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On the four-form realization of q-theory with a kinetic term for q

Master's Thesis of

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Abstract

We study various aspects of the four-form realization of q-theory with a kinetic term for the vacuum variable q. Specifically, we first discuss gauge-fixing procedures for the three-form gauge field, on which this four-form realization is based. Then, we give a simple way of deriving the field equations for a general action of the four-form realization of q-theory by comparing the nonfundamental pseudoscalar field q with a fundamental scalar field. This comparison allows us to discuss the properties of a propagating mode, which may be associated with q as a consequence of the kinetic term for q. Further, we show that the four-form realization of q-theory with a kinetic term for q does not suffer from the Ostrogradsky instability at the classical level, although a kinetic term for q leads to higher-order time derivatives in the Lagrangian. In addition, we propose a possible quantization procedure for the four-form realization of q-theory with a kinetic term for q.

Contents

1. Introduction

In this chapter, we will introduce the main concepts on which this master's thesis is based. In particular, we will briefly review the cosmological constant problem and give an introduction to the q-theory approach to the cosmological constant problem.

Throughout this master's thesis, we employ the convention $\epsilon_{0123} = -1$ for the Levi-Civita symbol $\epsilon_{\alpha\beta\gamma\delta}$, the metric signature $(- + ++)$, and natural units with $c = \hbar = 1$. We denote the metric by $g_{\alpha\beta}$ and follow Weinberg's book [1] with respect to the conventions for the Riemann tensor $R_{\alpha\beta\gamma\delta}$ and its contractions. The metric determinant is denoted by $g \equiv \det(g_{\alpha\beta})$. For flat spacetime, we use the standard Cartesian coordinates and the standard Minkowski metric,

$$(x^\alpha) = (x^0, x^a) = (x^0, x^1, x^2, x^3) = (t, x, y, z)\,, \tag{1.1a}$$

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} \equiv [\mathrm{diag}(-1, 1, 1, 1)]_{\alpha\beta}\,. \tag{1.1b}$$

We also define antisymmetrization without normalization factors, for example, $A_{[\alpha\beta]} \equiv A_{\alpha\beta} - A_{\beta\alpha}$.

1.1. Cosmological constant problem

The cosmological constant problem is most easily discussed in the context of semi-classical gravity, where quantized matter fields are coupled to a classical spacetime metric [2]. Specifically, semi-classical gravity implies that the classical energy-momentum tensor $T_{\alpha\beta}$ on the right-hand side of the classical Einstein field equations,

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R - \lambda\, g_{\alpha\beta} = -8\pi G_N\, T_{\alpha\beta}\,, \tag{1.2}$$

is replaced by the expectation value of the quantized energy-momentum tensor $\hat{T}_{\alpha\beta}$,

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R - \lambda\, g_{\alpha\beta} = -8\pi G_N\, \left\langle \hat{T}_{\alpha\beta} \right\rangle\,. \tag{1.3}$$

In Eqs. (1.2) and (1.3), λ denotes the bare cosmological constant and G_N is Newton's gravitational constant.

Instead of writing the bare cosmological constant λ on the left-hand side of Eq. (1.3), we could also have absorbed the $\lambda\, g_{\alpha\beta}$ term into the energy-momentum tensor on the right-hand side of Eq. (1.3). Specifically, consider the energy-momentum tensor for a general perfect fluid,

$$\left\langle \hat{T}_{\alpha\beta} \right\rangle = p\, g_{\alpha\beta} + (p + \rho)\, u_\alpha u_\beta\,, \tag{1.4}$$

where p and ρ are, respectively, the pressure and the energy density of the fluid and u^α is the 4-velocity of the fluid. It follows that the $\lambda\, g_{\alpha\beta}$ term on the left-hand side of Eq. (1.3) can be absorbed into the energy-momentum tensor of a perfect fluid (1.4) by replacing the pressure p and the energy density ρ by an effective pressure p_{eff} and an effective energy density ρ_{eff}, respectively,

$$\rho_{\text{eff}} \equiv \rho + \frac{\lambda}{8\pi G_N}\,, \tag{1.5a}$$

$$p_{\text{eff}} \equiv p - \frac{\lambda}{8\pi G_N}\,. \tag{1.5b}$$

From now on, we will always take the bare cosmological constant λ to be absorbed into the energy-momentum tensor as in Eq. (1.5).

If we take the quantum vacuum to be described by an energy-momentum tensor corresponding to a Lorentz-invariant perfect fluid, the dependence of this energy-momentum tensor on u^α has to drop out. We therefore obtain for the effective energy density $\rho_{\text{vac,eff}}$ and the effective pressure $p_{\text{vac,eff}}$ of the Lorentz-invariant vacuum

$$p_{\text{vac,eff}} = -\rho_{\text{vac,eff}}\,, \tag{1.6a}$$

$$\rho_{\text{vac,eff}} \equiv \rho_{\text{vac}} + \frac{\lambda}{8\pi G_N}\,, \tag{1.6b}$$

$$p_{\text{vac,eff}} \equiv p_{\text{vac}} - \frac{\lambda}{8\pi G_N}\,, \tag{1.6c}$$

where ρ_{vac} and p_{vac} denote the contributions to $\rho_{\text{vac,eff}}$ and $p_{\text{vac,eff}}$, respectively, which are not due to the bare cosmological constant λ. For the particular form of the energy-momentum tensor corresponding to Eq. (1.6), the Bianchi identities imply that $\rho_{\text{vac,eff}} = -p_{\text{vac,eff}}$ must be constant [1]. Therefore, using Eq. (1.6a) in Eq. (1.4), we see that the bare cosmological constant λ and the effective vacuum energy-momentum tensor enter Eq. (1.3) in the same way, namely as a constant times the metric tensor. This, in turn, allows us to introduce an effective cosmological constant,

$$\lambda_{\text{eff}} \equiv 8\pi G_N\, \rho_{\text{vac,eff}}\,. \tag{1.7}$$

Assuming a Lorentz-invariant quantum vacuum is well-motivated by measurements indicating that a potential Lorentz-violating energy scale may even exceed the Planck scale [3].

Experimental results are also available regarding the effective cosmological constant λ_{eff}. In particular, λ_{eff} has been found to be nonzero in 1998 [4, 5] and recent measurements confirm this nonzero value with an energy scale of the associated $\rho_{\text{vac,eff}}$ of the order of 10^{-3} eV [6, 7]. The main cosmological constant problem then arises from quantum field theoretic estimates of ρ_{vac}, suggesting that ρ_{vac} is wildly larger than the measured value of $\rho_{\text{vac,eff}}$. Hence, these estimates prevent a naturally small cosmological constant and make it necessary to excessively fine-tune the bare cosmological constant λ in order to obtain the measured value of $\rho_{\text{vac,eff}}$.

We will now have a closer look at how to estimate ρ_{vac} from Quantum Field Theory. It turns out that there are various contributions to consider, each of which requires excessive fine-tuning in λ on its own. One particular contribution is due to zero-point energies. For example, we obtain for the theory of a free real scalar field with mass m in flat Minkowski spacetime,

$$\rho_{\text{zero-point}} \equiv \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3} \sqrt{\vec{k}^2 + m^2} \,. \tag{1.8}$$

Supposing that this theory of a free real scalar field remains valid up to an energy scale given by $\Lambda \gg m$ and introducing a corresponding cut-off in the integral of Eq. (1.8) gives

$$\rho_{\text{zero-point}} \sim \Lambda^4 \,. \tag{1.9}$$

In particular, if the expression (1.8) remains valid up to the Planck scale $E_{\text{P}} \sim 1/\sqrt{G_N}$, we see that ρ_{vac} and $\lambda/8\pi G_N$ need to cancel each other to roughly 120 decimal places in order to give the measured value of $\rho_{\text{vac,eff}}$. Even if we are more conservative and take Λ at the electroweak scale, $\Lambda \sim 100\,\text{GeV}$, there are still more than 50 decimal places to cancel.

Note that the estimate (1.9) was obtained in a rather naive way. For example, an explicit momentum cut-off violates Lorentz invariance and we should rather use a Lorentz-invariant regularization procedure, if we assume a Lorentz-invariant quantum vacuum. As a result, the analogue of the explicit cut-off Λ used above might appear as $m^2\Lambda^2$ instead of as Λ^4 in the leading term of the zero-point contribution to ρ_{vac} [8, 9]. The estimate (1.9) is also invalidated in supersymmetric models, since unbroken supersymmetry leads to exact cancellations between bosonic and fermionic fields, $\rho_{\text{zero-point}} = 0$ [10]. However, supersymmetry must be broken in realistic scenarios such that ρ_{vac} receives contributions from the scale at which this breaking occurs. Consequently, if we consider realistic values of the scales involved, also more careful considerations lead to the conclusion that the theoretical estimates of the zero-point energy densities are naturally much larger than the measured value of $\rho_{\text{vac,eff}}$. In particular, we still expect $\rho_{\text{vac}} \gtrsim (100\,\text{GeV})^4$.

Another contribution to ρ_{vac} arises from spontaneous symmetry breaking. Consider, for example, the potential of the Higgs doublet φ in the Standard Model with $\mu^2 > 0$ and $g > 0$,

$$V(\varphi) = V_0 - \mu^2(\varphi^\dagger\varphi)^2 + g(\varphi^\dagger\varphi)^4 \,. \tag{1.10}$$

Besides the zero-point energies associated with the Higgs doublet φ, Eq. (1.10) leads to an additional contribution to ρ_{vac} in the broken phase, which corresponds to the minimum value of the potential $V(\varphi)$ [11],

$$\rho_{\text{broken-phase}} \equiv V_0 - \frac{\mu^4}{4g} \approx V_0 - (100\,\text{GeV})^4 \,. \tag{1.11}$$

As already mentioned above, $(100\,\text{GeV})^4$ is large compared to the measured value of $\rho_{\text{vac,eff}}$, implying that fine-tuning is needed for at least one of λ and V_0.

In addition to the main cosmological constant problem outlined above, the measured value of λ_{eff} brings about several other puzzles. For example, given the small value of λ_{eff}, we might wonder why it does not vanish completely. We can also observe that the measured cosmological matter and vacuum energy densities are of the same order of magnitude [6, 7]. Given that the matter energy density scales with the cosmic scale factor $a(t)$ as a^{-3}, while the vacuum energy density stays constant, we might then wonder why we should live in such a special epoch in the history of the universe, where the matter and vacuum energy densities are comparable in magnitude.

