed that a special *Ansatz* with a degenerate metric allows a singularity-free vacuum solution to the Einstein equations on the same[1] topologically nontrivial spacetime. In this thesis, the term "degenerate" denotes a metric tensor $g_{\mu\nu}$ that has no inverse $g^{\mu\nu}$ at some points. Both ideas were brought together in Refs. [9, 10], and nonsingular solutions for an SO(3) Skyrme field on this spacetime were found.

In this thesis, further consequences of a degenerate metric are discussed and the solution space of the model is investigated.

[1] To be precise, the spacetimes of Refs. [4] and [9, 10] are homeomorphic but not diffeomorphic, i.e. they have the same topology but unequal differential manifold structure, cf. Ref. [11] and subsection 2.1.2.

In chapter 2, the main steps of the theory are repeated, the spacetime M_4 and the metric are defined, and an introduction into the $\mathrm{SO}(3)$ Skyrme model is given. Since the term "mass" is very subtle in general relativity (GR), different conceptions and definitions of mass are reviewed: For example, active gravitational mass determines how strong an object attracts (for positive mass) resp. repels (for negative mass) a test particle.

In standard GR (i.e. with a non-degenerate metric), the model would satisfy the premises for the positive mass theorems [12–15], which show that the active gravitational mass is positive, and the topological censorship theorem [7], which would imply the presence of an event horizon or a singularity. The singularity-free solutions in Refs. [8–10] show that the latter theorem can be circumvented by a degenerate metric.

Although it seems like a small change to general relativity(GR), a degenerate metric indeed violates one of the axioms of GR, namely the equivalence principle. Since the determinant $g(x)$ of the metric is a scalar density, the statement $g(x) = 0$ is coordinate invariant. Therefore it is impossible to find a local inertial coordinate system in which the metric takes the Minkowski form $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ at such a point.

In Chapter 3, further consequences of a degenerate metric are investigated. Multiplying with the inverse metric is equivalent to solving a linear system of equations. If the inverse metric does not exist, such systems are underdetermined, which causes ambiguities in some definitions. Furthermore, some elements of the inverse metric tend to infinity in the neighborhood of a point where the metric is degenerate. This causes divergences of other quantities, e.g. covariant derivatives and the variation (functional derivative) of the Action. Therefore, the Action principle does not imply the Einstein equations at such points.

These results show that the definition of the theory needs to be extended. In Refs. [8–10], a continuous extension of the field equations was implicitly assumed. But there are other possibilities, e.g. an extension by *pseudoinverses* (see section 3.3). For both extensions, the extended field equations are satisfied automatically if the standard Einstein and Euler-Lagrange equations are satisfied at all non-degenerate points.

In chapter 4, numerical solutions are calculated. The numerical methods are optimized versions of the ones used in Refs. [4, 10] and allow a parameter scan of the solution space. The results show, among other things, that there are solutions with negative mass. This means that the positive mass theorem is also circumvented by the degenerate metric.

In chapter 5, some physical implications of negative masses are discussed and chapter 6 provides an outlook to possible future research.

2 SO(3) Skyrmion spacetime defect

2.1 Structure of the spacetime defect

2.1.1 Topology

The 3+1 dimensional spacetime considered in this thesis has the product topology

$$M_4 = \mathbb{R} \times M_3 \ , \tag{2.1.1}$$

with a three-dimensional hypersurface M_3, which can be obtained by surgery from the Euclidean space $\mathbb{R}^3$. An open ball with radius b around the origin is removed, and antipodal points of the boundary of this ball are identified, using the quotient topology:

$$\begin{aligned} M_3 &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, x_1^2 + x_2^2 + x_3^2 \geq b^2 \right\} \Big/ \sim \\ (x_1, x_2, x_3) &\sim (-x_1, -x_2, -x_3) \quad \text{for} \quad x_1^2 + x_2^2 + x_3^2 = b^2 \ . \end{aligned} \tag{2.1.2}$$

As topological spaces, M_3 and M_4 are Hausdorff and nonsimply connected [4]. The one-point-compactification $M_3 \cup \{\infty\}$ is homeomorphic to the three-dimensional projective space $\mathbb{RP}^3$ and the matrix Lie group $\mathrm{SO}(3)$. In subsection 2.1.3, another way of constructing this topology from a wormhole spacetime is discussed.

It is important to note that there is no *interior* of this topological defect since the points inside of the removed ball are indeed non-existent. Topologically—before defining a metric on the manifold—a point on the "defect surface" cannot even be distinguished from any other point of the manifold. After defining a metric, there will be noncontractible loops of minimal length πb, and b will be called the *defect size*.

2.1.2 Coordinate charts and metric

Since M_4 is nonsimply connected, an *atlas* consisting of multiple overlapping coordinate charts is needed to describe the manifold structure. These charts are homeomorphisms between subsets of M_4 and $\mathbb{R}^4$ (see e.g. [16]).

From the definition (2.1.2), every point $p \in M_3$ can be described by a tuple of cartesian coordinates $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$; but some points correspond to two different sets of coordinates (namely $\vec{x}$ and $-\vec{x}$ for $|\vec{x}|^2 = x_1^2 + x_2^2 + x_3^2 = b^2$). Nevertheless, if the coordinates are restricted to the region $|\vec{x}|^2 > b^2$, we obtain a valid coordinate chart. Similarly, we can define a spherical coordinate system with coordinates $(r, \theta_\mathrm{S}, \phi_\mathrm{S})$ on M_3, with the restriction $r > b$. By adding a time coordinate $T \in \mathbb{R}$, we get a coordinate chart of M_4.

The cartesian and spherical coordinate systems are suited for investigating the asymptotic behavior at spacelike infinity, $|\vec{x}| = r \to \infty$, but they explicitly exclude points with $r = b$. This problem was solved in Refs. [4] and [17], by introducing modified spherical coordinates[1] (T, Y, θ, ϕ). Each chart region surrounds one of the Cartesian coordinate axes on both sides of the cut-out ball but does not intersect the plane of the other two axes. For example, the chart surrounding the x_2-axis is given by

$$\begin{pmatrix} Y \\ \theta \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathrm{sign}(Y)\, r(Y) \cdot \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} ,$$
$$Y \in (-\infty, \infty) \,, \quad \theta \in (0, \pi) \,, \quad \phi \in (0, \pi) \,, \tag{2.1.3}$$

where $\mathrm{sign}(Y)$ denotes the signum function, $\mathrm{sign}(Y) = 1$ for $Y \geq 0$ and $\mathrm{sign}(Y) = -1$ for $Y < 0$. Because of definition (2.1.3) and $r(0) = b$, this map is indeed well-defined and continuous. The quasi-radial coordinate Y is zero at the defect core $r = b$, positive on one side of the x_1-x_3 plane and negative on the other side of the plane. The other two charts can be obtained by cyclic permutation of x_1, x_2, and x_3 in equation (2.1.3).

[1] The notation differs from Refs. [4, 17]. In this thesis, the coordinates of all three charts are called (T, Y, θ, ϕ). Since the metric takes the same form in all three charts, they are not distinguished here.

The general static, spherically symmetric, asymptotically flat *Ansatz* for the metric on such a coordinate chart is given by the line element

$$\begin{aligned} \mathrm{d}s^2 &= g_{TT}(Y)\,\mathrm{d}T^2 + g_{YY}(Y)\,\mathrm{d}Y^2 + r^2(Y)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\right), \\ &\text{with } g_{TT}(\pm\infty) = g_{YY}(\pm\infty) = 1\ . \end{aligned} \tag{2.1.4}$$

Away from the defect core, the function $r(Y)$ is defined like the Schwarzschild radial coordinate: The locus of constant $r > b$ is a sphere with surface area $4\pi r^2$.

In Ref. [4], the relation $r(Y) = b + |Y|$ leads to a curvature singularity, $R \propto \delta(Y)$, since the metric is not continuously differentiable. The existence of a singularity is not an artifact of the *Ansatz*, but follows from the Friedman-Schleich-Witt topological censorship theorem (see next subsection 2.3.2).

In Refs. [10, 17], the *Ansatz* for the metric is

$$\begin{aligned} \mathrm{d}s^2 &= -\mu^2\big(W(Y)\big)\,\mathrm{d}T^2 + \sigma^2\big(W(Y)\big)\left(1 - \frac{b^2}{W(Y)}\right)\mathrm{d}Y^2 + W(Y)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\right) \\ &\text{with } r^2(Y) = W(Y) = Y^2 + b^2\ . \end{aligned} \tag{2.1.5}$$

This metric is continuously differentiable, but since $1 - b^2/W(Y) = 0$ for $Y = 0$, the metric tensor is *degenerate*, i.e. $\det(g) = 0$. This also implies that there is no diffeomorphic transformation to a local Minkowski coordinate system—the equivalence principle is violated [8, 11]. With this non-standard *Ansatz*, some singularity theorems are circumvented. For example, geodesic incompleteness no longer implies the existence of singularities, since there are geodesics that cannot be extended uniquely, but in multiple different ways (see subsection 3.1.5). We will also see that the positive-mass theorem can be violated.

Mathematically, the charts with $r(Y) = b + |Y|$ resp. $r(Y) = \sqrt{b^2 + Y^2}$ define two unequal *differential structures* that are not diffeomorphic to each other, although the (topological) manifold M_4 itself is the same. Therefore, some tensors, like $g_{\mu\nu}$, are differentiable in one of the structures, but not in the other. It was shown in Ref. [11] that this fact also effects differences in the physics on these manifolds.

2.1.3 Relationship to wormholes

The *Ansatz* (2.1.4) closely resembles that of a symmetric Morris-Thorne wormhole [18] connecting two asymptotically flat spacetimes. The line element for these wormholes (again, only one of the coordinate charts is used) can be cast into the form

$$\begin{aligned} &\mathrm{d}s^2 = g_{TT}(Y)\,\mathrm{d}T^2 + g_{YY}(Y)\,\mathrm{d}Y^2 + r^2(Y)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\right) \\ &Y \in (-\infty, \infty), \quad \theta \in (0, \pi), \quad \phi \in (0, 2\pi) \quad \text{with } r(Y) = r(-Y)\ . \end{aligned} \tag{2.1.6}$$

The wormhole has an involutory symmetry $(Y, \theta, \phi) \leftrightarrow (-Y, \pi - \theta, \phi \pm \pi)$ without fixed points and the defect manifold M_4 can be seen as the quotient manifold of the wormhole with respect to that symmetry. In other words, a double covering of the defect manifold is created by mapping each pair of points (T, Y, θ, ϕ) and $(T, -Y, \pi - \theta, \phi \pm \pi)$ of the wormhole to only one point of the defect manifold.

The two asymptotically flat regions of the wormhole manifold are thereby mapped to the one asymptotically flat region of the defect manifold; the wormhole *throat* at $Y = 0$ is mapped to the defect surface $Y = 0$.

The hedgehog *Ansatz* (2.2.8) in the next section can also be extended to the wormhole manifold in a manner compatible with the symmetry

$$\Omega(Y, \theta, \phi) = \Omega(-Y, \pi - \theta, \phi \pm \pi)\ . \tag{2.1.7}$$

This covering map can therefore be used to transfer results from wormhole research to the topological defect. For example, it has also been suggested that wormholes could be stabilized by matter fields with nontrivial topology [19].

2.2 SO(3) Skyrme model

2.2.1 Action, field equations and symmetries

Because the one-point compactification of the spacelike hypersurface M_3 has the same topology as SO(3), Schwarz [4] suggested to use general relativity with an

SO(3)-valued scalar matter field $\Omega(x)$ from a modified Skyrme model:

$$S[g,\Omega] = \int \left(\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{matter}}\right) \sqrt{-g}\, \mathrm{d}^4x \ , \tag{2.2.1a}$$

$$\mathcal{L}_{\text{grav}} = \frac{1}{16\pi G_N} R \ , \tag{2.2.1b}$$

$$\mathcal{L}_{\text{matter}} = \frac{f^2}{4}\, \mathrm{tr}(\omega_\mu \omega^\mu) + \frac{1}{16e^2}\, \mathrm{tr}([\omega_\mu, \omega_\nu][\omega^\mu, \omega^\nu]) \ , \tag{2.2.1c}$$

$$\omega_\mu(x) = \Omega^{-1}(x)\, \frac{\partial \Omega(x)}{\partial x_\mu} \in \mathrm{so}(3) \ , \tag{2.2.1d}$$

where R is the Ricci scalar, G_N is Newton's gravitational constant, and e and f are parameters of the Skyrme model. Here, so(3) denotes the Lie algebra of SO(3), which consists of real, antisymmetric matrices and is generated by

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \ , \quad S_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \quad S_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{2.2.2}$$

The original Skyrme model [5] consists of an SU(2)-valued scalar field and (up to numerical factors) the same Lagrange density $\mathcal{L}_{\text{matter}}$ in Minkowski space. It was developed to describe interacting pions and has topological soliton solutions, called *Skyrmions*, that can be interpreted as baryons. Although not involving quarks or gauge fields, the Skyrme model can be regarded as an effective theory of QCD in the limit $N_c \to \infty$, where N_c is the number of quark colors [20].