Over the years, there have been many attempts to solve the main cosmological constant problem as well as the associated puzzles discussed in the previous paragraph. One particular class of attempts at solving the main cosmological constant problem involves a scalar field, which is supposed to dynamically obtain a vacuum expectation value that cancels the other contributions to λ_{eff}. More specifically, consider a fundamental scalar field φ whose source is proportional to the trace of the energy-momentum tensor,

$$\Box\, \varphi \propto T^{\alpha}_{\alpha} \,. \tag{1.12}$$

The idea is that the scalar field φ evolves until it reaches a constant value φ_0 corresponding to an equilibrium configuration with vanishing T^{α}_{α}. With the constant value φ_0 of φ, the standard Minkowski metric (1.1b) then solves the Einstein field equations. In contrast, a nonzero effective cosmological constant prevents such flat spacetime solutions [2]. In other words, a scalar field φ with the properties described above would dynamically adjust, so as to cancel all other contributions to the cosmological constant.

However, Weinberg argues in Ref. [2] that a scalar field theory, in which Eq. (1.12) holds and which possesses an equilibrium solution, cannot be achieved without fine-tuning in the Lagrangian. In the following, this result will be referred to as Weinberg's no-go theorem. More generally, one might suspect that Eq. (1.12) (or a generalized version of Eq. (1.12) as given by Eq. (6.4) in Ref. [2]) necessarily holds for any theory of fundamental scalar fields, which allows for a flat spacetime solution with all fields constant. In this case, Weinberg's no-go theorem implies that any theory of fundamental scalar fields allowing for such an equilibrium solution requires fine-tuning in the Lagrangian.

1.2. q-theory

Let us now introduce the approach to the cosmological constant problem, which we will be concerned with in this master's thesis. This approach was introduced in Ref. [12] and goes by the name of q-theory.

The q-theory approach to the cosmological constant problem is based on Lorentz invariance and thermodynamics. Assuming a Lorentz-invariant quantum vacuum is well-motivated by experimental bounds on Lorentz-violating theories, as already mentioned in the previous section. Further, the link of q-theory to thermodynamics comes from considering the thermodynamics of self-sustained media, i.e., media that have a macroscopic volume even in the absence of an environment. An example of such a self-sustained medium is a droplet of water in empty space. Typically, self-sustained media are characterized by a conserved extensive quantity and may have a vanishing relevant vacuum energy density due to a thermodynamic identity [13]. In the case of the water droplet in empty space, this conserved quantity is given by the particle number.

One of the main ideas of q-theory is to take also the quantum vacuum to be such a self-sustained medium with an associated conserved quantity. Compared with the case of the water droplet in empty space, this conserved quantity of q-theory must additionally respect the Lorentz invariance of the quantum vacuum.

1.2.1. Thermodynamics and nullification of the cosmological constant

In order to explore the implications of the previous paragraph, suppose that a portion of the vacuum in equilibrium in a spatial volume V is described by a vacuum variable q. For simplicity, we take q to be spacetime-independent in this subsection. Following Ref. [12], we will continue without discussing possible origins of q for the moment. We will only assume that the total "charge" $Q \equiv qV$ is conserved.

In the presence of an external pressure p_{ext}, the relevant thermodynamic potential is the Gibbs free energy $W = E + p_{\text{ext}}V$, where E denotes the energy [14]. Here, we consider the following simple example with an energy density $\epsilon(q)$:

$$W = E + p_{\text{ext}}V = \int d^3\vec{x}\,\epsilon(Q/V) + p_{\text{ext}}V\,. \tag{1.13}$$

The equilibrium value q_0 of q then satisfies an equation, which is obtained from variation of (1.13) over the free parameter V,

$$p_{\text{ext}} = -\epsilon(q) + q\left.\frac{d\epsilon(q)}{dq}\right|_{q=q_0}\,. \tag{1.14}$$

Eq. (1.14) corresponds to an integrated version of the Gibbs-Duhem equation, which, at zero temperature and in its simplest form $N d\mu = V dp_{\text{ext}}$, relates an infinitesimal change $d\mu$ in the chemical potential to an infinitesimal change dp_{ext} in the pressure [15]. In our case, the conserved particle number N is replaced by Q and $d\epsilon/dq$ corresponds to the chemical potential μ.

Further, Eq. (1.14) implies that the thermodynamically relevant vacuum energy density $\epsilon_{\text{th}}(q)$ is given by

$$\epsilon_{\text{th}}(q) \equiv \epsilon(q) - q\frac{d\epsilon(q)}{dq}\,. \tag{1.15}$$

Note that the equilibrium value $\epsilon_{\text{th}}(q_0)$ may be determined by a completely different energy scale than $\epsilon(q_0)$. This is because $\epsilon_{\text{th}}(q_0)$ is controlled by the external pressure p_{ext} according to Eq. (1.14), while the scale of $\epsilon(q_0)$ is presumably given by the estimates discussed in Sec. 1.1.

Another thermodynamic quantity of interest is the inverse isothermal compressibility $\chi^{-1} \equiv -V dp_{\text{ext}}/dV$ [14]. In the case of the equilibrium of q-theory, this quantity is given by

$$\chi_{\text{vac}}^{-1} = q^2\left.\frac{d^2\epsilon(q)}{dq^2}\right|_{q=q_0}\,. \tag{1.16}$$

If we require that the stationary point q_0 of (1.13) corresponds to a minimum of (1.13), we obtain a condition on χ_{vac}, namely

$$\chi_{\text{vac}}^{-1} \geq 0\,. \tag{1.17}$$

In the following, we will require Eq. (1.17) to hold, since this condition is necessary for the stability of the vacuum (cf. Sec. 21 of Ref. [14]).

We will now apply the thermodynamic arguments from above to the main cosmological constant problem. To this end, consider the energy-momentum tensor of the Lorentz-invariant vacuum as discussed in Sec. 1.1. In particular, this energy-momentum tensor is described by an effective pressure $p_{\text{vac,eff}}$ and an effective energy density $\rho_{\text{vac,eff}}$ with $\rho_{\text{vac,eff}} = -p_{\text{vac,eff}}$. Our first step is now to identify the external pressure p_{ext} with the internal pressure $p_{\text{vac,eff}}$. It then follows from the Gibbs-Duhem relation (1.14) that the energy density ϵ_{th}, which is relevant for thermodynamics, is also the energy density that plays the role of an effective cosmological constant,

$$\rho_{\text{vac,eff}} = \frac{\lambda_{\text{eff}}}{8\pi G} = \epsilon - q\frac{d\epsilon(q)}{dq}\bigg|_{q=q_0} = -p_{\text{vac,eff}} = -p_{\text{ext}}\,. \tag{1.18}$$

In other words, it is the energy density ϵ_{th} from Eq. (1.15) rather than ϵ that enters the Einstein field equations.

As a second step, we employ the idea that the vacuum can be described as a self-sustained medium. In particular, consider the case without an environment such that there is zero external pressure, $p_{\text{ext}} = 0$. Eq. (1.18), as deduced from the thermodynamic identity (1.14), then implies that the effective cosmological constant λ_{eff} vanishes in equilibrium. This nullification of the relevant vacuum energy density has an analog in the case of a liquid Helium droplet, where a thermodynamic identity similar to (1.14) is also responsible for the vanishing relevant vacuum energy density in equilibrium (cf. Sec. 3.3.4 of Ref. [13]).

We would like to stress that, in the case of the cosmological constant as well as in the case of the liquid Helium droplet, the nullification of the relevant vacuum energy density occurs without fine-tuning. It suffices that the system can be described as a self-sustained medium which is in an equilibrium state at zero temperature.

Moreover, the thermodynamic arguments above did not depend on the details of the correct microscopic theory at the high-energy scale. In the case of the liquid Helium droplet, this implies that we can deduce the vanishing of the relevant vacuum energy density without solving the microscopic theory at the level of individual atoms. Similarly, in the case of the effective cosmological constant, this suggests that the cosmological constant problem can be understood in terms of a low-energy effective theory.

1.2.2. Origin of q and four-form realization

So far, we have not discussed possible origins of the vacuum variable q. And as already noted above, the discussion of the equilibrium of q-theory in the previous section has been completely independent of this origin of q.

However, if we try to go beyond a description of the equilibrium itself and aim at describing the dynamics of the process of equilibration, we also need to know the dynamics of q. To this end, various suggestions for possible realizations of q-theory have been made. Examples are given by a realization based on an aether-type velocity field [12, 16], a realization based on the Gluon condensate of QCD [17], and the so-called brane realization [18]. However, we will mostly be concerned with another realization of q-theory in this master's thesis, namely the four-form realization [12, 19–21]. Therefore, we will now give a brief introduction to this particular realization.

The four-form realization of q-theory is based on a three-form gauge field A and the associated four-form field strength F. More concretely, let us introduce a rank-three antisymmetric tensor A with components $A_{\alpha\beta\gamma}$, from which a rank-four antisymmetric field-strength tensor F with components $F_{\alpha\beta\gamma\delta}$ can be obtained,

$$F_{\alpha\beta\gamma\delta}(x) \equiv \nabla_{[\alpha}A_{\beta\gamma\delta]}(x) = \partial_{[\alpha}A_{\beta\gamma\delta]}(x) \,. \tag{1.19}$$

Note that the covariant derivatives ∇_α in Eq. (1.19) can be replaced by partial derivatives ∂_α due to the antisymmetrization, which is employed in order to obtain an antisymmetric field strength tensor F. In a four-dimensional spacetime, every rank-four antisymmetric tensor, such as the field-strength tensor F, must be proportional to the Levi-Civita symbol. This observation allows us to define the q-variable of the four-form realization of q-theory,

$$F_{\alpha\beta\gamma\delta}(x) \equiv q(x)\,\sqrt{-g(x)}\,\epsilon_{\alpha\beta\gamma\delta} \,. \tag{1.20}$$

This q-variable is a pseudoscalar, since F is a tensor and the components $\sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$ correspond to a pseudotensor. Using Eq. (1.19), we can also express the q-variable from Eq. (1.20) directly in terms of A,

$$q(x) = -\frac{1}{\sqrt{-g(x)}}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha A_{\beta\gamma\delta} \,. \tag{1.21}$$

A three-form gauge field has already been discussed many years ago in the context of the cosmological constant problem [2, 22, 23]. The reason can be understood by considering the following action:

$$S_1 = -\int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} + \epsilon(q)\right) \,. \tag{1.22}$$

The equations of motion for A as well as the gravitational energy-momentum tensor implied by the action S_1 can, for example, be obtained by a procedure described in Chapter 3. Explicitly, the equations of motion for A read

$$\partial_\alpha\left(\frac{d\epsilon}{dq}\right) = 0 \,, \tag{1.23}$$

and the corresponding gravitational energy-momentum tensor is given by

$$T_{\alpha\beta} = -g_{\alpha\beta}\left(\epsilon(q) - \frac{d\epsilon}{dq}\,q\right) \,. \tag{1.24}$$

Now consider the case of a potential $\epsilon(q) \propto q^2$ as discussed in Refs. [2, 22, 23]. Then, Eq. (1.23) requires q to be constant, such that the energy-momentum tensor (1.24) is given by a constant times the metric. Hence, the integration constant associated with Eq. (1.23) contributes to the effective cosmological constant λ_{eff}, which is a type of contribution to λ_{eff} distinct from the contributions to λ_{eff} discussed in Sec. 1.1.