The Euler-Lagrange equation for the field Ω has the form of a conserved-current equation:

$$\partial_\mu V^\mu = 0 \ , \quad V^\mu = -\frac{f^2}{2}\sqrt{-g}\, g^{\mu\nu}\, \omega_\nu + \frac{1}{4e^2}\sqrt{-g}\, g^{\rho\sigma}\, g^{\mu\nu}\, [\omega_\rho, [\omega_\sigma, \omega_\nu]] \ . \tag{2.2.3}$$

Since SO(2) is a double cover of SO(3), both groups have the same Lie algebra and both models have the same *local* behavior. In the next subsection, however, we will see that the global behavior is governed by the topology and therefore different in both models.

The SO(3) Skyrme action has a global SO(3) × SO(3) symmetry

$$\Omega(x) \mapsto O_1\,\Omega(x)\,O_2^{-1}\ , \quad O_1, O_2 \in \mathrm{SO}(3)\ . \tag{2.2.4}$$

This symmetry is spontaneously broken to SO(3) by the choice of a boundary condition $\Omega(x) \to \mathbb{1}_3$ for $|\vec{x}| \to \infty$, in order to give a finite energy.

2.2.2 Homotopy classes and topological charge

Because of the boundary condition $\Omega(\infty) = \lim_{r\to\infty} \Omega(x) = \mathbb{1}$, every finite-energy field Ω can be extended to a continuous map on the one-point compactification of M_3,

$$\Omega(x) : M_3 \cup \{\infty\} \simeq \mathrm{SO}(3) \to \mathrm{SO}(3)\ . \tag{2.2.5}$$

The interesting feature of such a model is the topological structure of the configuration space. Two field configurations are in the same *homotopy class* if one can be "continuously deformed" into the other. Therefore, a continuous physical process with finite energy cannot change the homotopy class of a configuration. In particular, a nontrivial field configuration cannot simply decay into the vacuum solution $\Omega(x) \equiv \mathbb{1}$. However, such a topological conserved quantity does not guarantee the existence of stable solutions (see Ref. [21] for an introduction into topological solitons and the Skyrme model).

In the Skyrme model, the homotopy classes can be described by the *topological degree*, *generalized winding number*, or *topological charge*

$$B(T) = \int B^0(T, \vec{x})\, \mathrm{d}^3\vec{x}\ , \tag{2.2.6a}$$

$$B^\mu = \frac{1}{96\pi^2} \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}(\omega_\nu \omega_\rho \omega_\sigma)\ . \tag{2.2.6b}$$

The normalisation factor $1/96\pi^2$ depends on the group and differs from the corresponding definition in SU(2). For topological reasons, $B(T)$ is always an integer and uniquely determines the homotopy class of Ω. Because $B(T)$ cannot continuously change to another integer, it is a (topological) conserved quantity. The charge density B^μ is also locally conserved, i.e. $\partial_\mu B^\mu = 0$, regardless of the field equations.

The reason why we use an SO(3) field instead of an SU(2) field can be explained by using the double cover homomorphism $p : \mathrm{SU}(2) \to \mathrm{SO}(3)$ in the following diagram:

$$\begin{array}{ccc} & & \mathrm{SU}(2) \\ & \overset{\widetilde{\Omega}}{\nearrow} & \big\downarrow p \\ M_3 \cup \{\infty\} \simeq \mathrm{SO}(3) & \xrightarrow{\ \Omega\ } & \mathrm{SO}(3) \end{array}$$

For every SU(2)-valued field $\widetilde{\Omega}$ with arbitrary topological charge $\tilde{B}$, there is a corresponding SO(3)-valued field $\Omega = p \circ \widetilde{\Omega}$ with even topological charge $B = 2\tilde{B}$. If the space M_3 were simply connected, the converse would also be true by virtue of the *lifting property of covering maps*: Every SO(3)-field on a topologically trivial spacetime has even topological charge and the phenomenology can be mimicked by an SU(2) field. But since the defect space M_3 is nonsimply connected, this statement does not hold: Finite-energy SO(3) fields with odd topological charge are possible on M_3 (for $b > 0$), but not on $\mathbb{R}^3$.

Since M_3 is homeomorphic to $\mathbb{R}^3$ for a vanishing defect size $b = 0$, an SO(3) field with odd topological charge could possibly stabilize the spacetime defect against gravitational collapse.

2.2.3 Hedgehog Ansatz

Since both the spacetime and the Skyrme model have an SO(3) symmetry, we make a static *Ansatz* with the symmetry

$$\Omega(Ox) = O\,\Omega(x)\,O^{-1} \text{ for all } O \in \mathrm{SO}(3)\ , \tag{2.2.7}$$

in which a rotation in space is compensated by a rotation in internal space. The general *Ansatz* with this symmetry is the so-called *hedgehog Ansatz*

$$\begin{aligned} \Omega(x) &= \exp\left(-F\left(r^2\right)\hat{x}\cdot\vec{S}\right) \\ &= \cos\left(F(r^2)\right)\mathbb{1} + \left(1-\cos\left(F(r^2)\right)\right)\hat{x}\hat{x}^T - \sin\left(F(r^2)\right)\hat{x}\cdot\vec{S}\ , \end{aligned} \tag{2.2.8}$$

where $\hat{x} = \vec{x}/|\vec{x}|$ is the outward radial unit vector. $\Omega(x)$ is a rotation matrix about the $\hat{x}$-axis through the angle $-F(r^2)$.

For $\Omega(x)$ to be well-defined and continuous, it must be equal at $\vec{x}$ and $-\vec{x}$ for $r = b$. This condition is fulfilled if and only if $F(b^2)$ is a multiple of π. Without loss of generality, we use the boundary conditions

$$F(b^2) = B\,\pi \ , \quad F(\infty) = 0 \ . \tag{2.2.9}$$

The integer B is indeed the topological charge of the hedgehog field [4].

For SU(2) Skyrmions, it is believed [21] that the $B = \pm 1$ Skyrmions with minimal energy are indeed spherically symmetric, whereas the minimum-energy solutions for $|B| > 1$ are known to be less symmetric. For example, $B = \pm 2$ Skyrmions have an axial symmetry and Skyrmions for $|B| > 2$ have discrete symmetry groups.

Therefore, we only consider $B = 1$ hedgehog fields in this thesis. The argument in subsection 2.2.2 shows that such a field could stabilize the spacetime defect.

2.3 Mass and singularity theorems

2.3.1 Definitions of mass

2.3.1.1 Active, passive and inertial mass

To be precise when talking about “mass”, we distinguish three different conceptions: Inertial mass, active gravitational mass, and passive gravitational mass (cf. [22]).

In terms of Newtonian mechanics, the inertial mass of an object is the proportionality factor in Newton’s second law or in the definition of momentum,

$$F = m_{\text{inertial}}\, a \ , \quad p = m_{\text{inertial}}\, v \ . \tag{2.3.1}$$

If the mass is not time-dependent, both descriptions are equivalent, since forces are time-derivatives of momenta. Inertial mass is sometimes described as an object’s *resistance to acceleration*.

To see the difference between active and passive gravitational mass, we rewrite Newton's law of gravity in the following way:

$$|F_1| = G_\mathrm{N} \frac{m_{1,\mathrm{pass}}\, m_{2,\mathrm{act}}}{r^2} , \qquad |F_2| = G_\mathrm{N} \frac{m_{2,\mathrm{pass}}\, m_{1,\mathrm{act}}}{r^2} , \tag{2.3.2}$$

where F_1 resp. F_2 is the force acting on the first resp. second body. In this sense, the passive mass measures *susceptibility to gravitational forces* and the active mass describes, in modern terms, the *strength of the gravitational field* emitted by the body.

Experimentally, all three kinds of mass are proportional to each other, with material-independent proportionality constants. This allows us to choose a system of units in which they are equal to each other. The proportionalities correspond to the following physical principles:

1. Newton's third law $F_1 = -F_2$ or, equivalently, momentum conservation implies that active and passive gravitational masses are proportional to each other: $m_\mathrm{act} \propto m_\mathrm{pass}$.

2. The *weak equivalence principle* or *universality of free fall* states that all bodies in a given gravitational field accelerate in the same way if friction and other forces are neglected. This is equivalent to the proportionality of passive gravitational mass and inertial mass: $m_\mathrm{pass} \propto m_\mathrm{inertial}$.

In general relativity, the *strong equivalence principle*[2] implies the weak equivalence principle for small test masses, if the gravitation of the test mass itself is neglected.

Nevertheless, the definition of masses in GR is much more subtle than in Newtonian mechanics. It makes no sense to talk about the position or distance between two massive objects "at a certain time". Even the concept of forces is problematic since gravitational acceleration is coordinate dependent and multiple sources of gravity cannot be superposed since GR is a *nonlinear* theory.

[2] i.e. "the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system" [23]

2.3.1.2 Active gravitational mass in general relativity

Newtonian gravity can be obtained as a non-relativistic weak-field limit of GR. Therefore, we can try to determine the active gravitational mass of an object by making measurements far away from it. In an asymptotically flat spacetime, there are at least three common definitions of mass, namely the Komar [24], ADM [25], and Bondi mass [26].

The Komar mass is defined only for stationary spacetimes. Since there is a continuous time-translation symmetry, the Noether theorem gives a corresponding conserved energy—the Komar energy or mass. The ADM formalism is a Hamiltonian formulation of GR. The integral of the Hamilton density over a spacelike hypersurface is called the ADM energy or mass. Both of these volume integrals can also be expressed as a surface integral, e.g. over a sphere with radius $R \to \infty$. For this reason, the Komar and ADM masses only depend on the asymptotic fields. In an asymptotically flat spacetime, the ADM mass is a conserved quantity and for static spacetimes, ADM and Komar mass are equivalent.

The Bondi mass is also defined as a volume or surface integral, but on hypersurfaces that are asymptotically lightlike and approach a fixed *asymptotic retarded time*. Unlike the ADM mass, the Bondi mass is therefore time-dependent. The Bondi mass measures the *remaining* energy at that time, excluding energy that is carried away to infinity e.g. by gravitational waves that never cross that hypersurface. The ADM energy includes such waves and measures the *total* energy contained in the spacetime.

The Schön-Yau positive mass theorem [12] states that the ADM mass is non-negative, assuming the dominant energy condition. The same statement is also true for the Bondi mass [13–15]. Energy conditions are used in theorems about GR to make sure that the matter is “not exotic”. For example, the weak energy condition states that an observer with velocity v^μ measures a non-negative energy density:

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \text{ for all timelike vectors } v^\mu \ . \tag{2.3.3}$$

The dominant energy condition additionally demands that the measured energy-momentum flux is not faster than light:

$$T^{\mu}_{\nu} v^{\nu} \text{ is not spacelike for all timelike vectors } v^{\mu} \ . \tag{2.3.4}$$

The Skyrme model considered in this thesis satisfies both of these energy conditions, but the theorems are not valid for a degenerate metric. Chapter 4 shows that there are indeed solutions with negative mass.

For the spacetime defined in section 2.1, it is instructive to analyze the vacuum solution

$$\mu^2(r^2) = 1 - \frac{r_S}{r} \ , \quad \sigma^2(r^2) = \frac{r}{r - r_S} \ . \tag{2.3.5}$$

Outside of the topological defect, the metric is exactly that of the Schwarzschild spacetime with Schwarzschild radius $r_S = 2G_N\, m_S$. In a non-vacuum solution, the function σ can still be written as

$$\sigma^2(r^2) = \frac{r}{r - 2\,G_N\, m_S(r)} \ , \tag{2.3.6}$$

with an r-dependent mass function $m_S(r)$. Intuitively, the mass function measures the mass contained within a sphere of surface $4\pi r^2$ around the spacetime defect. It can be shown that $m_S(r)$ increases monotonically with r. Since the spacetime is asymptotically flat, the mass function converges against a finite *asymptotic Schwarzschild mass* $m_S(\infty)$. This definition of mass is equivalent to the Komar, ADM, and Bondi mass.

It is important to note that the asymptotic Schwarzschild mass provides a coordinate-independent definition of mass, even though mass function $m_S(r)$ is coordinate dependent. In general, gravitational energy cannot be defined in a coordinate independent way, since gravitational effects at a given point can be made to vanish in an appropriate local inertial system.

2.3.2 Topological censorship theorem

There are several restrictive theorems about nontrivial topology in general relativity, e.g. Gannon's singularity theorem [6] and the topological censorship theorem [7]. Since the validity of these theorems in the spacetime M_4 was already checked in Ref. [4], only a brief introduction is given here.

Nevertheless, we first need to define some technical terms. A spacetime is *globally hyperbolic* if it has a global Cauchy surface, i.e. a spacelike hypersurface that is intersected by every inextensible causal curve exactly once. In the spacetime defined in section 2.1, every surface of constant T is a global Cauchy surface. Such hypersurfaces are important in the proofs since every solution to the Einstein equations is uniquely determined by initial conditions on the Cauchy surface. *Geodesic completeness* means that every geodesic can be extended in both directions indefinitely. Reciprocally, if a spacetime is geodesically incomplete, it must intuitively have either a boundary or a hole which can be reached in a finite proper time.

Gannon's theorem shows geodesic incompleteness for a spacetime which is globally hyperbolic, has a nonsimply connected spacelike hypersurface, and satisfies the weak energy condition. Simply speaking, it states that a spacetime with nontrivial topology has a singularity somewhere.

The statement of the topological censorship theorem is stronger. It presupposes global hyperbolicity, asymptotic flatness and the averaged null energy condition. The statement of the theorem can be expressed as follows: An observer cannot probe nontrivial spacetime topology by sending out test particles. Such test particles either stay in his topologically trivial neighborhood or they cannot return to him, either because they pass through an event horizon or hit a singularity. In the spacetime of Ref. [4], the latter is the case, because there is a curvature singularity at $Y = 0$.

The degenerate metric (2.1.5) circumvents this theorem because it is smooth everywhere and the test particle can simply move over this surface. Gannon's theorem is circumvented by the fact that there are some geodesics that are ambiguous (see subsection 3.1.5), i.e. they cannot be extended *uniquely*.

3 Extensions of general relativity for a degenerate metric

In the *Ansatz* (2.1.5), the metric $g_{\mu\nu}$ is degenerate for $Y = 0$. This metric has been chosen in Ref. [10] in order to remove resp. regularize the curvature singularity that is predicted by the topological censorship theorem. The regularisation can be explained by a non-diffeomorphic coordinate transformation: The spacetimes of Refs. [4] and [10] have the same topology, but unequal differential manifold structure. In other words, the term "differentiable" has different meanings on both spacetimes.

In a sense, a curvature singularity has been traded in for a degenerate metric. But there is a caveat: Since the inverse of a degenerate metric does not exist, the model specified in chapter 2 is not directly applicable. We need to give an extended definition of the theory, and there is some freedom of choice in doing so.

In sections 3.1 and 3.2, the main problems of general relativity with a degenerate metric are analyzed: Ambiguities and divergences.

From the view of differential geometry, the metric tensor $g_{\mu\nu}$ is the fundamental quantity that describes the geometry of spacetime. It defines proper lengths of curves and therefore proper distances on the spacetime manifold. The inverse metric $g^{\mu\nu}$ only plays a secondary role; it is needed to solve equations by eliminating $g_{\mu\nu}$.

But what happens if the metric *has no inverse*? In this case, such equations are underdetermined and may have multiple solutions or no solutions at all. In section 3.1, this issue is carefully analyzed, and the usage of a pseudoinverse $\tilde{g}^{\mu\nu}$ as a replacement for the true inverse $g^{\mu\nu}$ is motivated. Some singularity theorems can be circumvented by the fact that geodesics are not unique (see subsection 3.1.5).

Another problem is posed by the divergence of the inverse metric. Since an infinitesimal change to the metric causes infinitely large changes of the inverse metric, we cannot directly apply variational calculus to the action functional (see section 3.2).

In section 3.3, two possible extended theories are laid out which are both applicable to the degenerate *Ansatz* metric (2.1.5). In the first extension, the field equations are defined at $Y = 0$ by continuous extension from their limits $Y \to 0$. In the second extension, the inverse metric is replaced by a pseudoinverse. For the given *Ansatz* metric and fields, the Lagrange density has indeed a well-defined value at $Y = 0$, as long as the boundary conditions $\mu(b^2) > 0$ and $\sigma(b^2) > 0$ hold. The same boundary conditions are also implied by the continuous extension.

In this chapter, the term "standard GR" denotes the usual formulation of general relativity with a non-degenerate metric, in contrast to the "extended theories".

3.1 Ambiguities and pseudoinverses

3.1.1 Ambiguous raising of indices

If the metric is non-degenerate, raising of indices is a well-defined procedure, because for a given covariant vector b_μ, there is exactly one contravariant vector v^μ that solves the equations

$$g_{\mu\nu}\, v^\nu = b_\mu \tag{3.1.1}$$

and we usually write this solution as $v^\mu = w^\mu := g^{\mu\nu} b_\nu$. If $g_{\mu\nu}$ is degenerate, Eq. (3.1.1) is an underdetermined, inhomogeneous linear system. In order to find its solution space, it will be convenient to interpret $g_{\mu\nu}$ as a linear map $g : v^\mu \mapsto g_{\mu\nu} v^\nu$ which maps contravariant to covariant vectors. From the definitions of the image and kernel of this map,

$$\mathrm{im}(g) = \left\{ b_\mu \,\middle|\, \exists v^\mu : b_\mu = g_{\mu\nu} v^\nu \right\} \ , \tag{3.1.2a}$$

$$\ker(g) = \left\{ k^\mu \,\middle|\, g_{\mu\nu} k^\nu = 0 \right\} \ , \tag{3.1.2b}$$

it is clear that the index on b_μ can only be raised if $b_\mu \in \mathrm{im}(g)$. Since $g_{\mu\nu}$ is a symmetric matrix, this condition is equivalent to

$$b_\mu k^\mu = 0 \quad \text{for all} \quad k^\mu \in \ker(g) \ . \tag{3.1.3}$$

If this is the case, the solution space of (3.1.1) is given by

$$v^\mu \in v^{(0)\mu} + \ker(g) \ , \tag{3.1.4}$$

with an arbitrary, but fixed particular solution $v^{(0)}$.

In the spacetime defect metric (2.1.5), the kernel is one-dimensional and contains only multiples of the Y-coordinate vector. Therefore, Eq. (3.1.3) is equivalent to $b_Y = 0$.

3.1.2 Definition of pseudoinverses

Even if the inverse metric $g^{\mu\nu}$ doesn't exist, it will be convenient to define a generalized inverse or *pseudoinverse* $\tilde{g}^{\mu\nu}$ that shares some properties with $g^{\mu\nu}$. In particular, we want $v^\nu := \tilde{g}^{\nu\rho} b_\rho$ to be a solution of Eq. (3.1.1), if the system is solvable at all. Using (3.1.2a) and (3.1.3), this condition is equivalent to

$$g_{\mu\nu}\, \tilde{g}^{\nu\rho}\, g_{\rho\sigma} = g_{\mu\sigma} \ . \tag{3.1.5}$$

It is a well-known result [27, 28] that such *pseudoinverse* matrices $\tilde{g}^{\nu\rho}$ always exist and are not uniquely determined by the equation (3.1.5). Most authors employ additional conditions to ensure uniqueness, but these are not coordinate covariant and therefore not sensible in general relativity.

Instead, we will keep in mind that $\tilde{g}^{\mu\nu}$ is ambiguous and merely a convenient notation for raising indices, which is also ambiguous. Any physical quantities must not depend on the choice of $\tilde{g}^{\mu\nu}$. Note that if the inverse metric $g^{\mu\nu}$ exists, the pseudoinverse is identical to it. Therefore, Eq. (3.1.5) indeed generalizes the notion of the inverse metric.

3.1.3 Well-definedness of traces

In general relativity, the inverse metric often appears in (partial) traces $g^{\mu\nu} B_{\mu\nu\Sigma}$ of tensors or tensor-like objects, where Σ denotes an arbitrary number of co- and contravariant indices. If the metric is degenerate, such terms are undefined. But if (and only if) the condition

$$B_{\mu\nu\Sigma}\, k^{\mu} = 0,\ B_{\mu\nu\Sigma}\, k^{\nu} = 0 \quad \text{for all} \quad k^{\mu} \in \ker(g) \tag{3.1.6}$$

holds, we can raise the two indices[1] μ and ν on the tensor B and define the partial trace in the following way:

$$\mathrm{tr}_{\mathrm{partial}}(B)_{\Sigma} := g_{\rho\sigma} A^{\rho\sigma}{}_{\Sigma} \quad \text{for} \quad g_{\mu\rho}\, g_{\nu\sigma}\, A^{\rho\sigma}{}_{\Sigma} = B_{\mu\nu\Sigma}\ . \tag{3.1.7}$$

The condition (3.1.6) ensures the existence of at least one such tensor A, and the definition does not depend on the choice of A. To see this, let $A'^{\rho\sigma}_{\ \ \Sigma} = A^{\rho\sigma}{}_{\Sigma} + \delta A^{\rho\sigma}{}_{\Sigma}$ be another solution. Equation 3.1.7 then implies

$$\begin{aligned}
& & g_{\mu\rho}\, g_{\nu\sigma}\, \delta A^{\rho\sigma}{}_{\Sigma} &= 0 \\
\Longrightarrow & & \left(g_{\mu\rho}\, \delta A^{\nu\mu}{}_{\Sigma}\right) \left(g_{\nu\sigma}\, \delta A^{\rho\sigma}{}_{\Sigma}\right) &= 0 \\
\Longrightarrow & & g_{\mu\rho}\, \delta A^{\nu\mu}{}_{\Sigma} = g_{\nu\sigma}\, \delta A^{\rho\sigma}{}_{\Sigma} &= 0 \\
\Longrightarrow & & g_{\rho\sigma} A'^{\rho\sigma}{}_{\Sigma} - g_{\rho\sigma} A^{\rho\sigma}{}_{\Sigma} = g_{\rho\sigma}\, \delta A^{\rho\sigma}{}_{\Sigma} &= 0\ ,
\end{aligned} \tag{3.1.8}$$

so A and A' have the same trace.

The trace can also be calculated with an arbitrary pseudoinverse $\tilde{g}^{\mu\nu}$:

$$\tilde{g}^{\mu\nu}\, B_{\mu\nu\Sigma} = g_{\mu\rho}\, \tilde{g}^{\mu\nu}\, g_{\nu\sigma}\, A^{\rho\sigma}{}_{\Sigma} = g_{\rho\sigma} A^{\rho\sigma}{}_{\Sigma} = \mathrm{tr}_{\mathrm{partial}}(B)_{\Sigma}\ . \tag{3.1.9}$$

This result shows that it is sensible to replace the inverse metric in terms such as $g^{\mu\nu} B_{\mu\nu\Sigma}$ by a pseudoinverse, as long as the condition (3.1.6) holds. However, there is a

[1] Raising only one index does not suffice for a well-defined trace. Consider for example the tensor $B_{\mu\nu} = 0$. We can choose $A^{\mu}{}_{\nu} = 0$ and $A'^{\mu}{}_{\nu} = k^{\mu} v_{\nu}$ with $k^{\mu} \in \ker(g)$ and $A'^{\mu}{}_{\mu} = k^{\mu} v_{\mu} \neq 0 = A^{\mu}{}_{\mu}$, despite $g_{\mu\rho} A^{\rho}{}_{\nu} = g_{\mu\rho} A'^{\rho}{}_{\nu} = B_{\mu\nu}$.

caveat: Traces of continuous tensors can fail to be continuous. For example, the trace of the metric $\tilde{g}^{\mu\nu} g_{\mu\nu} = \dim\big(\mathrm{im}(g)\big)$ has a jump discontinuity where g is degenerate.

3.1.4 Violation of the fundamental theorem of Riemannian geometry

The defining conditions of the Levi-Civita connection

$$\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu} \qquad \text{(torsion free)} \; , \tag{3.1.10a}$$

$$g_{\mu\nu;\rho} = 0 \qquad \text{(preserves metric)} \; , \tag{3.1.10b}$$

lead to the defining equation for the Christoffel symbols,

$$\Gamma^{\sigma}_{\mu\nu}\, g_{\sigma\rho} = \Gamma_{\rho\mu\nu} := \frac{1}{2}\left(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right) \; , \tag{3.1.11}$$

where a comma denotes a partial derivative and a semicolon denotes a covariant derivative. The results of subsection 3.1.1 show that $\Gamma^{\sigma}_{\mu\nu}$ exists if and only if

$$k^{\rho}\Gamma_{\rho\mu\nu} = 0 \quad \text{for all} \quad k^{\mu} \in \ker(g) \tag{3.1.12}$$

and even if this is the case, $\Gamma^{\sigma}_{\mu\nu}$ is not uniquely determined. With an arbitrary, but fixed pseudoinverse $\tilde{g}^{\mu\nu}$ satisfying Eq. (3.1.5), the general solution is

$$\Gamma^{\sigma}_{\mu\nu} = \tilde{g}^{\sigma\rho}\Gamma_{\rho\mu\nu} + C^{\sigma}_{\mu\nu} \quad \text{with} \quad g_{\rho\sigma}C^{\sigma}_{\mu\nu} = 0, \quad C^{\sigma}_{\mu\nu} = C^{\sigma}_{\nu\mu} \; . \tag{3.1.13}$$

The kernel of the *Ansatz* metric (2.1.5) is one-dimensional, so there are ten degrees of freedom left for the choice of $\Gamma^{\sigma}_{\mu\nu}$ in this case.

This result is a significant difference to Riemannian or Lorentzian geometry, in which the *fundamental theorem of Riemannian geometry* ensures existence and uniqueness of the Levi-Civita connection.