From the perspective of q-theory, there is even more motivation for studying the four-form realization. In particular, the transmutation from the energy density $\epsilon(q)$ in the action (1.22) to the energy density $\epsilon - q\, d\epsilon/dq$ in the energy-momentum tensor (1.24) precisely resembles the transmutation from the energy density $\epsilon(q)$ to the thermodynamically relevant energy density $\epsilon_{\text{th}}(q)$ discussed in Sec. 1.2.1. Also, the quantity $d\epsilon/dq$ is constant as a result of Eq. (1.23) such that, following the discussion below Eq. (1.14), $d\epsilon/dq$ indeed plays the role of a chemical potential.

It is important to note that q as defined in Eq. (1.20) is a nonfundamental field. The fundamental fields in the four-form realization of q-theory are the metric and the three-form gauge field. This property allows the four-form realization of q-theory to account for a vanishing effective cosmological constant without fine-tuning in the Lagrangian, which is in contrast to theories of fundamental scalar fields, as discussed in Refs. [16, 18]. In order to understand this difference between a fundamental and a nonfundamental (pseudo-)scalar field regarding the cosmological constant problem, consider a fundamental scalar field φ with an action

$$S_2 = -\int d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} + \epsilon_2(\varphi) \right) . \tag{1.25}$$

The associated equation of motion for the fundamental scalar field φ reads

$$\frac{d\epsilon_2}{d\varphi} = 0 , \tag{1.26}$$

and the gravitational energy-momentum tensor is given by

$$T_{\alpha\beta} = -g_{\alpha\beta}\, \epsilon_2(\varphi) . \tag{1.27}$$

Consequently, in order to have a solution of the φ field equations with a vanishing associated effective cosmological constant, the following equations must hold:

$$\frac{d\epsilon_2}{d\varphi} = 0 , \tag{1.28a}$$

$$\epsilon_2(\varphi) = 0 . \tag{1.28b}$$

Since these are two equations for one quantity, a vanishing effective cosmological constant is possible only if the potential ϵ_2 is fine-tuned.

In contrast, together with the requirement of a vanishing effective cosmological constant, the field equations in the case of the nonfundamental pseudoscalar field q imply

$$\partial_\alpha \left(\frac{d\epsilon}{dq} \right) = 0 , \tag{1.29a}$$

$$\epsilon(q) - \frac{d\epsilon}{dq} q = 0 . \tag{1.29b}$$

Note that Eq. (1.29b) corresponds to the thermodynamic equilibrium condition (1.14) with vanishing external pressure. Now consider any constant value q_0 of q which solves Eq. (1.29b). Then, Eq. (1.29a) is automatically satisfied without requiring any fine-tuning in the potential ϵ. Therefore, we conclude that nonfundamental (pseudo-)scalar fields can indeed account for a vanishing effective cosmological constant without fine-tuning in the Lagrangian. This result is not in conflict with Weinberg's no-go theorem discussed in Sec. 1.1, since this theorem applies to *fundamental* scalar fields with field equations as in Eq. (1.28a) but not to *nonfundamental* (pseudo-)scalar fields with field equations as in Eq. (1.29a) [16, 18].

Above, we have referred to A as a gauge field. The reason is that the q-variable (1.20) is invariant under gauge transformations of the following form:

$$A_{\alpha\beta\gamma}(x) \to A'_{\alpha\beta\gamma}(x) = A_{\alpha\beta\gamma}(x) + \partial_{[\alpha}\lambda_{\beta\gamma]}(x)\,, \tag{1.30}$$

for arbitrary (not necessarily infinitesimal) gauge functions $\lambda_{\beta\gamma}(x)$. Therefore, any action which depends on A only indirectly through q is invariant with respect to the gauge transformations (1.30) as well. An example of such a gauge-invariant action is given by the action S_1 from Eq. (1.22).

Considering a gauge-invariant theory, rather than a gauge-noninvariant theory, of the three-form field A may help in avoiding unacceptable modifications of the Newtonian limit of the Einstein field equations in equilibrium [24]. This is suggested by the example of Dolgov theory [25, 26], which utilizes a gauge-noninvariant theory of a vector field in order to dynamically nullify the effective cosmological constant. In the original Dolgov theory, the Newtonian limit is spoiled due to a large zero-component of the vector field [27] and significant modifications are needed in order to fix the Newtonian limit [28, 29]. In contrast, individual components of a gauge field, such as the three-form gauge field A, cannot be physically significant on their own. Hence, we will always consider the three-form field A as a gauge field in this master's thesis.

1.2.3. Kinetic term for q

In Eq. (1.22) above, we have already introduced one particular action for the four-form realization of q-theory. In the following chapters, we will mainly study a generalization of the action (1.22), which was first introduced in Ref. [30],

$$S_{\text{kin}} = -\int d^4x\,\sqrt{-g}\,\left(\frac{R}{16\pi G(q)} + \frac{1}{2}\,C(q)\,\nabla_\beta\, q\,\nabla^\beta q + \epsilon(q)\right), \tag{1.31}$$

where we take $G(q)$, $\epsilon(q)$, and $C(q)$ to be even functions of q, in order to have a manifestly parity-conserving theory.

In the context of this master's thesis, the most important difference between the action S_{kin} from Eq. (1.31) and the action S_1 from Eq. (1.22) is that S_{kin} contains a term with derivatives of q. In the following, we will refer to this term with derivatives of q as the kinetic term for q. Another difference between S_1 and S_{kin}, which will be of minor importance in this master's thesis, is that Newton's gravitational constant G_N has been replaced by a q-dependent gravitational coupling parameter $G(q)$.

One consequence of having a kinetic term for q is given by the q-ball solution, which has $q \approx q_0 \neq 0$ in the interior region and $q = 0$ in the exterior region corresponding to absolutely empty space, as discussed in Ref. [30]. Additionally, a kinetic term for q gives rise to rapid oscillations of q, which can behave like dark matter [31, 32]. Interestingly, rapid oscillations of q, which may contribute to the observed dark matter density, are also a consequence of a nontrivial gravitational coupling parameter $G(q)$ [19].

In the following, we will not be directly concerned with either the q-ball solution or with how oscillations of q can mimic dark matter. Instead, we will mostly consider other implications of having a kinetic term for the q-variable of the four-form realization of q-theory. Specifically, the outline of this master's thesis is as follows. We first consider gauge-fixing procedures for the three-form gauge field A in Chapter 2. Then, we discuss the structure of the field equations associated with a general action of the four-form realization of q-theory in Chapter 3. In Chapter 4, we study whether or not there are propagating degrees of freedom associated with the three-form gauge field A both in the case of the action S_1 from Eq. (1.22) and in the case of the action S_{kin} from Eq. (1.31). In Chapter 5, we consider the higher-order time derivatives associated with the kinetic term for q. In particular, we determine whether or not these higher-order time derivatives lead to the Ostrogradsky instability. In Chapter 6, we propose a possible quantization procedure for the four-form realization of q-theory with a kinetic term for q. Finally, we conclude in Chapter 7.

2. Gauge fixing

As discussed in Sec. 1.2.2, the four-form realization of q-theory gives rise to a gauge theory. We will now consider two different types of gauge-fixing procedures for the associated three-form gauge field A.

2.1. Complete gauge fixing

The three-form gauge field A has four independent components, i.e., components which are not related to each other by the antisymmetry of A. These independent components can be taken to be, for example, A_{123}, A_{023}, A_{013}, and A_{012}.

We start by observing that we can get rid of any three of these four components by a gauge transformation (1.30). For example, suppose we choose

$$\lambda_{\beta\gamma}(x) = -\frac{1}{2!}\int_{x_0^0}^{x^0} d\tilde{x}^0\, A_{0\beta\gamma}(\tilde{x}^0, x^1, x^2, x^3)\,, \tag{2.1}$$

where x stands for the collection of all coordinates (x^0, x^1, x^2, x^3) and x_0^0 lies inside the allowed range of values of the coordinate x^0. Let us further refer to the three-form field obtained from A by a gauge transformation with gauge functions (2.1) as A'. Then, the components $A'_{0\beta\gamma}$ of A' vanish,

$$A'_{0\beta\gamma} = 0\,. \tag{2.2}$$

As a result, the component A'_{123} is the only nonvanishing component of A' and the expression (1.21) of the gauge-invariant quantity q in terms of A' gives

$$q = -\frac{3!}{\sqrt{-g}}\,\partial_0 A'_{123}\,. \tag{2.3}$$

Eq. (2.3) implies that the most general form of A'_{123} is given by

$$-3!\,A'_{123}(x) = \int_{x_0^0}^{x^0} d\tilde{x}^0\, \sqrt{-g(y)}\,q(y)\Big|_{y=(\tilde{x}^0,\,x^1,\,x^2,\,x^3)} - A_0(x^1, x^2, x^3)\,, \tag{2.4}$$

with arbitrary x^0-independent A_0.

We will now perform another gauge transformation (1.30) with gauge functions

$$\lambda_{23}(x) = -\lambda_{32}(x) = -\frac{1}{2!\cdot 3!}\int_{x_0^1}^{x^1} d\tilde{x}^1\, A_0(\tilde{x}^1, x^2, x^3)\,, \tag{2.5a}$$

$$\lambda_{0\alpha} = \lambda_{\alpha 0} = \lambda_{1\alpha} = \lambda_{\alpha 1} = 0\,, \tag{2.5b}$$

where x_0^1 lines inside the allowed range of values of the coordinate x^1. Then, similar to A', the only nonvanishing component of the resulting three-form field A'' is the component A''_{123}. But in contrast to A'_{123}, A''_{123} is uniquely determined by the gauge-invariant quantity q, since the effect of the gauge transformation with gauge functions (2.5) is to remove the function A_0. In other words, the following choice completely fixes the gauge and can be obtained by the procedure described above:

$$A_{0\beta\gamma} = 0\,, \tag{2.6a}$$

$$A_{123}(x) = -\frac{1}{3!}\int_{x_0^0}^{x^0} d\tilde{x}^0\;\sqrt{-g(y)}\,q(y)\Big|_{y=(\tilde{x}^0,\,x^1,\,x^2,\,x^3)}\,. \tag{2.6b}$$

Further, Eq. (2.3) still applies in the gauge (2.6),

$$q = -\frac{3!}{\sqrt{-g}}\,\partial_0 A_{123}\,. \tag{2.7}$$

In the following, the gauge choice (2.6) will be referred to as the A_{123} gauge.

Since there is nothing special about the component A_{123} of A, the above considerations can be generalized to other components of A. For example, the following choice completely fixes the gauge as well and can be obtained by a similar procedure as the A_{123} gauge (2.6),

$$A_{1\beta\gamma} = 0\,, \tag{2.8a}$$

$$A_{023}(x) = \frac{1}{3!}\int_{x_0^1}^{x^1} d\tilde{x}^1\;\sqrt{-g(y)}\,q(y)\Big|_{y=(x^0,\,\tilde{x}^1,\,x^2,\,x^3)}\,. \tag{2.8b}$$

Using Eq. (2.8), q can be written as

$$q = \frac{3!}{\sqrt{-g}}\,\partial_1 A_{023}\,. \tag{2.9}$$

In the following, the gauge choice (2.8) will be referred to as the A_{023} gauge.

Note that the gauges (2.6) and (2.8) can be obtained independently of the choice of a specific action. It suffices to assume invariance with respect to the gauge transformations (1.30). Furthermore, the gauge-fixing procedures discussed in this section do not depend on the behavior of A at the spacetime boundaries, since all the integrals needed in order to obtain the gauges (2.6) and (2.8) run over finite intervals only.

Nevertheless, there may be subtleties in obtaining the gauges (2.6) and (2.8) in a general spacetime. For example, consider a spacetime requiring multiple charts in order to cover the whole spacetime manifold. Since neither Eq. (2.6) nor Eq. (2.8) are covariant, both Eq. (2.6) and Eq. (2.8) may hold in one particular chart only, if we take A to be a single-valued tensor on the whole spacetime manifold. In contrast, suppose we define components $A_{\alpha\beta\gamma}$ by Eq. (2.6) or Eq. (2.8) in every chart. Then, these components $A_{\alpha\beta\gamma}$ do not in general agree with each other in overlapping regions of two different charts, i.e., they are not in general related by a change of coordinates. However, all $A_{\alpha\beta\gamma}$ give rise to the same gauge-invariant quantity q on the whole spacetime manifold. Therefore, components $A_{\alpha\beta\gamma}$ defined in two different charts differ at most by a gauge transformation in any overlapping region of these charts (cf. the treatment of Dirac monopoles in Sec. 2.7 of Ref. [33]). In the following chapters, these subtleties will not play an important role.

2.2. Generalized Lorenz gauge

In the following chapters, we will mainly employ the gauge choices (2.6) and (2.8) discussed in the previous section. Still, we would like to consider one additional gauge choice, which, in flat Minkowski spacetime, is defined by

$$\partial_\alpha A^{\alpha\beta\gamma} = 0\,. \tag{2.10}$$

This gauge choice can be regarded as a generalized Lorenz gauge due to the similarity between Eq. (2.10) and the Lorenz gauge condition for the vector field A^{ED} in electrodynamics,

$$\partial_\alpha A^{\alpha,\mathrm{ED}} = 0\,. \tag{2.11}$$

Obtaining the gauge (2.10) is equivalent to solving the following equations for the gauge functions $\lambda^{\beta\gamma}$:

$$\begin{aligned} 0 &= \partial_\alpha A^{\alpha\beta\gamma} + \partial_\alpha \partial^{[\alpha}\lambda^{\beta\gamma]} \\ &= \partial_\alpha A^{\alpha\beta\gamma} + 2!\left(\partial_\alpha\partial^\alpha\lambda^{\beta\gamma} - \partial_\alpha\partial^\beta\lambda^{\alpha\gamma} + \partial_\alpha\partial^\gamma\lambda^{\alpha\beta}\right)\,. \end{aligned} \tag{2.12}$$

One particular possibility to satisfy Eq. (2.12) is

$$\Box\,\lambda^{\beta\gamma} = -2!\,\partial_\alpha A^{\alpha\beta\gamma}\,, \tag{2.13a}$$

$$\partial_\beta\lambda^{\beta\gamma} = 0\,. \tag{2.13b}$$

If $\partial_\alpha A^{\alpha\beta\gamma}$ vanishes sufficiently fast for $|\vec{x}|, |t| \to \infty$, Eq. (2.13a) can be solved using, for example, the standard retarded Green's function for the d'Alembert operator,

$$\lambda^{\beta\gamma}(t,\vec{x}) \propto \int d^3\vec{x}'\,\frac{(\partial_\alpha A^{\alpha\beta\gamma})(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}\,. \tag{2.14}$$

In Appendix A.1, it is shown that Eq. (2.13b) is then automatically fulfilled due to the antisymmetry of A.

However, in contrast to the A_{123} gauge and the A_{023} gauge discussed in the previous section, it is not immediately clear, whether or not the generalized Lorenz gauge (2.10) is still applicable, if $\partial_\alpha A^{\alpha\beta\gamma}$ does not vanish sufficiently fast for $|\vec{x}|, |t| \to \infty$. Another difference is that the gauge choices (2.6) and (2.8) fix the gauge completely, while Eq. (2.10) leaves a residual gauge freedom.

More precisely, consider two gauge-equivalent three-form fields A and A'. If both A and A' satisfy either Eq. (2.6) or Eq. (2.8), they necessarily have to be identical. In contrast, if both A and A' satisfy Eq. (2.10), A and A' can differ. To see this, first note that the condition that A and A' are gauge-equivalent implies

$$A^{\alpha\beta\gamma\prime} = A^{\alpha\beta\gamma} + \partial^{[\alpha}\lambda^{\beta\gamma]}\,, \tag{2.15}$$

for some gauge functions $\lambda^{\beta\gamma}$. As both A and A' satisfy the generalized Lorenz gauge condition (2.10), Eq. (2.15) then gives

$$0 = \Box\,\lambda^{\beta\gamma} - \partial_\alpha\partial^\beta\lambda^{\alpha\gamma} + \partial_\alpha\partial^\gamma\lambda^{\alpha\beta}\,. \tag{2.16}$$

Hence, A and A' can differ by a gauge transformation with gauge functions $\lambda^{\beta\gamma}$ satisfying Eq. (2.16), the reason being that Eq. (2.10) does not completely fix the gauge.

This last result is analogous to the case of electrodynamics, where two gauge-equivalent vector fields satisfying Eq. (2.11) can differ by a gauge transformation $A^{\mathrm{ED}}_{\alpha} \to A^{\mathrm{ED}}_{\alpha} + \partial_{\alpha}\omega$ with

$$\Box\,\omega = 0\,. \tag{2.17}$$

This phenomenon of incomplete gauge fixing is discussed in Ref. [34] both for electrodynamics and for the more general case of non-Abelian gauge theories of a vector field. One particular issue with incomplete gauge fixing is that, when quantizing a theory using path integrals, one needs to be careful not to double-count gauge-equivalent configurations. A relatively straightforward solution in the case of electrodynamics in the Lorenz gauge (2.11) is restricting the theory to fields with finite spacetime support [34].

A similar fix may or may not apply to the three-form gauge field A in the generalized Lorenz gauge (2.10). However, we will not further investigate this issue of incomplete gauge fixing in the generalized Lorenz gauge, since we will mostly employ the A_{123} gauge (2.6) as well as the A_{023} gauge (2.8) in this master's thesis.

3. Structure of the field equations

In Sec. 1.2.2, we had a first look at the field equations of a fundamental scalar field as compared with those of the nonfundamental pseudoscalar q-variable of the four-form realization of q-theory. In Sec. 3.1, we will now further study the general structure of the field equations of the four-form realization of q-theory by comparing the nonfundamental pseudoscalar field q with a fundamental scalar field. In Sec. 3.2, we will generalize the results from Sec. 3.1 for the four-form realization of q-theory to certain other realizations of q-theory.

3.1. Four-form realization

Let us start by introducing two actions, which we will compare with each other further below. First, consider the action S_{f} of a fundamental scalar field ϕ which is given by the spacetime integral over a Lagrangian density $\mathcal{L}$ and an additional term linear in ϕ,

$$S_{\mathrm{f}}[\phi, \mathsf{g}] = \int d^4x \sqrt{-g}\, \left(\mathcal{L}[\phi, \mathsf{g}] + \mu_{\mathrm{f}}\, \phi\right) . \tag{3.1}$$

Here, μ_{f} is an arbitrary constant and the metric with tensor indices omitted for readability is denoted by g in order to distinguish the metric from its determinant g. Second, consider the action S_{nf} of the nonfundamental pseudoscalar field q of the four-form realization of q-theory,

$$S_{\mathrm{nf}}[q(A, \mathsf{g}), \mathsf{g}] = \int d^4x \sqrt{-g}\, \mathcal{L}[q(A, \mathsf{g}), \mathsf{g}] , \tag{3.2}$$

where $\mathcal{L}$ is the same functional as in Eq. (3.1) and A stands for the collection of all components $A_{\alpha\beta\gamma}$. Note that ϕ must have the same mass dimension as q as a consequence of using the same functional $\mathcal{L}$ in both S_{f} and S_{nf}. In particular, this implies that ϕ has mass dimension 2 instead of 1, which would be the usual mass dimension of a fundamental scalar field. The reason for using the same $\mathcal{L}$ in both actions is that ϕ and q can be more easily compared with each other in this case.

As a further preparation, let us define the quantity μ,

$$\mu \equiv -\frac{1}{\sqrt{-g}} \frac{\delta S_{\mathrm{nf}}}{\delta q} . \tag{3.3}$$

In the case of the action S_1 from Eq. (1.22), μ is given by

$$\mu = \frac{d\epsilon}{dq} , \tag{3.4}$$

which can be interpreted as a chemical potential, as discussed in Sec. 1.2.1. In the following, we will therefore refer to μ as a chemical potential also for more general actions like S_{nf}. Below, we will show that the equations of motion for A in the case of the general action S_{nf} lead to the emergence of an integration constant. In the particular case of the action S_1, Eq. (1.23) implies that this integration constant is determined by μ as given in Eq. (3.4). Similarly, it turns out that the integration constant in the case of the general action S_{nf} is determined by μ as given in Eq. (3.3). In other words, μ is required to be constant by the equations of motion for A. Note, however, that we employ the definition (3.3) independently of whether or not the equations of motion are imposed.