3.1.5 Ambiguous geodesics circumvent singularity theorems

A similar ambiguity appears when we derive the geodesic equation from the variational principle:

$$g_{\mu\sigma}\frac{\mathrm{d}^2\gamma(\tau)^\sigma}{\mathrm{d}\tau^2} = -\Gamma_{\mu\nu\rho}\frac{\mathrm{d}\gamma(\tau)^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}\gamma(\tau)^\rho}{\mathrm{d}\tau} \ . \tag{3.1.14}$$

If Levi-Civita-like connections exist, i.e. Eq. (3.1.12) is satisfied, the second derivative on the left-hand site is only determined up to some vector $k^\sigma \in \ker(g)$.

In standard GR, a geodesic can always be uniquely extended until it hits a singularity or the boundary of the spacetime manifold. If the metric is degenerate, there is the additional possibility that the geodesic can be extended in multiple different ways.

Therefore, *geodesic incompleteness does not imply the existence of singularities* if the metric is allowed to be degenerate. This means that the Gannon topological censorship theorem [6] (see 2.3.2) and similar theorems are circumvented in our model.

Similar ambiguities appear if we analyze the properties of minimal surfaces instead of minimal curves. This explains why the Schön-Yau positive mass theorems (see subsection 2.3.1), whose proofs rely on minimal surfaces, can be violated in such a spacetime. The numerical solutions in chapter 4 show that negative masses are indeed possible in our model.

3.2 Divergences

3.2.1 Riemann and Ricci curvature

If the metric $g_{\mu\nu}$ is two times continuously differentiable (i.e. second *partial* derivatives exist and are continuous), then the Christoffel symbols of the first kind, $\Gamma_{\mu\nu\rho}$, are continuously differentiable (see Eq. (3.1.11)), but Christoffel symbols of the second kind, $\Gamma^\mu_{\nu\rho}$, can diverge in the limit $\det(g) \to 0$. Therefore, the covariant derivative

$$v^\mu{}_{;\nu} = v^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu}\, v^\rho \tag{3.2.1}$$

of a smooth vector field v^μ is not necessarily differentiable nor continuous. But note that

$$(g_{\rho\sigma}v^\sigma)_{;\mu} = (g_{\rho\sigma}v^\sigma)_{,\mu} - \Gamma_{\sigma\rho\mu}v^\sigma \tag{3.2.2}$$

is, nevertheless, continuously differentiable.

Therefore, we can define the covariant Riemann tensor in the following way:

$$\begin{aligned}(g_{\rho\sigma}v^\sigma)_{;\mu\nu} - (g_{\rho\sigma}v^\sigma)_{;\nu\mu} &= -v^\sigma (\Gamma_{\sigma\rho\mu,\nu} - \Gamma_{\sigma\rho\nu,\mu} + g_{\alpha\beta}\Gamma^\alpha_{\rho\nu}\Gamma^\beta_{\sigma\mu} - g_{\alpha\beta}\Gamma^\alpha_{\rho\mu}\Gamma^\beta_{\sigma\nu}) \\ &= -v^\sigma \underbrace{(\Gamma_{\sigma\rho\mu,\nu} - \Gamma_{\sigma\rho\nu,\mu} + \tilde{g}^{\alpha\beta}\Gamma_{\alpha\rho\nu}\Gamma_{\beta\sigma\mu} - \tilde{g}^{\alpha\beta}\Gamma_{\alpha\rho\mu}\Gamma_{\beta\sigma\nu})}_{=:R_{\sigma\rho\mu\nu}}\end{aligned} \tag{3.2.3}$$

The last identity holds for *any* pseudoinverse $\tilde{g}^{\mu\nu}$, as long as equations (3.1.5) and (3.1.12) are satisfied.

In general, $R_{\sigma\rho\mu\nu}$ need not be continuous and $R^\lambda{}_{\rho\mu\nu}$ is ambiguous and possibly divergent. From subsection 3.1.3, we know that partial traces of $R_{\sigma\rho\mu\nu}$, like the Ricci tensor, Ricci scalar, and Kretschmann scalar, exist if $k^\sigma R_{\sigma\rho\mu\nu}$ vanishes for all $k^\mu \in \ker(g)$.

If $g_{\mu\nu}$ is non-degenerate, the equations

$$(g_{\rho\sigma}\, v^\sigma)_{;\mu\nu} = g_{\rho\sigma}\, v^\sigma{}_{;\mu\nu}\ , \tag{3.2.4a}$$

$$R_{\sigma\rho\mu\nu} = g_{\sigma\lambda}\, R^\lambda{}_{\rho\mu\nu}\ , \tag{3.2.4b}$$

are valid as in standard general relativity.

3.2.2 Einstein-Hilbert action and Einstein equations

In standard GR, the Einstein equations can be derived from the Einstein-Hilbert action together with a matter action

$$S = \int \left(\frac{1}{16\pi G_N}R + \mathcal{L}_{\mathrm{matter}}\right)\sqrt{-g}\,\mathrm{d}^4x\ . \tag{3.2.5}$$

According to the principle of least action, we are interested in the variation of this action with respect to the metric. For a non-degenerate metric $g_{\mu\nu}$ and a variation $\delta g_{\mu\nu}$, the variation of the inverse metric is given by

$$\delta g^{\mu\nu} = -g^{\mu\rho}\,\delta g_{\rho\sigma}\,g^{\sigma\nu}\ . \tag{3.2.6}$$

Hence, the variation of the matter and gravitational action is

$$\begin{aligned}\delta\left(\int \mathcal{L}_{\mathrm{matter}}\sqrt{-g}\,\mathrm{d}^4x\right) &= \frac{1}{2}\int \delta g_{\mu\nu}\,T^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x\\ &= -\frac{1}{2}\int \delta g^{\mu\nu}\,T_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x\ ,\end{aligned} \tag{3.2.7a}$$

$$\begin{aligned}\delta\left(\int R\sqrt{-g}\,\mathrm{d}^4x\right) &= -\int \delta g_{\mu\nu}\,G^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x\\ &= \int \delta g^{\mu\nu}\,G_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x\ ,\end{aligned} \tag{3.2.7b}$$

where $T^{\mu\nu}$ is the energy-momentum tensor and $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}R\,g^{\mu\nu}$ is the Einstein tensor (the derivation can be found in any standard GR textbook, e.g. [29]).

If the metric is degenerate, equation (3.2.6) implies that a generic *infinitesimal* variation of the metric leads to an *infinitely* large variation of the inverse in a neighborhood of a degenerate point. Since the variation of the action is generally a diverging integral, it cannot be expressed by tensors $T^{\mu\nu}$ resp. $G^{\mu\nu}$. Only for a special class of variations $\delta g_{\mu\nu}$, the variation of the action is finite.

Furthermore, the derivation of equation (3.2.7b) cannot be generalized to our model. The left-hand side contains a total derivative term $\sqrt{g}(g^{\sigma\nu}\,\delta\Gamma^{\rho}_{\sigma\nu} - g^{\rho\sigma}\,\delta\Gamma^{\mu}_{\mu\sigma})_{;\rho}$ which does not contribute to the integral because of Stokes' theorem. But if the metric is degenerate, the term in the brackets is divergent and the derivative does not exist.

In conclusion: If we adhere to the action principle, the Einstein equations are not valid at points where the metric is degenerate. The methods of variational calculus fail even if the set of such points has measure zero.

3.3 Possible extensions of the theory

Because of the problems stated in the last two sections, we have to extend general relativity to make it applicable to the *Ansatz* (2.1.5) and (2.2.8). A sensible extension should be invariant under local diffeomorphisms and therefore be equivalent to standard GR in regions where the metric is non-degenerate. This section shows two possible extensions and the boundary conditions arising from them.

3.3.1 Continuous extension of field equations

Since only the hyperplane $Y = 0$ causes problems, we can define the values of relevant quantities at this hyperplane by calculating their limits as $Y \to 0$. If these limits exist and are independent of the direction of approach, we obtain a *continuous extension*.

For $Y \neq 0$, the inverse $g^{\mu\nu}$ of the *Ansatz* metric (2.1.5) is

$$g^{\mu\nu} = \mathrm{diag}\left(-\frac{1}{\mu^2(W(Y))}, \frac{W(Y)}{Y^2\sigma^2(W(Y))}, \frac{1}{W(Y)}, \frac{1}{W(Y)\sin^2(\theta)}\right) . \tag{3.3.1}$$

Under the assumption $b > 0$, $\mu(b^2) \neq 0$, and $\sigma(b^2) \neq 0$, all matrix elements of $g^{\mu\nu}$ apart from the YY-element can be continuously extended to $Y = 0$.

The Euler-Lagrange equation (2.2.3) for the Skyrme field takes the form of a conserved current equation $\partial_\mu V^\mu$ of the Lie-Algebra valued current density

$$V^\mu = -\frac{f^2}{2}\sqrt{-g}\, g^{\mu\nu}\, \omega_\nu + \frac{1}{4e^2}\sqrt{-g}\, g^{\rho\sigma}\, g^{\mu\nu}\, [\omega_\rho, [\omega_\sigma, \omega_\nu]] . \tag{3.3.2}$$

A straightforward calculation shows that V^μ can be extended continuously and that the extension is continuously differentiable at $Y = 0$. By continuity, the Euler-Lagrange equation $\partial_\mu V^\mu$ is automatically satisfied if it is satisfied for $Y \neq 0$.

To understand this result, note that the factor Y in the current component

$$\omega_Y = -2YF'(W(Y))\, \hat{x} \cdot \vec{S} \tag{3.3.3}$$

cancels the divergent factor Y^{-2} in g^{YY} if the inverse metric is contracted two times with ω_μ.

In the remaining terms, ω appears as a contravariant current density

$$\sqrt{-g}\,\omega^Y = \sqrt{-g}\,g^{YY}\omega_Y = -2\frac{W(Y)\,\mu(W(Y))\,\sin(\theta)}{\sigma(W(Y))}\,\frac{|Y|}{Y}F'(W(Y))\,\hat{x}\cdot\vec{S}\ , \tag{3.3.4}$$

which is continuously differentiable since $(|Y|/Y)\,\hat{x}\cdot\vec{S} = \mathrm{sign}(Y)\,\hat{x}\cdot\vec{S}$ is continuously differentiable (see equation (2.1.3)).

By a similar argument, the covariant Energy-momentum tensor $T_{\mu\nu}$ and Einstein tensor $G_{\mu\nu}$ have a continuous extension to $Y = 0$ and the extended Einstein equations are automatically satisfied if they are satisfied for all $Y \neq 0$.

3.3.2 Pseudoinverse extension of the action

The results from section 3.1 suggest another possible extension in which we replace every occurrence of the inverse metric $g^{\mu\nu}$ by a pseudoinverse metric $\tilde{g}^{\mu\nu}$. Since pseudoinverses are ambiguous, we must check that the results do not depend on the choice of $\tilde{g}^{\mu\nu}$ (see subsection 3.1.3).

One pseudoinverse of the *Ansatz* metric is given by

$$\tilde{g}^{\mu\nu}\Big|_{Y=0} = \mathrm{diag}\left(-\frac{1}{\mu^2(W(Y))}, 0, \frac{1}{W(Y)}, \frac{1}{W(Y)\sin^2(\theta)}\right)\ . \tag{3.3.5}$$

The extended definitions of the Skyrme Lagrange density, Riemann tensor, Ricci tensor, and Ricci scalar are given by

$$\mathcal{L}_{\text{matter}} = \frac{f^2}{4}\tilde{g}^{\mu\nu}\,\mathrm{tr}(\omega_\mu\omega_\nu) + \frac{1}{16e^2}\tilde{g}^{\mu\nu}\tilde{g}^{\rho\sigma}\,\mathrm{tr}([\omega_\mu,\omega_\rho][\omega_\nu,\omega_\sigma])\ , \tag{3.3.6a}$$

$$R_{\sigma\rho\mu\nu} = \Gamma_{\sigma\rho\mu,\nu} - \Gamma_{\sigma\rho\nu,\mu} + \tilde{g}^{\alpha\beta}\Gamma_{\alpha\rho\nu}\Gamma_{\beta\sigma\mu} - \tilde{g}^{\alpha\beta}\Gamma_{\alpha\rho\mu}\Gamma_{\beta\sigma\nu}\ , \tag{3.3.6b}$$

$$R_{\mu\nu} = \tilde{g}^{\rho\sigma}R_{\mu\rho\nu\sigma}\ , \tag{3.3.6c}$$

$$R = \tilde{g}^{\mu\nu}R_{\mu\nu}\ . \tag{3.3.6d}$$

and these definitions are well-defined if

$$k^\rho \Gamma_{\rho\mu\nu} = 0 \ , \tag{3.3.7}$$

$$k^\rho R_{\rho\sigma\mu\nu} = 0 \ , \tag{3.3.8}$$

$$k^\mu \omega_\mu = 0 \ , \tag{3.3.9}$$

$$\text{for all} \quad k^\mu \in \ker(g) \ . \tag{3.3.10}$$

These conditions can be checked by direct calculation and they are indeed satisfied by the *Ansatz* (2.1.5) and (2.2.8) if $b > 0$, $\mu(b^2) \neq 0$, and $\sigma(b^2) \neq 0$.

Since $\mathcal{L}_{\text{matter}}$ and $\mathcal{L}_{\text{grav}}$ have well-defined, finite values, the integrand in the action vanishes for $Y = 0$,

$$\sqrt{-g}(\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{grav}})\Big|_{Y=0} = 0 \ . \tag{3.3.11}$$

The same result can also be obtained by continuous extension of the Lagrange density.

For $Y \neq 0$, the Lagrange density remains unchanged and the Einstein equations are valid in this region. If the Einstein equations are fulfilled for all $Y \neq 0$, the action is stationary under variations that do not change the *locus* of $g = 0$. This approach corresponds to minimizing the action under the constraint of a constant defect size b.

4 Numerical methods and results

In chapter 2, a static, spherically symmetric *Ansatz* for the Skyrmion spacetime defect have been made. The metric and the matter is described by three *Ansatz* functions $F(w), \sigma(w), \mu(w)$ in one variable $w \in \left[y_0^2, \infty\right)$. In chapter 3, two extensions of the theory were discussed, both of which resulted in the additional boundary conditions $\mu(b^2) \neq 0$, and $\sigma(b^2) \neq 0$.

This chapter describes the numerical solutions for this model and the methods used to calculate them. A discussion of the physical ramifications, especially of the negative-mass solutions, follows in chapter 5.

In section 4.1, prerequisites for numerical solutions are discussed. The Einstein and Euler-Lagrange equations reduce to a system of three ordinary differential equations (ODEs) with boundary conditions at $w = y_0^2$ and $w \to \infty$. A weak-field approximation allows an analysis of the asymptotic behavior (i.e. far away from the topological defect) and provides a way to implement the boundary conditions at infinity, while still having a finite domain for the numerical integration.

In section 4.2, two numerical methods are defined. The first method is an optimized version of the methods used in Refs. [4, 10], which uses the weak-field approximation to implement the boundary conditions and adaptively chooses the upper bound for the numerical integration. As a so-called *shooting method*, it needs multiple iterations until all boundary conditions are met. The second method is a further improvement on the first one. It provides a direct solution of the boundary value problem in a single run and is much faster and presumably more stable than the shooting method.

In section 4.3, numerical solutions are shown and the solution space is discussed. Two features of this solution space are remarkable. First, there are solutions with a negative total mass, despite having an everywhere positive energy density. The asymptotic Schwarzschild mass of the solutions does not even seem to be bounded from below. Second, there is a two-dimensional solution space for every choice of the model parameter $\tilde{\eta}$ rather than just two branches of solutions, as it would be the case in a spacetime without topological defect [30]. In Ref. [10], the existence of this bigger class of solutions was probably not recognized due to numerical issues that are discussed in subsection 4.3.1.

4.1 Prerequisites

4.1.1 Differential equations

In this chapter, the dimensionless variables and parameters

$$\tilde{\eta} = 8\pi G_N f^2 \ , \tag{4.1.1a}$$

$$w = (ef)^2 W = y_0^2 + y^2 \ , \tag{4.1.1b}$$

$$y = efY \ , \tag{4.1.1c}$$

$$y_0 = efb \ , \tag{4.1.1d}$$

are used, so all lengths are expressed in units of $1/ef$, the length scale of the Skyrme model. The definition (4.1.1a) absorbs the factor f^2 into the gravitational constant. In this way, the parameters e and f of the Skyrme model do not appear in the field equations, and $\tilde{\eta}$ is the only free parameter of the theory.

After inserting the *Ansatz* (2.1.5), (2.2.8), the matter Lagrange density (2.2.1c) for $Y \neq 0$ becomes[1]

$$\mathcal{L}_{\text{matter}} = T^0_0 = -2\frac{e^2 f^4}{w^2}\left(\mathcal{A}\Big(F(w), w\Big) + \frac{F'(w)^2}{\sigma(w)^2}\mathcal{B}\Big(F(w), w\Big)\right) \ , \tag{4.1.2}$$

where $\mathcal{A}$ and $\mathcal{B}$ are shorthands for the nonnegative terms

$$\mathcal{A}\Big(F(w), w\Big) := 2\sin^2\left(\frac{F(w)}{2}\right)\left(\sin^2\left(\frac{F(w)}{2}\right) + w\right) \tag{4.1.3a}$$

$$\mathcal{B}\Big(F(w), w\Big) := w^2\left(4\sin^2\left(\frac{F(w)}{2}\right) + w\right) = w^2(2 - 2\cos(F(w)) + w) \ . \tag{4.1.3b}$$

[1] $F'(w)$, $\sigma'(w)$, and $\mu'(w)$ denote derivatives with respect to w.

The TT and YY-components of the Einstein equations are equivalent to

$$\sigma'(w) = \sigma(w)\left(-\frac{1}{4w}(\sigma(w)^2-1)+\frac{\tilde{\eta}}{2}\frac{\sigma(w)^2}{w^2}\left(\mathcal{A}+\frac{F'(w)^2}{\sigma(w)^2}\mathcal{B}\right)\right) \tag{4.1.4a}$$

$$\mu'(w) = -\mu(w)\left(-\frac{1}{4w}(\sigma(w)^2-1)+\frac{\tilde{\eta}}{2}\frac{\sigma(w)^2}{w^2}\left(\mathcal{A}-\frac{F'(w)^2}{\sigma(w)^2}\mathcal{B}\right)\right)\ . \tag{4.1.4b}$$

These two equations can be used to eliminate derivatives of μ and σ in the Euler-Lagrange equations (2.2.3). The resulting field equation for $F(w)$ is

$$\begin{aligned} F''(w) = &+\frac{1}{2\mathcal{B}}\sigma(w)^2\sin\big(F(w)\big)\left(2\sin^2\!\left(\frac{F(w)}{2}\right)+w\right) \\ &-\frac{1}{2w^2}\sigma(w)^2F'(w)\left(w-2\tilde{\eta}\sin^2\!\left(\frac{F(w)}{2}\right)\left(2\sin^2\!\left(\frac{F(w)}{2}\right)+2w\right)\right) \\ &-\frac{1}{\mathcal{B}}w^2F'(w)\left(F'(w)\,\sin\big(F(w)\big)+1\right)\ . \end{aligned} \tag{4.1.4c}$$

Equations (4.1.4a), (4.1.4b), and (4.1.4c) constitute a set of three ODEs for three functions $\sigma(w), \mu(w), F(w)$ in one variable. These three equations are sufficient since an explicit calculation shows that their solutions also satisfy the omitted components of the Einstein equations.

Two remarks are in order: First, the above equations contain y and y_0 only in the combination $w = y_0^2 + y^2$, and are the same as for Skyrmions in a simply connected spacetime (cf. subsection 2.1.2). In this sense, all information about the topology and the defect size is encoded in the boundary conditions, not in the ODEs.

Second, without defining an extended theory for a degenerate metric, the ODEs are only valid in the region $Y \neq 0$ resp. $w > y_0^2$. The physics at $Y = 0$ may restrict the boundary conditions. For the extended theories discussed in section 3.3, the *Ansatz* automatically fulfill the equations at $Y = 0$ under the conditions $\mu(y_0^2) \neq 0$ and $\sigma(y_0^2) \neq 0$.

4.1.2 Monotonicity of the Skyrmion profile function $F(w)$

The field equation (4.1.4c) was derived in a straightforward but tedious way: First, derive the Euler-Lagrange-equations (2.2.3) from the Lagrange density; second, insert the *Ansatz* into these equations. By invoking the principle of symmetric criticality (see Ref. [21]), we can basically swap these two steps and simplify the calculation: First, calculate the energy for the *Ansatz* field, second, derive the the Euler-Lagrange equations to minimize this energy with respect to the *Ansatz* functions.

As it was laid out in subsection 2.2.3, the hedgehog *Ansatz* is the general static Skyrme field with the specified global SO(3) symmetry. Since SO(3) is a compact subgroup of symmetries of the Lagrange density, the principle of symmetric criticality states: If the energy is minimized among the symmetric field configurations, it is also minimal among *all* (symmetric or non-symmetric) field configurations, i.e. it is indeed a static solution.

For a given metric (2.1.5), the static hedgehog field (2.2.8) is a solution if it minimizes the energy

$$\begin{aligned} E_{\mathrm{M}} &= \int_{-\infty}^{\infty}\int_{0}^{\pi}\int_{0}^{\pi} -\sqrt{-g}\,\mathcal{L}_{\mathrm{M}}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}Y \\ &= \int_{-\infty}^{\infty}\int_{0}^{\pi}\int_{0}^{\pi} -\mathcal{L}_{\mathrm{M}}\,\mu(W)\,\sigma(W)\,W\frac{Y}{\sqrt{W}}\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}Y \\ &= 2\pi\int_{-\infty}^{\infty} -\mathcal{L}_{\mathrm{M}}\,\mu(W)\,\sigma(W)\,\sqrt{W}\,Y\,\mathrm{d}Y \\ &= \frac{2\pi}{(ef)^3}\int_{y_0^2}^{\infty}\underbrace{-\mathcal{L}_{\mathrm{M}}\,\mu(w)\,\sigma(w)\,\sqrt{w}}_{:=\mathcal{H}_{\mathrm{eff}}}\,\mathrm{d}w\;, \end{aligned} \tag{4.1.5}$$

with the matter Lagrange density $\mathcal{L}_{\mathrm{M}}$ given by equation (4.1.2). The Euler-Lagrange equation

$$\frac{\partial\mathcal{H}_{\mathrm{eff}}}{\partial F(w)} - \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\partial\mathcal{H}_{\mathrm{eff}}}{\partial F'(w)}\right) = 0 \tag{4.1.6}$$

is equivalent to equation (4.1.4c).

It is instructive to investigate the properties of the effective Hamilton density. It can be written as a function

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_{\text{eff}}\left(\sin^2\left(\frac{F(w)}{2}\right), F'^2(w), w\right) \tag{4.1.7}$$

that is strictly monotonic in $\sin^2\left(F(w)/2\right)$ and $F'^2(w)$. These properties can be used to show the following

Lemma: For a given topological charge $B \in \mathbb{Z}$, let $F(w)$ be a continuous, piecewise continuously differentiable function that satisfies the boundary conditions $F(y_0^2) = B\pi$, $F(\infty) = 0$, and minimizes the energy E_M among all such functions. Then, $F(w)$ is a monotonic function.

In short, the lemma ensures that every static, spherically symmetric, finite-energy solution for the Skyrmion spacetime defect has a monotonic Skyrmion profile function $F(w)$. The appendix contains a proof of this lemma.

4.1.3 Boundary conditions and weak-field solutions

As laid out in chapters 2 and 3, the boundary conditions for the *Ansatz* functions are

$$\begin{aligned} &F(y_0^2) = \pi \ , \quad \sigma(y_0^2) \neq 0 \ , \quad \sigma(y_0^2) \neq 0 \\ &F(\infty) = 0 \ , \quad F'(\infty) = 0 \ , \quad \mu(\infty) = 1 \ , \quad \sigma(\infty) = 1 \ . \end{aligned} \tag{4.1.8}$$

For $w \to \infty$, a weak-field approximation can be made: After linearizing the field equations (4.1.4) around the limiting values at $w \to \infty$, they have the general solutions

$$F(w) = \frac{k}{w} + d\sqrt{w} \quad \text{and} \quad \mu(w) = \sigma^{-1}(w) = \sqrt{1 - \frac{l}{\sqrt{w}}} \ . \tag{4.1.9}$$

The weak-field solutions for $\mu(w)$ and $\sigma(w)$ are the vacuum solutions and correspond exactly to the Schwarzschild metric with radial coordinate $r = \sqrt{w}/ef$ and Schwarzschild radius $r_s = l/ef$. This means that the spacetime approaches the

Schwarzschild spacetime asymptotically (i.e. far away from the spacetime defect), which is the expected behavior of a localized lump of energy. The parameter l will be called *(dimensionless) asymptotic Schwarzschild radius* (cf. subsection 2.3.1).

The weak-field solution of the Skyrme profile function $F(w)$ is described by the parameters k and d. The latter can be used as an error measure since the boundary condition $F(\infty) = 0$ requires $d = 0$. The solution $F(w) = k/w$ corresponds to a dipole field with dipole moment $\propto k$, as it is also the case for Skyrmion solutions in flat space [21].

The weak-field approximation provides a convenient way to implement the boundary conditions *at infinity* in the numerical methods. For $w \in [y_0^2, w_{\max}]$, the nonlinear field equations (4.1.4) are integrated numerically, whereas in the region $w \in [w_{\max}, \infty)$, the weak-field solution (4.1.9) is used, with a continuous resp. continuously differentiable transition at $w_{\max}$.

The choice of $w_{\max}$ is a compromise between two demands. First, $w_{\max}$ has to be large enough, so that $F(w_{\max})$, $\sigma(w_{\max})$, and $\mu(w_{\max})$ are sufficiently close to their limiting values and the weak-field approximation is justified. Second, as Eq. (4.1.9) suggests, numerical errors (e.g. from initially small rounding errors) can grow proportionally to $\sqrt{w_{\max}}$, which makes the problem ill-conditioned if $w_{\max}$ is too large.