Our main result regarding the field equations associated with S_{nf} and S_{f} is now given by the following proposition:

Proposition 1

1. *The equations of motion for A as derived from S_{nf} are equivalent to μ being a constant,*

$$0 = \partial_\alpha \left(-\frac{1}{\sqrt{-g}} \frac{\delta S_{nf}}{\delta q} \right) = \partial_\alpha \, \mu \, . \tag{3.5}$$

2. *For a fixed constant value of μ with $\mu = \mu_f$, the equations of motion for A and g as derived from S_{nf} have the same formal structure as the equations of motion for ϕ and g as derived from S_f. More precisely,*

$$-\frac{1}{\sqrt{-g}} \frac{\delta S_{nf}}{\delta q} = \mu = \mu_f \, , \; \frac{\delta S_{nf}}{\delta g_{\alpha\beta}} = 0 \quad \Leftrightarrow \quad \left. \frac{\delta S_f}{\delta \phi} \right|_{\phi = q} = 0 \, , \; \left. \frac{\delta S_f}{\delta g_{\alpha\beta}} \right|_{\phi = q} = 0 \, . \tag{3.6}$$

A proof of this proposition is given in Appendix B.1. Note that Proposition 1 does not say that S_{nf} and S_{f} have *equivalent* associated field equations. This is because Eq. (3.3) implies that the integration constant μ transforms nontrivially under parity, since q is a pseudoscalar. In contrast, μ_{f} is simply a parameter in the action S_{f} which does not transform under parity at all.

We can now make use of Proposition 1 in order to obtain the equations of motion for both the three-form gauge field A and the metric g in the case of the action S_{kin} from Eq. (1.31), i.e., in the case of an action which includes a kinetic term for q as well as a nontrivial gravitational coupling parameter $G(q)$.[1] First, according to (3.5), the equations of motion for A are solved by having an arbitrary constant value of μ. In particular, the following quantity must be constant:

$$\mu = -\frac{1}{\sqrt{-g}} \frac{\delta S_{\text{kin}}}{\delta q} = \frac{d\epsilon(q)}{dq} - \frac{1}{2} \frac{dC(q)}{dq} \nabla_\beta \, q \, \nabla^\beta q - C(q) \, \Box \, q + \frac{R}{16\pi} \frac{dG^{-1}(q)}{dq} \, . \tag{3.7}$$

Next, according to (3.6), the equations of motion for g associated with the action S_{kin} are given by

$$0 = \left. \frac{\delta S_{\text{kin,f}}}{\delta g_{\alpha\beta}} \right|_{\phi = q} \, . \tag{3.8}$$

[1]Using a different approach, these equations of motion were first derived in Ref. [35].

Here, $S_{\text{kin,f}}$ is obtained from S_{kin} by replacing q by ϕ and adding the spacetime integral of $\mu\phi$, treating μ as a field-independent constant. Explicitly, we find

$$\begin{aligned}0 =& \frac{1}{8\pi G(q)}\left(R_{\alpha\beta} - \frac{1}{2}R\, g_{\alpha\beta}\right) + \frac{1}{8\pi}\left(\nabla_\alpha\nabla_\beta\, G^{-1}(q) - g_{\alpha\beta}\,\Box\, G^{-1}(q)\right)\\ &- g_{\alpha\beta}\left(\epsilon(q) - \mu\, q\right)\\ &- g_{\alpha\beta}\,\frac{1}{2}C(q)\,\nabla_\gamma\, q\,\nabla^\gamma q + C(q)\nabla_\alpha\, q\,\nabla_\beta\, q\,.\end{aligned} \tag{3.9}$$

Up to the μq term, Eq. (3.9) is what we would have obtained by simply treating q as a fundamental (pseudo-)scalar field and completely forgetting about the three-form gauge field A. As a result, perturbations of q around the equilibrium value q_0 can behave like dark matter, as argued in Ref. [31]. In particular, having derivative terms with the same formal structure as in the case of a fundamental scalar field is important for this result of Ref. [31].

It may be convenient to rephrase Proposition 1 in a particular way, which we will illustrate using the action S_{kin} with the associated Eqs. (3.7) and (3.9). Specifically, consider a fixed constant value of the integration constant μ. Then, Proposition 1 is equivalent to the statement that Eqs. (3.7) and (3.9) can be obtained by replacing the potential $\epsilon(q)$ in S_{kin} with an effective potential $\epsilon_{\text{eff},\mu}(q)$,

$$\epsilon_{\text{eff},\mu}(q) \equiv \epsilon(q) - \mu\, q\,, \tag{3.10}$$

and subsequently treating q as a fundamental (pseudo-)scalar field when deriving the equations of motion. The reason is that the $\mu_{\text{f}}\,\phi$ term in the action S_{f} from Eq. (3.1) can always be absorbed into a potential term for ϕ, if such a potential term exists.

We note that the above-discussed connection between the field equations of the four-form realization of q-theory and those of a fundamental scalar field was already noted in Sec. II of Ref. [19] in the case of the action S_{kin} without a kinetic term for q, $C(q) = 0$.

3.2. Generalization to other realizations of q-theory

Although we will exclusively be concerned with the four-form realization of q-theory in the following chapters, we would like to note that Proposition 1 can be generalized to certain other realizations of q-theory.

In particular, consider a scalar or pseudoscalar q-variable $\overline{q} = \overline{q}[\overline{A}, g]$, which is constructed as a local functional from a fundamental bosonic field $\overline{A}$ and the metric g and satisfies $\overline{q}|_{\overline{A}=0} = 0$. Similar to Eqs. (3.1) and (3.2), we further define

$$\overline{S}_{\text{f}}[\phi, g] = \int d^4x\sqrt{-g}\,\left(\mathcal{L}[\phi, g] + \overline{\mu}_{\text{f}}\,\phi\right)\,, \tag{3.11a}$$

$$\overline{S}_{\text{nf}}[\overline{q}(\overline{A}, g), g] = \int d^4x\sqrt{-g}\,\mathcal{L}[\overline{q}(\overline{A}, g), g]\,. \tag{3.11b}$$

Then, the following Proposition, which is proven in Appendix B.2, holds:

Proposition 2
Suppose that, for any $\overline{S}_{nf}$, the equations of motion for $\overline{A}$ as derived from $\overline{S}_{nf}$ are equivalent to requiring that the quantity $\overline{\mu}$,

$$\overline{\mu} \equiv -\frac{1}{\sqrt{-g}}\frac{\delta \overline{S}_{nf}}{\delta \overline{q}}\,, \tag{3.12}$$

is constant,

$$\partial_\alpha \overline{\mu} = 0\,. \tag{3.13}$$

Then, the equations of motion for any given $\overline{S}_{nf}$ and $\overline{\mu}$ have the same formal structure as those of the corresponding $\overline{S}_f$ with $\overline{\mu} = \overline{\mu}_f$. More explicitly, the second part of Proposition 1 holds with q, A, S_{nf}, μ, S_f, and μ_f replaced by $\overline{q}$, $\overline{A}$, $\overline{S}_{nf}$, $\overline{\mu}$, $\overline{S}_f$, and $\overline{\mu}_f$, respectively.

Let us make a few remarks regarding Proposition 2. First, the discussion below Eq. (3.4) carries over to realizations of q-theory to which Proposition 2 applies. In particular, the quantity $\overline{\mu}$ from Eq. (3.12) can be interpreted as a chemical potential. Second, the proof of Proposition 2 shows that any $\overline{q}$ to which the proposition applies, is necessarily of the form

$$\overline{q} = \nabla_\alpha V^\alpha[\overline{A}, g]\,, \tag{3.14}$$

for some (pseudo-)vector $V^\alpha[\overline{A}, g]$. Conversely, having $\overline{q} = \nabla_\alpha V^\alpha$ is not sufficient for Proposition 2 to apply, since counterexamples can be constructed. And third, if the $\overline{\mu}_\mathrm{f}\,\phi$ term in $\overline{S}_\mathrm{f}$ is omitted, the derivative terms in the equations of motion associated with $\overline{S}_\mathrm{f}$ do not change. Therefore, these derivative terms are still the same as those in the equations of motion of $\overline{S}_\mathrm{nf}$. This suggests that any realization of q-theory to which Proposition 2 applies, can lead to dark-matter-like behavior in the sense of Refs. [31, 32].

4. Propagating degrees of freedom

In this chapter, we will discuss whether or not there are propagating degrees of freedom associated with the three-form gauge field A. Our main interest will be the action S_{kin} from Eq. (1.31) with a nonvanishing kinetic term for q, $C \neq 0$. But before discussing the effects of a kinetic term for q, we will first review the case without such a kinetic term, $C = 0$, in Sec. 4.1. In Sec. 4.2, we will then consider the case with $C \neq 0$.

Here, we will not discuss the gravitationally-induced dynamics of A, which arise if q is coupled to the Ricci scalar R through a nontrivial gravitational coupling parameter $G(q)$ [19, 36]. Instead, we are interested in the dynamics associated with A itself. Therefore, we take $G(q) = G_N$ in this chapter.

4.1. Without kinetic term for q

Let us now consider the case without a kinetic term for q and with $G(q) = G_N$. This choice of parameters corresponds to considering the action S_1 from Eq. (1.22) instead of the action S_{kin} from Eq. (1.31). For a quadratic potential $\epsilon(q)$, this action S_1 has already been discussed in Refs. [22, 23] regarding propagating degrees of freedom associated with the three-form gauge field A. We will briefly review the arguments from Refs. [22, 23] in Secs. 4.1.1 and 4.1.2. In Sec. 4.1.3, we will additionally discuss an analogy with electrodynamics, confirming the standard result that there are no propagating degrees of freedom associated with A, if we employ the action S_1 without a kinetic term for q.