4.2 Numerical methods

4.2.1 Method 1: Optimized shooting method

The first method is an optimized version of the one used in Ref. [10]. The numerical integration of the ODEs (4.1.4) is started at $w = y_0^2$ with some initial values

$$F(y_0^2) = \pi, \quad F'(y_0^2) = \alpha, \quad \sigma(y_0^2) = \beta, \quad \mu(y_0^2) = \gamma \tag{4.2.1}$$

and stopped (i.e. $w_{\max}$ is chosen during numerical integration) as soon as the conditions

$$|F(w)| < \pi/50 \text{ and } |1 - \sigma(w)| < 1/50 \tag{4.2.2}$$

are met, or $F'(w)$ changes signs. The threshold value $1/50$ is chosen empirically. Varying it in the range $[1/10, 1/100]$ does not notably change the numerical solutions.

If $F'(w)$ changes signs, $F(w)$ is not monotonic and therefore does not provide a solution with minimal energy (see subsection 4.1.2). If condition (4.2.1) is fulfilled, the weak-field approximation is considered valid and the parameters k, d, and l are calculated from the numerical values of $F(w_{\mathrm{max}})$, $F'(w_{\mathrm{max}})$ and $\sigma(w_{\mathrm{max}})$.

Then, the whole procedure is repeated with varying initial conditions α, β, to achieve $d \approx 0$; this step is called *shooting*. The initial value γ is not involved in the shooting since the function $\mu(w)$ can simply be rescaled to $\mu(w_{\mathrm{max}}) = \sigma(w_{\mathrm{max}})^{-1}$.

The parameter d provides a sensible error measure for this method since it is independent of the chosen value of w_{max}, as long as the fields are weak enough. Because w_{max} is chosen during numerical integration, this method adapts to the size of the Skyrmion and suffers less from numerical instabilities than a method with fixed w_{max}.

4.2.2 Method 2: Direct solution without shooting

In the second method, the numerical integration runs *backward*. For given input parameters k and l, the upper bound

$$w_{\mathrm{max}} = \max\left(50\frac{k}{\pi}, \left(\frac{1}{2}\frac{l}{50}\right)^2\right) \tag{4.2.3}$$

is chosen to ensure the validity of the weak-field approximation (cf. condition (4.2.2) in the first method). The initial values $F(w_{\mathrm{max}})$, $F'(w_{\mathrm{max}})$, $\sigma(w_{\mathrm{max}})$, $\mu(w_{\mathrm{max}})$ are calculated from equation (4.1.9) with $d = 0$ and the numerical integration of the ODEs is carried out for decreasing w until $F(w) = \pi$.

It may happen that the numerical integration fails before $F(w) = \pi$ is reached, either because the solution contains a coordinate singularity at $w > 0$, or because $w = 0$ is reached before $F(w)$ sufficiently increases. In both cases, the boundary conditions cannot be met and the solution is considered invalid.

Since the ODEs (4.1.4) allow numerical integration without knowing the defect size *a priori*, y_0 can be determined from the condition $F(y_0^2) = \pi$. In this way, all boundary conditions are satisfied explicitly and no shooting is needed.

For this reason, method 2 is much faster than method 1 and is especially suited for parameter scans. Additionally, the input parameters k and l are easy to interpret as a dipole strength and mass (see subsection 4.1.3).

4.3 Numerical solutions and structure of the solution space

4.3.1 Degrees of freedom in shooting methods

The monotonicity of $F(w)$ gives an important restriction on the initial values for method 1. In order to satisfy the boundary conditions (4.1.8), the function $F(w)$ has to be monotonically decreasing, which implies[2] $\alpha = F'(y_0^2) < 0$.

Figure 4.1 shows solution sets for the first numerical method for $\tilde{\eta} = 1/20$ and some values of y_0. The shooting has been done for both parameters separately, keeping $\alpha = F'(y_0^2)$ fixed and varying $\beta = \sigma(y_0^2)$ or vice versa.

Apparently, there is exactly one solution $\beta(\tilde{\eta}, y_0, \alpha) > 0$ of the shooting problem for each given values of $\tilde{\eta} > 0$, $y_0 > 0$, $\alpha < 0$. Therefore, the solution set for a given model parameter $\tilde{\eta} > 0$ can uniquely be parametrized by y_0 and α; it is two-dimensional.

In Refs. [4] and [10], only two branches of solutions (i.e. exactly two solutions for each $\tilde{\eta} < \tilde{\eta}_{\text{crit}}$ and $y_0 > y_{0,\text{crit}}$), were found for essentially the same model, and the existence of a lower bound $y_{0,\text{crit}} > 0$ for the defect size was conjectured. The above results instead suggest $y_{0,\text{crit}} = 0$ and exhibit an additional degree of freedom.

This discrepancy can be explained by the additional boundary condition, equivalent to $\sigma(y_0^2) = 1$, that was imposed in Ref. [4]. From figure 4.1, it is clear that there are exactly two solutions with $\sigma(y_0^2) = 1$ if the defect size is big enough. Therefore, one obtains two branches of solutions that coalesce at $y_0 = y_{0,\text{crit}} > 0$, and no solutions for $y_0 < y_{0,\text{crit}}$.

[2] α has to be strictly negative, because $\alpha = 0$ leads to a constant profile function $F(w) = \pi$. This is the non-vacuum solution in Ref. [9], which has infinite energy and is not asymptotically flat.

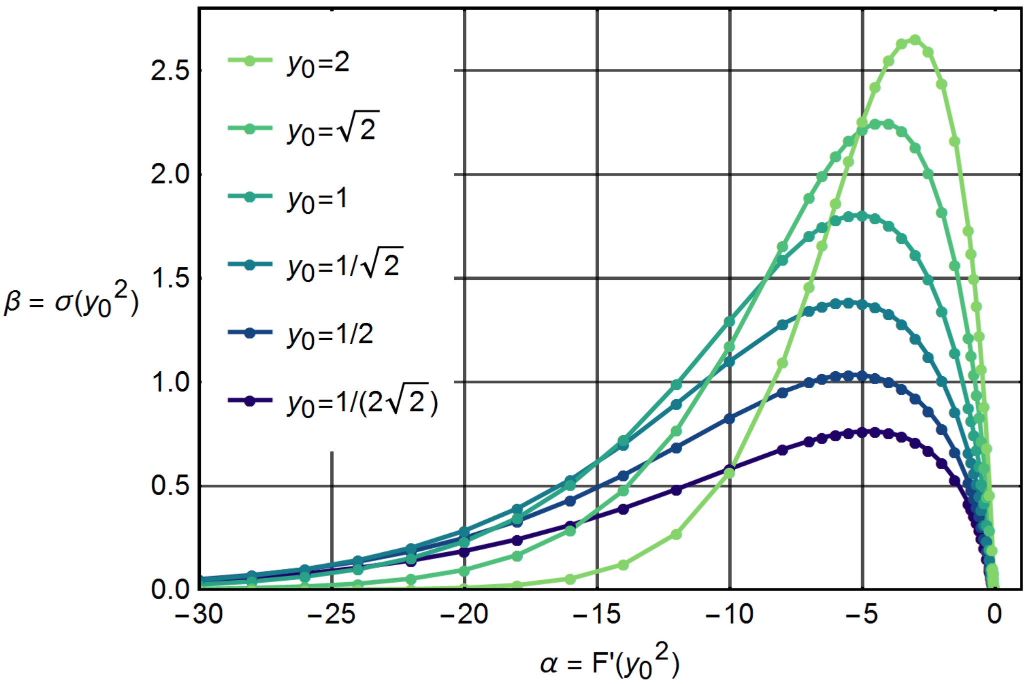


Figure 4.1: Solution set for the shooting problem (method 1) for model parameter $\tilde{\eta} = 1/20$ and various dimensionless defect sizes y_0. Dots indicate sampled numerical solutions, lines indicate the continuum of solutions. Solutions with $\beta < 0$ have a negative Schwarzschild mass of the defect core.

In Ref. [10], the condition $\sigma(y_0^2) = 1$ was not imposed, but one of the boundary conditions, $F(\infty) = 0$, was instead implemented by two conditions, $F(w_{\max}) \approx 0$ and $F'(w_{\max}) \approx 0$. Presumably, one degree of freedom was lost as a result of this implementation.

For the SU(2) Einstein-Skyrme model in a topologically trivial spacetime, the solution set indeed consists of two branches [30]. These solutions have to satisfy an additional boundary condition “to ensure regularity at the origin”. This condition is not necessary for the defect spacetime, and the defect size is an additional free parameter. Naively counting the degrees of freedom, we should expect a 3-dimensional solution set (parametrized by one model parameter and two solution parameters), which exactly matches the numerical results discussed above.

4.3.2 Upper and lower bounds for solution parameters

As already mentioned in subsection 4.2.2, the second method is much faster than the shooting method and better suited to investigate the structure of the solution space. For this purpose, numerical solutions have been calculated for approx. $600\,000$ tuples $(\tilde{\eta}, k, l)$ with $\tilde{\eta} \in [0, 5]$, $l \in [-5, 30]$, $k \in [10^{-4}, 900]$. The dimensionless defect size y_0 is obtained from the numerical solutions. If the boundary condition $F(y_0) = \pi$ cannot be satisfied, the solution is considered invalid. This happened for approx. 30% of the points.

The solution space can be described in terms of upper and lower bounds for l: Exactly those parameter tuples that satisfy

$$\begin{aligned} & \tilde{\eta} > 0 \,, \quad k > 0 \,, \quad l \in \Big(l_{\text{min}}(\tilde{\eta}, k) \,,\ \infty \Big) \,, \\ \text{or} \quad & \tilde{\eta} = 0 \,, \quad k > 0 \,, \quad l \in \Big(l_{\text{min}}(0, k) \,,\ l_{\text{max}}(0, k) \Big) \,, \end{aligned} \tag{4.3.1}$$

lead to valid solutions. The condition $k > 0$ follows from the monotonicity of $F(w)$ and the boundary conditions[3].

In the case $\tilde{\eta} > 0$, this description can be reversed: Parameter tuples satisfying

$$\tilde{\eta} \in (0, \tilde{\eta}_{\text{max}}(k, l)) \,, \quad k > 0 \tag{4.3.2}$$

lead to valid solutions, while there is no valid solution with $\tilde{\eta} \geq \tilde{\eta}_{\text{max}}(k, l)$. Note that the interval $(0, \tilde{\eta}_{\text{max}}(k, l))$ may be empty. For certain values of k and l, there are no solutions at all, regardless of $\tilde{\eta}$. Figure 4.2 shows the numerical value for $\tilde{\eta}_{\text{max}}(k, l)$. Apparently, there is no global upper limit for $\tilde{\eta}$.

4.3.3 Boundaries and limiting behavior

It is instructive to investigate the boundaries of the solution space. In most cases, the limiting behavior cannot be calculated directly but must be extrapolated from the sampled numerical solutions.

[3] The case $k = 0$ leads to a solution with constant $F(w) = 0$, equivalent to a vacuum solution with no Skyrme field. Such solutions are discussed e.g. in [8, 17].

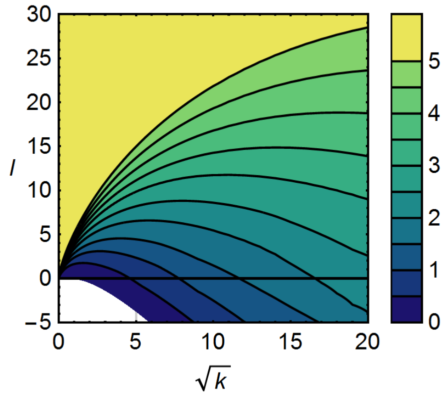


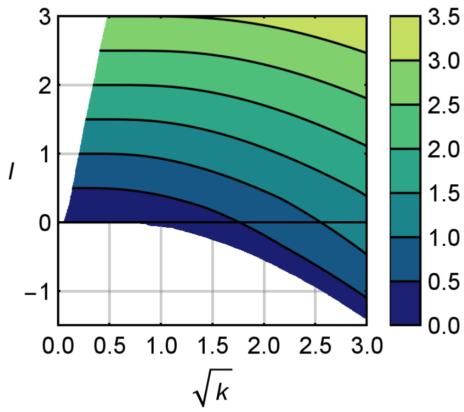
Figure 4.2: Numerical values of $\tilde{\eta}_{\mathrm{max}}(k,l)$. k is the dimensionless diople strength of the asymptotic Skyrme field and l is the dimensionless asymptotic Schwarzschild mass. In the white region below the $\tilde{\eta}_{\mathrm{max}} = 0$ line, there are no solutions for any choice of $\tilde{\eta} \geq 0$. In the bright region above the $\tilde{\eta}_{\mathrm{max}} = 5$ line, the maximal value could not be determined since numerical solutions were only sampled for $\tilde{\eta} \in [0, 5]$.