4.1.1. Equations-of-motion argument

The argument from Ref. [23] regarding propagating degrees of freedom associated with the three-form gauge field A is based on an analogy between A in four spacetime dimensions and a vector gauge field A^{S} in two spacetime dimensions. In order to understand this analogy, consider the field strength tensor F^{S} associated with A^{S},

$$F^{\text{S}}_{\alpha\beta} \equiv \partial_\alpha A^{\text{S}}_\beta - \partial_\beta A^{\text{S}}_\alpha \,. \tag{4.1}$$

Since F^{S} is an antisymmetric rank-two tensor in two spacetime dimensions, it must be proportional to the two-dimensional Levi-Civita symbol $\epsilon_{\alpha\beta}$. Similar to the definition of q in Eq. (1.20), we can then define a quantity q^{S},

$$F^{\text{S}}_{\alpha\beta} \equiv \sqrt{-g_2}\, q^{\text{S}}\, \epsilon_{\alpha\beta} \,, \tag{4.2}$$

where g_2 is the determinant of the two-dimensional spacetime metric $g_{2,\alpha\beta}$. For an action which contains a potential $\epsilon^{\text{S}}(q^{\text{S}})$ and no further coupling of q^{S} to other fields, the A^{S} field equations and the gravitational energy-momentum tensor associated with A^{S} read

$$0 = \partial_\alpha \left(\frac{d\epsilon^{\mathrm{S}}}{dq^{\mathrm{S}}} \right) , \tag{4.3a}$$

$$T^{\mathrm{S}}_{\alpha\beta} = -g_{2,\alpha\beta} \left(\epsilon^{\mathrm{S}}(q^{\mathrm{S}}) - q^{\mathrm{S}} \frac{d\epsilon^{\mathrm{S}}}{dq^{\mathrm{S}}} \right) . \tag{4.3b}$$

Eq. (4.3) now establishes the analogy of the theory of A^{S} with a potential ϵ^{S} to the three-form gauge field A with an action S_1. In particular, Eq. (4.3a) corresponds to the A field equations (1.23) and Eq. (4.3b) corresponds to the gravitational energy-momentum tensor (1.24) associated with A.

Further, it follows from Eq. (4.3a) that the gauge-invariant quantity q^{S} must be constant. The gravitational energy-momentum tensor (4.3b) is then given by a constant times the metric. More generally, this constant value of q^{S} completely determines all gauge-invariant quantities. Therefore, the theory contains no propagating degrees of freedom, the only physical degree of freedom is the integration constant of Eq. (4.3a).

The above argument holds, as long as $d\epsilon^{\mathrm{S}}/dq^{\mathrm{S}}$ is not a q^{S}-independent constant for any finite interval of values of q^{S}. This is because the equations of motion (4.3a) do not restrict A^{S} at all, if q^{S} stays inside an interval where $d\epsilon^{\mathrm{S}}/dq^{\mathrm{S}}$ is constant. Here, we will not investigate potentials leading to such pathological behavior any further and assume that q^{S} is required to be constant by Eq. (4.3a). In other words, we assume that ϵ^{S} cannot be written as $\epsilon^{\mathrm{S}}(q^{\mathrm{S}}) = \Lambda + a\, q^{\mathrm{S}}$ for any finite interval of values of q^{S} as well as for any constants Λ and a.

We can now argue that there are no propagating degrees of freedom associated with the three-form gauge field A with an action S_1 due to the above-mentioned analogy [23]. Here, we assume that, similar to $\epsilon^{\mathrm{S}}(q^{\mathrm{S}})$, $\epsilon(q)$ is not a polynomial of degree one in q for any finite interval of values of q. As a result, q is required to be constant by the A field equations and the above arguments regarding the absence of propagating degrees of freedom associated with A^{S} carry over to the three-form gauge field A.

4.1.2. Gauge-fixing argument

In Ref. [22], a path integral corresponding to a quantum theory of the three-form gauge field A is considered. Strictly speaking, only a quadratic potential, $\epsilon \propto q^2$, is considered. But in order to find the perturbative particle spectrum, the arguments from Ref. [22] apply as well to a potential which is expanded to quadratic order around, for example, an equilibrium configuration (1.29).

The argument regarding propagating degrees of freedom associated with A from Ref. [22] is then based on a gauge-fixing procedure for the above-mentioned path integral. This gauge-fixing procedure is introduced in Ref. [37] and can be described as a generalized Faddeev-Popov method [38], since ghost fields are introduced until all gauge degrees of freedom are cancelled. After this procedure is completed, the action inside the path integral contains a term proportional to $\bar{c}\,\Box\, c$ for each complex, anti-commuting ghost field and a term proportional to $\psi\,\Box\,\psi$ for each real, commuting field ψ, where ψ denotes either a component of A or one of the real, commuting ghost fields.

The number of particles in the perturbative particle spectrum is then obtained by counting each real, commuting field ψ as +1 and each complex, anti-commuting field c as -2. Including all the components of the three-form gauge field A as well as the various ghost fields, this counting yields zero [22], which suggests that there are no propagating degrees of freedom associated with A, if the action S_1 is employed.

4.1.3. Discussion

Above, we have reviewed two arguments which suggest that there are no propagating degrees of freedom associated with the three-form gauge field A, if the action S_1 is employed. In Sec. 4.1.1, the argument was that, classically, all gauge-invariant quantities are determined by the integration constant associated with the A field equations. In Sec. 4.1.2, the argument was that a certain path-integral quantization does not lead to any perturbative propagating degrees of freedom.

Thus, the main effect of the three-form gauge field A with the action S_1 from Eq. (1.22) seems to be to provide a constant q-field. In particular, no localized solutions of the gauge-invariant quantity q are allowed in the absence of external sources.

This last remark can be illustrated by considering an external source J which, in flat Minkowski spacetime and in the case of $\epsilon \propto q^2$, can be taken to act like

$$\partial_\alpha F^{\alpha\beta\gamma\delta}(x) = J^{\beta\gamma\delta}(x) \,. \tag{4.4}$$

By adjusting J, we can push $F_{\alpha\beta\gamma\delta}$, and therefore also q, to have arbitrary spacetime-dependence. Now suppose that the source J has acted for some time and has thereby brought q to a localized configuration. We will now consider what happens, when the source is smoothly switched off. If there were propagating degrees of freedom, we would expect that the localized configuration spreads out or relaxes to a constant configuration after the source has been switched off. However, this is not what happens here. After the source has been switched off, the equations of motion for A (1.23) require q to be constant. Hence, q necessarily relaxes to a constant value already during the switching-off of the source, indicating that there are no propagating degrees of freedom.

It may also be illustrative to compare the three-form gauge field A to the vector field A^{ED} from electrodynamics. The most general solution of the free vector potential in electrodynamics involves not only the usual expansion in plane waves, but also constant electric and magnetic fields, which cannot be expanded in plane waves, as noted in Ref. [12]. This means that, symbolically,

$$A^{\text{ED}}_\alpha = -\frac{1}{2}\bar{F}^{\text{ED}}_{\alpha\beta}x^\beta + (\text{plane waves}) \,, \tag{4.5}$$

where $\bar{F}^{\text{ED}}$ is a constant antisymmetric tensor representing the constant electric and magnetic fields mentioned above. Upon quantization, the plane-wave solutions give rise to an interpretation in terms of photons. In contrast, it may not be obvious how to correctly treat $\bar{F}^{\text{ED}}$ in a quantized theory. However, $\bar{F}^{\text{ED}}$ will most likely not correspond to additional propagating degrees of freedom, but rather represent constant background magnetic and electric fields.

In analogy with the $\bar{F}^{\text{ED}}$ term on the right-hand side of Eq. (4.5), the equations of motion for the three-form gauge field A (1.23) in flat Minkowski spacetime and for $\epsilon \propto q^2$ have solutions of the form

$$A_{\alpha\beta\gamma} = -\frac{1}{4!}\bar{F}_{\alpha\beta\gamma\delta}x^{\delta}\,, \tag{4.6}$$

where $\bar{F}$ is a constant rank-four antisymmetric tensor corresponding to a constant value of q. The analogy with electrodynamics now suggests that there would be propagating degrees of freedom, if there were solutions for A of the form

$$A_{\alpha\beta\gamma} = -\frac{1}{4!}\bar{F}_{\alpha\beta\gamma\delta}x^{\delta} + (\text{plane waves})\ . \tag{4.7}$$

Indeed, terms corresponding to plane waves in A lead to a nontrivial associated field strength. However, in Appendix A.2, plane-wave solutions are treated explicitly in the generalized Lorenz gauge (2.10) and it is shown that all plane-wave solutions can be gauged away, leaving no potentially propagating degrees of freedom. Plane-wave solutions can also be ruled out by simply noting that they are incompatible with constant q, which is required by the equations of motion for A.

We conclude that the three-form gauge field A with the action S_1 from Eq. (1.22) is not associated with any propagating degrees of freedom. This conclusion may not hold, if $d\epsilon/dq$ is constant for some finite interval of values of q. However, in this case, the A field equations do not restrict A at all, as long as q stays inside the interval where $d\epsilon/dq$ is constant. Therefore, we will not consider such pathological potentials in this master's thesis.

4.2. With kinetic term for q

In the previous section, we have seen that the three-form gauge field A without a kinetic term for q is not associated with any propagating degrees of freedom in the cases we are interested in. We will now consider a nonvanishing kinetic term for q, $C \neq 0$, keeping the gravitational coupling parameter $G(q)$ constant, $G(q) = G_N$. We will show that the kinetic term for q may lead to a propagating mode associated with q, as already discussed in Ref. [30]. Compared to the discussion in Ref. [30], we will allow for a more general form of the function $C(q)$.

Before going into the details, we note that the difference between a vanishing and a nonvanishing kinetic term for q regarding propagating degrees of freedom cannot be associated with gauge fixing, at least not if we use one of the gauge-fixing procedures discussed in Sec. 2.1. This is because the gauge-fixing procedures from Sec. 2.1 fix the gauge completely and are independent of the chosen action. Hence, the reason for the possibility of a propagating degree of freedom in the case of $C \neq 0$ is to be sought somewhere else.

We start by considering the equilibrium of q-theory for the action S_{kin} from Eq. (1.31). As Lorentz invariance requires the q-variable in equilibrium to be constant, Eq. (3.9) implies that Minkowski spacetime is obtained by the same condition (1.29b) as in the case of the action S_1 from Eq. (1.22),

$$\epsilon(q_0) - q\frac{d\epsilon(q)}{dq}\bigg|_{q=q_0} = 0\,. \tag{4.8}$$

This corresponds to an equilibrium value μ_0 of the quantity μ from Eq. (3.3) given by

$$\mu_0 = \frac{d\epsilon(q)}{dq}\bigg|_{q=q_0}\,. \tag{4.9}$$

For concreteness, we will consider quartic potentials $\epsilon(q)$ in this section,

$$\epsilon(q) = \Lambda - a\,q^2 + b\,q^4\,. \tag{4.10}$$

Here, Λ corresponds to a cosmological constant and $a > 0$, $b > 0$ are constants.

For $\Lambda \neq 0$, Eq. (4.8) only has solutions with $q = q_0 \neq 0$. These solutions correspond to the equilibrium discussed in Sec. 1.2.1, where a cosmological constant is dynamically cancelled by the q-field. For $\Lambda = 0$, in contrast, $q = 0$ is also a solution of Eq. (4.8). This solution with $\Lambda = 0$ and $q = 0$ corresponds to a completely empty universe, which does not contain any type of quantum field and is therefore also devoid of the associated quantum fluctuations [30]. In the following, we will refer to this case with $\Lambda = 0$ and $q = 0$ as the empty “vacuum”, where we have put quotation marks around the word vacuum for reasons that will be explained further below in Sec. 4.2.2.

Let us now consider a linear perturbation $\varphi(x)$ of the constant solution q_0, keeping the integration constant μ fixed at the value μ_0 from Eq. (4.9),

$$q(x) \equiv q_0 + \varphi(x)/\sqrt{|C(q_0)|}\,. \tag{4.11}$$

Here, we allow for both zero and nonzero values of q_0, corresponding to the different types of solutions discussed in the previous paragraph.

Expanding to leading order in φ, the A field equations (3.7) as well as the energy-momentum tensor derived from Eq. (3.9) are given by

$$0 = \Box\,\varphi - m^2(\mu_0)\,\varphi\,, \tag{4.12a}$$

$$T_{\alpha\beta} = -\eta_{\alpha\beta}\,\sigma_0\left(\frac{1}{2}\,m^2(\mu_0)\,\varphi^2 + \frac{1}{2}\,\partial_\gamma\,\varphi\,\partial^\gamma\,\varphi\right) + \sigma_0\,\partial_\alpha\,\varphi\,\partial_\beta\,\varphi\,, \tag{4.12b}$$

$$m^2(\mu_0) \equiv \frac{1}{C(q_0)}\,\frac{d^2\epsilon}{dq^2}\bigg|_{q=q_0}\,, \tag{4.12c}$$

where σ_0 denotes the sign of $C(q_0)$. The energy density derived from Eq. (4.12b) is then given by

$$T_{00} = \frac{1}{2}\,\sigma_0\left(m^2(\mu_0)\,\varphi^2 + (\partial_0\varphi)^2 + (\partial_a\varphi)^2\right)\,. \tag{4.13}$$

We see that $C(q_0)$ must be positive in order to ensure a positive energy density. In the following, we will therefore assume that $C(q) \geq 0$, which implies $\sigma_0 = 1$ for nonvanishing

$C(q_0)$. Further, since the energy-momentum tensor (4.12b) is quadratic in φ, we can neglect the effect of φ on the metric in linear approximation.

The main observation regarding Eq. (4.12) is now that φ satisfies a Klein–Gordon equation (4.12a) with mass-square (4.12c). The energy-momentum tensor (4.12b) also has precisely the same form as the energy-momentum tensor of a standard fundamental scalar field satisfying the Klein–Gordon equation (4.12a). In the remaining part of this section, we will expand on this observation.

4.2.1. Physical equilibrium

Let us first consider the case of $\Lambda \neq 0$ and $q_0 \neq 0$. As noted above, this case corresponds to the physical equilibrium of q-theory described in Sec. 1.2.1. Consequently, the inverse isothermal vacuum compressibility χ_{vac}^{-1} from Eq. (1.16) is positive. For $q_0 \neq 0$, positive vacuum compressibility implies that $d^2\epsilon/dq^2|_{q=q_0}$ is positive as well. Therefore, we have

$$m^2(\mu_0) \geq 0\,, \tag{4.14}$$

where we have used $C(q_0) \geq 0$. Alternatively, Eq. (4.14) can be derived using the explicit form of $\epsilon(q)$ from Eq. (4.10).

If we assume that $C(q_0) > 0$, $m^2(\mu_0)$ takes a finite value according to Eq. (4.12c). Then, perturbations around the equilibrium behave like a standard perturbative propagating degree of freedom with mass-square $m^2(\mu_0)$. Given a particular solution for φ, we can also find a corresponding solution for the three-form gauge field A using, for example, the A_{123} gauge (2.6). However, Eqs. (4.12) and (4.13) show that it is really q and not A with which the propagating degree of freedom is associated. Therefore, we are mainly interested in φ and not in A in this section. Eqs. (4.12) and (4.13) further suggest that φ rather than A will have the standard form with respect to creation and annihilation operators upon quantization.[1]

Moreover, if we assume the only scale in $\epsilon(q)$ as well as $C(q)$ to be given by the Planck scale $E_\text{P} \sim 1/\sqrt{G_N}$, we can estimate $m^2(\mu_0)$ to be determined by the Planck scale as well,

$$m^2(\mu_0) \sim 1/C(q_0) \sim (E_\text{P})^2\,. \tag{4.15}$$

Next, consider the case with $C(q_0) = 0$. According to Eq. (4.12c), $C(q_0) = 0$ corresponds to an infinite mass $m^2(\mu_0)$. Consequently, perturbations around the equilibrium do not correspond to a propagating degree of freedom for $C(q_0) = 0$. One might argue that the definition of the perturbation φ in Eq. (4.11) is not even well-defined for $C(q_0) = 0$, implying that the quantity $m^2(\mu_0)$ is not meaningful in this case. However, the result that $C(q_0) = 0$ corresponds to not having a propagating degree of freedom can also be obtained in a simpler way. This is because, at least for linear perturbations around q_0, $C(q_0) = 0$ corresponds to not having a kinetic term for q at all such that the results from the previous section apply.

To sum up, in the case of the physical equilibrium, there is a propagating mode with mass given by Eq. (4.12c) as long as $C(q_0) > 0$. As already mentioned at the beginning

[1]For a possible quantization procedure see Chapter 6.

of this section, we have arrived at this result without having to deal with gauge fixing. The reason can now be seen to be that this propagating mode is associated with the gauge-invariant quantity q rather than with the gauge field A.

4.2.2. Empty "vacuum"

We will now consider the case of the empty "vacuum", $\Lambda = 0$, $q_0 = 0$. For quartic potentials (4.10), we then have

$$\left.\frac{d^2\epsilon}{dq^2}\right|_{q=0} < 0\,. \tag{4.16}$$

Hence, using $C(q) \geq 0$, we obtain from Eq. (4.12c)

$$m^2(0) < 0\,. \tag{4.17}$$

Thus, $m^2(0)$ takes a finite negative value for $C(0) \neq 0$ such that perturbations around $q = 0$ correspond to an instability rather than a propagating degree of freedom [39]. To illustrate this point, consider $C = \text{const} > 0$ and a fixed constant value of μ, $\mu = \mu_0 = 0$. Then, the full, nonlinear equation of motion (3.7) written in terms of φ reads

$$0 = \Box\,\varphi - \frac{d\epsilon(\varphi/\sqrt{C})}{d\varphi}\,, \tag{4.18}$$

where ϵ is the quartic potential from Eq. (4.10). Note that Eq. (4.18) has the same formal structure as the equation of motion of a fundamental (pseudo-)scalar field φ with a standard kinetic term and a quartic potential $\epsilon(\varphi/\sqrt{C})$. We can now see that expanding around $q = 0$ or, equivalently, around $\varphi = 0$, corresponds to expanding around a maximum rather than a minimum of the potential ϵ. The reason is that $\epsilon(\varphi/\sqrt{C})$ has a maximum at $\varphi = 0$ and two degenerate minima for some nonzero values $\pm\varphi_{\text{min}}$ of φ. The negative sign of $m^2(0)$ is then a consequence of this fact that $\epsilon(q)$ has a maximum rather than a minimum at $q = 0$ and corresponds to the usual instability associated with an expansion around a maximum rather than a minimum of the potential. This also implies that it may not be justified to refer to the empty "vacuum" as a vacuum, which is the reason for the quotation marks in "vacuum".

Another case to consider is that of $C(0) = 0$. This case naturally occurs, if we choose $C(q)$ as in Ref. [30], namely $C(q) = q^2\,K(q)$ for some positive $K(q)$. The same arguments as given in Sec. 4.2.1 for the case of $C(q_0) = 0$ then imply that perturbations around $q = 0$ do not propagate for $C(0) = 0$.

We conclude that, in the case of the empty "vacuum", perturbations around $q = 0$ do not correspond to propagating degrees of freedom. Depending on the specific choice of $C(q)$, the reason is either that there is an instability or that the mass becomes infinite.

5. Absence of the Ostrogradsky instability

The kinetic term for q in the action S_{kin} from Eq. (1.31) leads to a higher-derivative theory, because q itself already contains derivatives of the three-form gauge field A according to Eq. (1.21). Typically, higher-derivative theories are pathological as they suffer from the Ostrogradsky instability [40, 41]. The purpose of this chapter is to determine whether or not the four-form realization of q-theory with a kinetic term for q is affected by the Ostrogradsky instability. To this end, we will mainly follow Ref. [42].[1]

As already indicated above, we will be concerned with the action S_{kin} from Eq. (1.31),

$$\begin{aligned} S_{\text{kin}} &= -\int d^4x\,\sqrt{-g}\,\left(\frac{R}{16\pi G(q)} + \frac{1}{2}\,C(q)\,\nabla_\beta\, q\,\nabla^\beta q + \epsilon(q)\right), \\ &\equiv \int dx^0\, L\left[A,\, \partial_0 A, \partial_0^2 A\right]\,. \end{aligned} \tag{5.1}$$

where, in the functional of the last expression, we keep spatial derivatives of A implicit and assume A without indices to stand for the collection of all components $A_{\alpha\beta\gamma}$. Recall that, by assumption, $G(q)$, $C(q)$, and $\epsilon(q)$ in the action (5.1) are even functions of q. Here, we also take $C(q) > 0$ and $G(q) > 0$. We further assume $\epsilon(q)$ to be a polynomial in q^2, which is bounded from below ($\epsilon \geq$ const) and nonconstant ($d\epsilon/dq \neq 0$). The gauge fields $A(x)$ are considered to have finite spacetime support (a physical context is provided by the q-ball solution mentioned in Sec. 1.2.3).

5.1. Ostrogradsky instability

Let us start with a brief review of the Ostrogradsky instability [40, 41]. To this end, consider a single higher-derivative harmonic oscillator as a model for typical higher-derivative theories, as discussed in Ref. [41]. Note that this model suffices to discuss the most important aspects of the Ostrogradsky instability. In particular, it is not necessary to consider a field-theoretic model, since the results are the same qualitatively. The reason is that the Ostrogradsky instability is exclusively associated with higher-order time derivatives and not with higher-order spatial derivatives [41].