The case $\tilde{\eta} = 0$ is special because the metric is not influenced by the matter. The Schwarzschild-like solution (4.1.9) for σ and μ is exact in this case. In the limit $l \to l_{\mathrm{max},\tilde{\eta}=0}(k)$, the defect size y_0 approaches the Schwarzschild radius l (cf. figure 4.3(a)). In the numerical solutions, the defect size cannot be smaller than the Schwarzschild radius, because there would be an event horizon[4] with $\sigma(w) \to \infty$. It could be possible to investigate this case in another coordinate system, e.g. the Painlevé-Gullstrand-like coordinates proposed in Ref. [10].

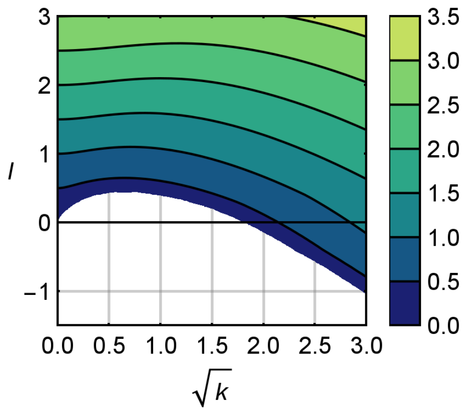
For $\tilde{\eta} > 0$ and $k \to 0$, the defect size also approaches the Schwarzschild radius (cf. figures 4.3(b), 4.3(c)), while the maximal value of the metric function $\sigma(w)$ approaches infinity. Interestingly, $\sigma(y_0^2)$ approaches zero, so the Schwarzschild mass of the core approaches negative infinity. The Energy of the Skyrme field is typically concentrated in a thin region. This large energy is (partly) compensated by the large negative mass of the topological defect itself. In other words: The Skyrme field, together with the negative mass of the defect prevents the formation of an event horizon. Figure 4.6 shows a typical solution with small k.

[4]Such “hairy black holes” with an SU(2) field were investigated in [30–32], without regarding the topology within the event horizon.

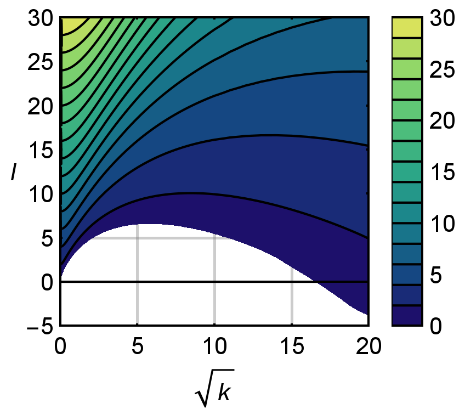
In the limit $l \to l_{\min}(\tilde{\eta}, k)$ resp. $\tilde{\eta} \to \tilde{\eta}_{\max}(k, l)$, while keeping the other two parameters fixed, the defect size $y_0(\tilde{\eta}, k, l)$ approaches zero monotonically. The Schwarzschild mass of the core approaches negative infinity.



(a) defect size y_0 for $\tilde{\eta} = 0$

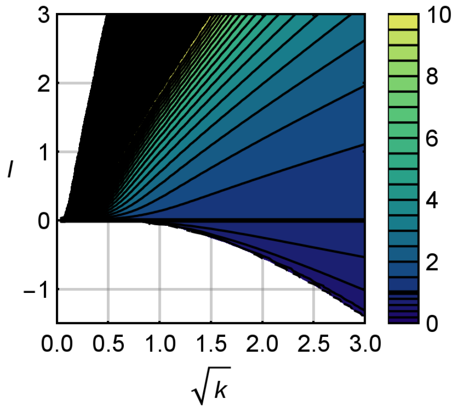


(b) defect size y_0 for $\tilde{\eta} = 1/20$

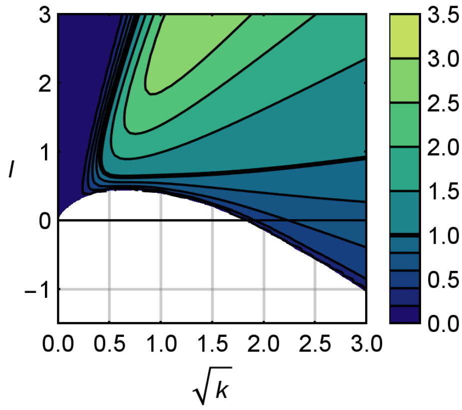


(c) defect size y_0 for $\tilde{\eta} = 2$

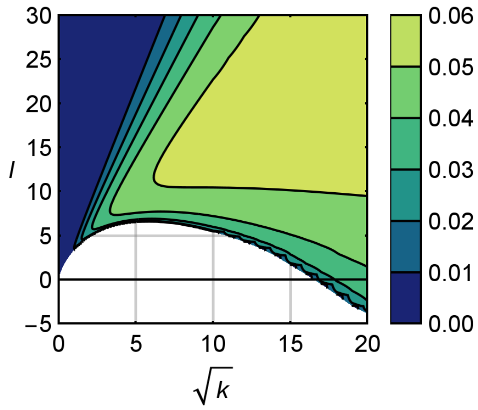
Figure 4.3: Dimensionless defect size y_0 for various values of $\tilde{\eta}$.
In the white regions, no valid solutions exist because the boundary conditions cannot be satisfied. In the limit $l \to l_{\min}(\tilde{\eta}, k)$, the defect size approaches zero. For $\tilde{\eta} > 0$ and $k \to 0$, the defect size approaches l. For $\tilde{\eta} = 0$, there is an additional excluded region because the metric decouples from the Skyrme matter and an event horizon appears for $y_0 \to l$.



(a) $\beta = \sigma(y_0^2)$ for $\tilde{\eta} = 0$



(b) $\beta = \sigma(y_0^2)$ for $\tilde{\eta} = 1/20$



(c) $\beta = \sigma(y_0^2)$ for $\tilde{\eta} = 2$

Figure 4.4: Boundary value $\beta = \sigma(y_0^2)$ for various values of $\tilde{\eta}$.
Since the metric decouples from the matter in the case $\tilde{\eta} = 0$, the value of $\sigma(y_0)$ approaches infinity as $y_0 \to l$ in the black region (see also figure 4.3(a)). For $\tilde{\eta} > 0$ the influence of the matter causes the limiting behavior $\sigma(y_0) \to 0$ for both $k \to 0$ and $l \to l_{\min}$. Note that a negative core mass, $\sigma(y_0) < 1$, is possible even if the asymptotic mass l is positive.

Table 4.1: Numerical properties of some solutions.

description		$\tilde{\eta}$	k	$l = l(\infty)$	$l(y_0^2)$	y_0	$F'(y_0^2)$
typical solution	Fig. 4.5	0.05	6.25	0.5	−0.363	1.132	−0.845
$k \ll l$	Fig. 4.6	0.05	0.10	2.0	-2.237×10^6	1.975	−25.499
$l \approx l_{\min}(\tilde{\eta}, k)$	Fig. 4.7	0.05	6.25	−0.5	−2.058	0.268	−0.930

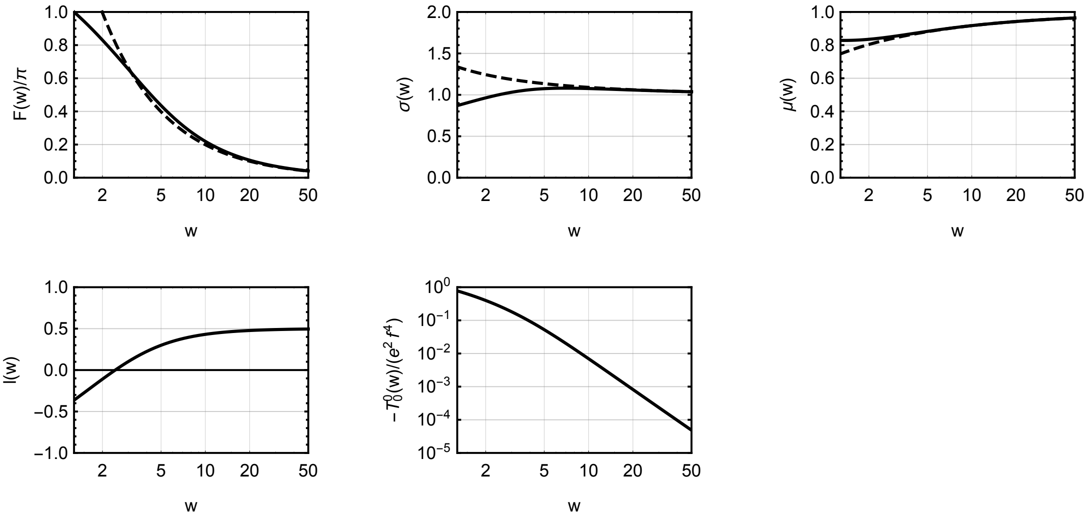


Figure 4.5: A typical solution to the ODEs (4.1.4). Values for the parameters are given in table 4.1. The variable w is given by $w = y^2 + y_0^2 = e^2\, f^2\, r^2$. Top row: Skyrme profile function $F(w)$ and metric functions $\sigma(w)$ and $\mu(w)$ (solid), together with their weak-field approximations (dashed). Bottom row: Dimensionless Schwarzschild mass function $l(w) = \sqrt{w}(1 - \sigma^{-2}(w))$ (cf. equation (2.3.6)) and energy density $-T_0^0$. Note that the mass of the core is negative even though the asymptotic mass is positive.

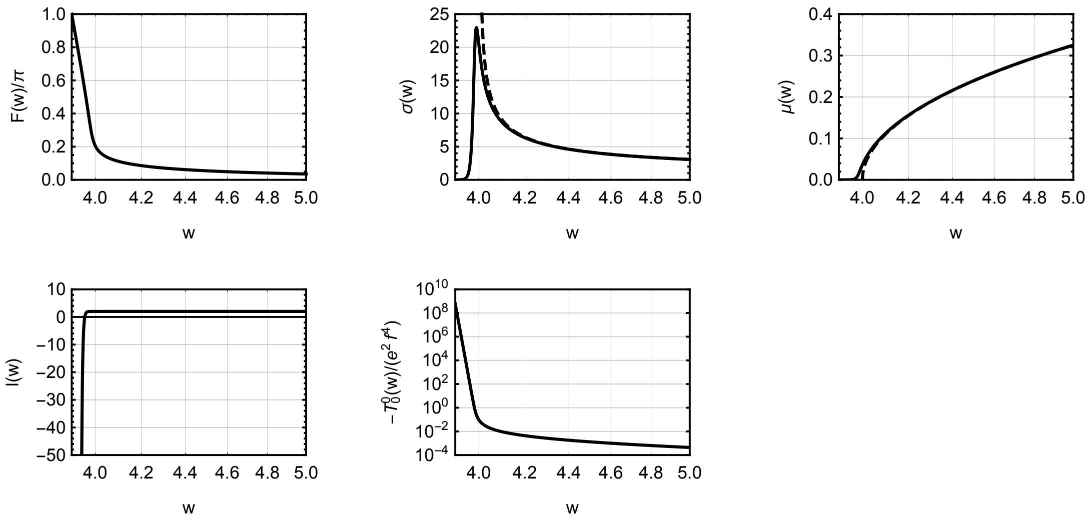


Figure 4.6: Same as in figure 4.5, for a solution with $k \ll l$ (parameter values are listed in table 4.1). The behavior is typical for the limit $k \to 0$. The energy of the Skyrme field is concentrated in a thin shell $0 \leq w < l^2$ and the mass function near the core has large negative values. For $w > l^2$, the solution is almost equal to the weak-field resp. vacuum solution.

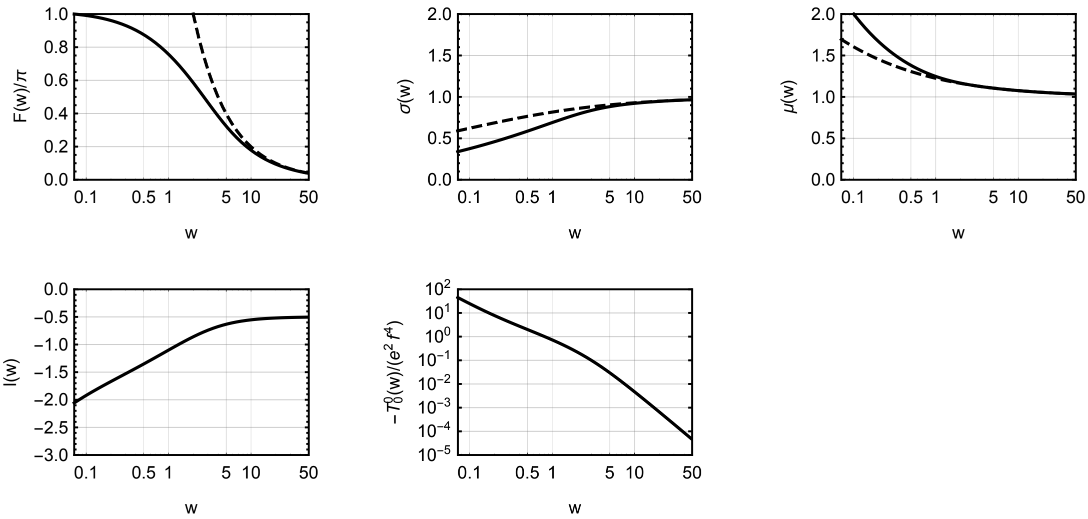


Figure 4.7: Same as in figure 4.5, for a solution with $l \approx l_{\mathrm{min}}(\tilde{\eta}, k)$ (parameter values are listed in table 4.1). In the limit $l \to 0$, the derivative of the Skyrme profile function $F'(w)$ tends to zero.

5 Discussion

5.1 Violation of the equivalence principle

Let $\mathcal{N} = \{x^\mu \,|\, g(x) = 0\}$ denote the set of spacetime points where the metric is degenerate. In the *Ansatz* (2.1.5), $\mathcal{N}$ is identical to the hypersurface $Y = 0$. At all points in $\mathcal{N}$, the equivalence principle is violated because there are no local inertial coordinate systems. While we can choose coordinate systems in which the first derivative of the metric vanishes at a given point[1], the property $g = 0$ is coordinate invariant since the determinant g of the metric is a scalar density. In other words, if the metric is degenerate at one point, it is degenerate at this point in every coordinate system.

The surface $\mathcal{N}$ is therefore not an artifact of coordinates, but a physical entity with some peculiar properties:

First, it has a finite, positive or negative Schwarzschild mass $m_{\mathrm{S}}(b)$, concentrated on a shell of zero thickness. In this sense, it has an *infinite energy density*, much like a curvature singularity, but the curvature and the energy-momentum tensor is finite everywhere. Observables like the Kretschmann scalar cannot be used to detect this effect.

Second, as noted in subsection 3.2.1, a generic, two-times differentiable covariant vector field (i.e. second *partial* derivatives exist) has a diverging *covariant* and no second covariant derivative at $\mathcal{N}$. This problem has been circumvented by differentiating $g_{\mu\nu}v^\nu \in \mathrm{im}(g)$ instead of v^μ, but it can also be interpreted as a "kink" in the spacetime.

These properties suggest the following interpretation: Using a coordinate system with a degenerate metric provides a regularization to a curvature singularity. In

[1] If Levi-Civita-like connections exist, (see subsection 3.1.4), such a coordinate system is given by the transformation $x^\mu \mapsto x^\mu + \frac{1}{2}\Gamma^\mu_{\rho\sigma}x^\rho x^\sigma$.

particular, the metric is differentiable and fields can be smoothly continued across the “singular” regions of spacetime. But the divergence resp. the non-existence of the inverse metric causes several other serious problems that are discussed in sections 3.1 and 3.2.

5.2 Negative mass and stability

The difference between inertial mass, active gravitational mass, and passive gravitational mass has been explained in section 2.3.1, as well as the ADM and Bondi mass.

The asymptotic Schwarzschild mass $m_S(\infty)$ is a measure of active gravitational mass. It determines how strong a test particle with negligible mass is attracted (for $m_S(\infty) > 0$) or repelled (for $m_S(\infty) < 0$) by the spacetime defect. In first order, if the test particle is far away and has negligible speed, the gravitational acceleration is the same as in Newtonian gravity: $a_{\text{grav}} = G_N m_{\text{act}}/r^2$.

The numerical solutions of chapter 4 show that a *negative active gravitational mass* is possible in our topologically nontrivial model, without having a curvature singularity. These solutions circumvent both the positive mass theorems and the topological censorship theorem since the theorems are only valid in non-degenerate Lorentzian spacetimes (see subsections 2.3.2 and 3.1.5 for details).

The ADM mass is the total energy of a gravitational system in ADM formalism. Since it only depends on asymptotic fields, it is a conserved quantity, even if the metric is degenerate in a bounded region in space. The Bondi mass is not conserved and excludes energy that is carried away to infinity by radiation.

In the absence of a lower bound for the ADM and Bondi mass, unphysical “runaway” processes could be possible in which the spacetime defect radiates away an infinite amount of energy by lowering its Bondi mass. Alternatively, the system could be metastable and react to perturbations by emitting some radiation and then settling to another metastable state.

The limiting behavior of the static solutions for $k \to 0$ (see subsection 4.3.3) suggests that the Skyrme field is able to prevent gravitational collapse. In the case $\tilde{\eta} > 0$, solutions exist for arbitrarily small values of k. In such solutions, the Skyrme field builds up a shell of high energy density around the would-be Schwarzschild radius. Together with a large negative core mass, this prevents the formation of an event horizon (see also figure 4.6). This behavior is in contrast with the case $\tilde{\eta} = 0$, in which the metric decouples from the Skyrme field and an event horizon can exist.

Before a thorough investigation of stability can be done, we need to know the exact field equations at the defect surface, where $\det(g)$ vanishes. The extended definitions made in 3.3 rely on some additional conditions to the metric and matter fields (e.g. the condition $\omega_Y = 0$ for $Y = 0$) and are only valid for a fixed defect size. The Einstein equations are not valid at the defect surface $\mathcal{N}$ and the time-dependence cannot be inferred from these extensions. Therefore, no statements about the passive gravitational mass or inertial mass of the solutions can be made at the moment.

5.3 Quantum effects and other possible models

As stated above, a degenerate metric can be interpreted as a regularized curvature singularity; it has a finite energy concentrated on a shell of zero thickness. In a (hypothetical) quantum theory of gravitation, the picture could be quite different. In the simplest case, quantum effects would only "smear out" the energy density over the Planck scale. The shell would no longer have a thickness of zero and the energy density would be finite, but possibly negative. In nongravitational quantum field theory, negative energy densities are possible in small regions of space (e.g. in the Casimir set-up), even if the classical theory would have a non-negative energy density.

On the classical level, we could try to imitate such a behavior by changing the matter sector. For example, higher order derivative terms in the Lagrange density would only change the behavior in the vicinity of the defect core and could have an effect similar to the degenerate metric. Since such a model must necessarily allow a negative energy density, neither the topological censorship theorem nor the positive mass theorem would be valid for such a model. There is no need for regularizing any curvature singularity and the metric could be non-degenerate everywhere.

In Ref. [17] it was conjectured that a topological defect could be produced by a topology-changing effect of quantum gravity. If the production process could be simulated in a classical theory, it could give indirect hints at the connection between classical GR and quantum gravity.

6 Summary and outlook

In this thesis, a numerical analysis of the Skyrmion spacetime defect model (cf. Ref. [10]) has been performed. Numerical calculations show that the space of static, spherically symmetric solutions is two-dimensional for a given model parameter $\tilde{\eta} = 8\pi G_N f^2$. This result is different from the case of gravitating Skyrmions in a topologically trivial spacetime [30] which have only two branches of solutions due to additional boundary conditions.

The degenerate metric (2.1.5) can be seen as a regularization to the curvature singularity that would otherwise be present due to the topological censorship theorem. Although this metric is smooth on the whole spacetime, many aspects of GR cannot be readily applied. Definitions using the inverse metric become ambiguous and/or divergent. These divergences affect the action integral even if the metric is only degenerate on a set of zero measure, e.g. a hypersurface. A thorough analysis of these problems has been done in chapter 3, and two possible extensions of the theory have been defined.

These extended theories can be consistently applied to the *Ansatz* (2.1.5) and (2.2.8), but they are not entirely satisfactory since the time-dependence could not be derived. The integral equations (3.2.7) suggest another possible extension: If the Einstein and energy-momentum tensors are interpreted as *distributions* rather than ordinary functions, the divergence problem (see subsection 3.2.2) could perhaps be solved.

As soon as time-dependent calculations become feasible, it is instructive to investigate the stability of this system. Because there is apparently no lower bound to the mass, the system cannot be stable, but it could be metastable or at least decay very slowly. In this case, one could try to determine the inertial and passive gravitational mass of the spacetime defect by calculating the reaction to an external field.

Answering these questions could lead to progress on a more important question: Which role does nontrivial spacetime topology ultimately play in physics?

Appendix

Proof that the Skyrmion profile function $F(w)$ is monotonic

As stated in subsection 4.1.2, the Skyrmion profile function $F(w)$ minimizes the energy

$$E_{\mathrm{M}} = \frac{2\pi}{(ef)^3} \int_{y_0^2}^{\infty} \mathcal{H}_{\mathrm{eff}}\left(\sin^2\left(\frac{F(w)}{2}\right), F'^2(w), w\right) \mathrm{d}w \ , \tag{.0.1}$$

where the effective Hamilton density $\mathcal{H}_{\mathrm{eff}}$ is strictly monotonically increasing in its arguments $\sin^2\big(F(w)/2\big)$ and $F'^2(w)$. The integral is defined for all continuous, piecewise continuously differentiable functions $F(w)$.

Lemma: For a given topological charge $B \in \mathbb{Z}$, let $F(w)$ be a continuous, piecewise continuously differentiable function that satisfies $F(y_0^2) = B\pi$, $F(\infty) = 0$, and minimizes the energy E_{M} among all such functions. Then, $F(w)$ is a monotonic function.

Proof: Assume that $F(w)$ minimizes the energy, but is not monotonic. Then, $F(w)$ has at least one local extremum $w_{\mathrm{ex}} \in (y_0^2, \infty)$, which is also a local maximum or minimum of $\sin^2\big(F(w)/2\big)$.

First case: If $w_{\rm ex}$ is a local maximum of $\sin^2\bigl(F(w)/2\bigr)$, we can find a neighbourhood $[w_1, w_2] \ni w_{\rm ex}$, with

$$\sin^2\left(\frac{F(w_{\rm ex})}{2}\right) \geq \sin^2\left(\frac{F(w)}{2}\right) \qquad \text{for all } w \in [w_1, w_2] \tag{.0.2}$$
$$\text{and } \sin^2\left(\frac{F(w_{\rm ex})}{2}\right) > \sin^2\left(\frac{F(w)}{2}\right) \qquad \text{for } w = w_1 \text{ or } w = w_2 \quad . \tag{.0.3}$$

By the intermediate value theorem, we can choose w_1 and w_2 in such a way that $F(w_1) = F(w_2)$. The new function

$$\tilde{F}(w) = \begin{cases} F(w) & , w \notin [w_1, w_2] \\ F(w_1) & , w \in [w_1, w_2] \end{cases} \tag{.0.4}$$

is continuous and piecewise continuously differentiable. By this construction, the new function satisfies $\sin^2\bigl(\tilde{F}(w)/2\bigr) \leq \sin^2\bigl(F(w)/2\bigr)$ and $\tilde{F}'(w)^2 \leq F'(w)^2$ wherever the derivative is defined. Since $\mathcal{H}_{\rm eff}$ is strictly monotonic,

$$\mathcal{H}_{\rm eff}\left(\sin^2\left(\frac{\tilde{F}(w)}{2}\right), \tilde{F}'^2(w), w\right) \leq \mathcal{H}_{\rm eff}\left(\sin^2\left(\frac{F(w)}{2}\right), F'^2(w), w\right) \quad . \tag{.0.5}$$

Since the inequalities are strict for a non-null subset of $[w_1, w_2]$, the energy of the new function $\tilde{F}(w)$ is strictly lower than that of $F(w)$, in contradiction to the assumption.

Second case: If $\sin^2\bigl(F(w)/2\bigr)$ has a local minimum at $w_{\rm ex}$, it must also have a local maximum at some $w_2 \in (w_{\rm ex}, \infty)$, since $\sin^2\bigl(F(\infty)/2\bigr) = 0 \leq \sin^2\bigl(F(w_{\rm ex})/2\bigr)$. Let w_2 be the smallest of these maxima. From the first case, we already know that w_2 cannot also be an extremum of $F(w)$; this is only possible if $\sin^2\bigl(F(w_2)/2\bigr) = 1$, so $F(w_2)$ is an odd multiple of π.

Now, let w_1 be the largest element of $[y_0^2, w_{\rm ex})$ that satisfies $\sin^2\bigl(F(w_1)/2\bigr) = 1$. If B is even, the existence of w_1 follows from the same argument as above, if B is odd, we can set $w_1 = y_0^2$.

By the intermediate value theorem, we can again choose w_1 and w_2 in such a way that $F(w_1) = F(w_2)$ and define the new function.

$$\tilde{F}(w) = \begin{cases} F(w) & , w \notin [w_1, w_2] \\ 2F(w) - F(w_1) & , w \in [w_1, w_2] \end{cases} \tag{.0.6}$$

the new function satisfies $\sin^2\big(\tilde{F}(w)/2\big) = \sin^2\big(F(w)/2\big)$ and $\tilde{F}'(w)^2 = F'(w)^2$ wherever the derivative is defined, since $F(w_1)$ is an odd multiple of π. Therefore, $\tilde{F}(w)$ has the same energy as $F(w)$, but it has a local extremum w_2 which is a maximum of $\sin^2\big(F(w)/2\big)$. Since $\tilde{F}(w)$ does not minimize the energy (see first case), $\tilde{F}(w)$ does also not minimize the energy, in contradiction to the assumption. □

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