Specifically, take the following Lagrangian [41] of a higher-derivative harmonic oscillator:

$$\overline{L} = -\frac{\varepsilon}{2}\,\frac{m}{\omega^2}\,(\ddot{x})^2 + \frac{m}{2}\,(\dot{x})^2 - \frac{m\,\omega^2}{2}\,x^2\,, \tag{5.2}$$

[1] Below, we will employ the gauge-fixing procedures from Sec. 2.1, which are slightly different from those employed in Ref. [42]. However, the arguments from Ref. [42] regarding the Ostrogradsky instability are unaffected by this.

where the overdot stands for the derivative with respect to time t and where ε, m, and ω are finite positive parameters.

The equation of motion from the Lagrangian (5.2) contains time derivatives of $x(t)$ up to fourth order,

$$\varepsilon \frac{m}{\omega^2} \ddddot{x} + m \ddot{x} + m \omega^2 x = 0 . \tag{5.3}$$

Therefore, four initial-data inputs are needed to uniquely specify a solution. This implies that there are four canonical variables which can be chosen as follows:

$$\overline{Q}_1 = x , \tag{5.4a}$$

$$\overline{P}_1 = \frac{\partial \overline{L}}{\partial \dot{x}} - \partial_0 \frac{\partial \overline{L}}{\partial \ddot{x}} = m \dot{x} + \varepsilon \frac{m}{\omega^2} \dddot{x} , \tag{5.4b}$$

$$\overline{Q}_2 = \dot{x} , \tag{5.4c}$$

$$\overline{P}_2 = \frac{\partial \overline{L}}{\partial \ddot{x}} = -\varepsilon \frac{m}{\omega^2} \ddot{x} . \tag{5.4d}$$

The canonical Hamiltonian is given by the usual Legendre transformation with respect to these canonical variables,

$$\begin{aligned} \overline{H} &= \overline{P}_1 \dot{\overline{Q}}_1 + \overline{P}_2 \dot{\overline{Q}}_2 - \overline{L} \\ &= \overline{P}_1 \overline{Q}_2 - \frac{\omega^2}{2 \varepsilon m} (\overline{P}_2)^2 - \frac{m}{2} (\overline{Q}_2)^2 + \frac{m \omega^2}{2} (\overline{Q}_1)^2 . \end{aligned} \tag{5.5}$$

From expression (5.5), it is clear why the higher-derivative harmonic oscillator is unstable. The Hamiltonian $\overline{H}$ is, namely, linear in the canonical momentum $\overline{P}_1$, which typically allows for runaway solutions as soon as we add interaction terms to $\overline{L}$ (for example, a term $-\lambda x^4$). The $\overline{P}_1 \overline{Q}_2$ term and the rest of the Hamiltonian $\overline{H}$ can then both grow arbitrarily large, while $\overline{H}$ stays constant. This is how the Ostrogradsky instability reveals itself at the classical level.

In order to formulate a quantized theory of the higher-derivative harmonic oscillator, note that the general solution of the classical equation of motion can be written as

$$x(t) = \alpha_+ e^{-ik_+ t} + \alpha_- e^{ik_- t} + \alpha_+^* e^{ik_+ t} + \alpha_-^* e^{-ik_- t} , \tag{5.6a}$$

$$k_\pm \equiv \omega \sqrt{\frac{1 \mp \sqrt{1 - 4 \varepsilon}}{2 \varepsilon}} , \tag{5.6b}$$

where $\alpha_\pm$ are arbitrary complex numbers. The Hamiltonian $\overline{H}$ reads [41]

$$\overline{H} = 2 m \sqrt{1 - 4 \varepsilon} \left(k_+^2 \, |\alpha_+|^2 - k_-^2 \, |\alpha_-|^2 \right) . \tag{5.7}$$

These last results show that quantization can proceed in the usual way by introducing creation and annihilation operators. There are then two degrees of freedom with opposite energies as can be seen from the expression (5.7). (A noncanonical quantization scheme [41] is not considered here, as it leads to problems with unitarity.) Now suppose that we add interactions to $\overline{L}$. Then, positive-energy and negative-energy degrees of freedom will inevitably interact with each other, as x carries both of them. Therefore, the vacuum will decay into pairs of positive-energy and negative-energy degrees of freedom. This is a manifestation of the Ostrogradsky instability at the quantum level.

5.2. Classical stability of higher-derivative q-theory

As mentioned in the previous section, the Ostrogradsky formalism is sensitive to higher-order time derivatives but not to higher-order spatial derivatives. The action (5.1), with a kinetic term proportional to $\nabla_\beta\, q\, \nabla^\beta q$, contains both higher-order time derivatives and higher-order spatial derivatives. However, the special form of q, namely $q \propto \epsilon^{\alpha\beta\gamma\delta}\partial_\alpha A_{\beta\gamma\delta}$, implies that the only time derivative in q is the one of the gauge-field component A_{123} (the other gauge-field components A_{213}, A_{231}, etc. are the same as A_{123} up to a factor ± 1). Therefore, the higher-order time derivatives in (5.1) are associated with A_{123} only. In short, we have

$$L\left[A,\, \partial_0 A,\, \partial_0^2 A\right] = L\left[A,\, \partial_0 A,\, \partial_0^2 A_{123}\right]\,, \tag{5.8}$$

where, again, spatial derivatives of A are kept implicit and A without indices stands for the collection of all components $A_{\alpha\beta\gamma}$. We can now proceed in two ways.

5.2.1. Gauge choice with nonvanishing A_{123}

As we are dealing with a gauge theory, we have to fix a gauge before we can start calculating physical quantities. To this end, we choose a gauge in which the component A_{123} does not vanish. Specifically, our gauge choice is the A_{123} gauge from Eq. (2.6), which completely fixes the gauge and leaves A_{123} as the only nonvanishing component of A. For simplicity, we first neglect gravity, $G(q) = 0$. Thus, according to Eq. (2.7),

$$q = -3!\; \partial_0 A_{123}\,. \tag{5.9}$$

The first step is now to identify the canonical variables and to calculate the Hamiltonian in terms of these canonical variables, with all explicit time derivatives eliminated [41]. There are two canonical coordinates Q_i with associated momenta P_i, for $i = 1, 2$, since the Lagrangian (5.8) depends on the first and second time derivative of A_{123}. Introducing factors of $-3!$ for convenience, we find

$$Q_1 = -3!\, A_{123}\,, \tag{5.10a}$$

$$P_1 = \frac{\partial L}{\partial(-3!\,\partial_0 A_{123})} - \partial_0 \frac{\partial L}{\partial(-3!\,\partial_0^2 A_{123})} = \frac{\delta S_{\text{kin}}}{\delta q} = -\mu\,, \tag{5.10b}$$

$$Q_2 = -3!\,\partial_0 A_{123} = q\,, \tag{5.10c}$$

$$P_2 = \frac{\partial L}{\partial(-3!\,\partial_0^2 A_{123})} = C(q)\,\partial_0 q\,, \tag{5.10d}$$

where $\delta S_{\text{kin}}/\delta q$ in (5.10b) follows from (5.1) with vanishing Ricci curvature scalar, $R = 0$, and the quantity μ was defined in Eq. (3.3). The canonical Hamiltonian now reads

$$H = \int d^3x \left(P_1\, \partial_0 Q_1 + P_2\, \partial_0 Q_2\right) - L\,[Q_2,\, P_2] \tag{5.11a}$$

$$= \int d^3x \left(P_1\, Q_2 + \frac{1}{2}\,\frac{1}{C(q)}\,(P_2)^2 + \frac{1}{2}\,C(q)\,(\partial_a Q_2)^2 + \epsilon(Q_2)\right) \tag{5.11b}$$

$$= \int d^3x \left(\epsilon(q) - \mu\, q + \frac{1}{2}\,C(q)\,(\partial_0 q)^2 + \frac{1}{2}\,C(q)\,(\partial_a q)^2\right) . \tag{5.11c}$$

The Hamiltonian H is conserved and coincides with the energy derived from the gravitational energy-momentum tensor.

We see that the Hamiltonian H from (5.11b) is linear in the canonical momentum P_1, just as the Hamiltonian $\overline{H}$ of the higher-derivative harmonic oscillator discussed in Sec. 5.1. For the present case, however, the result (5.10b) and the A field equations discussed in Sec. 3.1 imply that P_1 is constant,

$$\partial_\beta\, P_1 = \partial_\beta\, \frac{\delta S_{\text{kin}}}{\delta q} = \partial_\beta\, (-\mu) = 0\,. \tag{5.12}$$

Therefore, no runaway solutions are possible. More precisely, the conservation of H and the result (5.12) make that Q_2 and P_2 do not grow arbitrarily large in time. In order to see this explicitly, note that the Hamiltonian (5.11b) is bounded from below for a fixed constant value of P_1. In particular, following the definition (3.10), the terms $P_1\, Q_2$ and $\epsilon(Q_2)$ from the integrand of (5.11b) can be combined into an effective potential $\epsilon_{\text{eff},-P_1}(Q_2) = \epsilon(Q_2) + P_1 Q_2$, which is a polynomial in Q_2 and bounded from below. Recall that, by assumption, $\epsilon(Q_2)$ is a nonconstant polynomial in $(Q_2)^2$, which is bounded from below. Hence, if Q_2 or P_2 were to grow arbitrarily large, this would contradict the conservation of H. In contrast to Q_2 and P_2, the canonical coordinate Q_1 is allowed to grow arbitrarily, as it does not appear in H. However, no physical quantity will directly depend on Q_1, since $Q_1 \propto A_{123}$ is gauge-noninvariant. Consequently, the result (5.12) implies that the linear appearance of P_1 in H does not lead to a dynamical instability.

Next, consider the case with standard gravity, $G(q) = G_N \neq 0$. Similar arguments as the ones given above show that the Hamiltonian is linear in a single canonical momentum. It can be shown that this canonical momentum is proportional to $(1/\sqrt{-g})\; \delta S_{\text{kin}}/\delta q = -\mu$. Again, this is exactly what is required to be constant by the equations of motion for A, now with gravity present, as discussed in Sec. 3.1. Therefore, also for the case with gravity, the term of the Hamiltonian with the linearly appearing canonical momentum can be absorbed into a well-behaved effective potential, implying that the Ostrogradsky instability is absent.

The absence of the Ostrogradsky instability for the extended q-theory (5.1) in the A_{123} gauge can be illustrated by considering a modified version of the higher-derivative harmonic oscillator discussed in Sec. 5.1. Consider the modified Lagrangian $\overline{L}_{\text{mod}}$ which is obtained from $\overline{L}$, as given in (5.2), by omitting the term without time derivatives,

$$\overline{L}_{\text{mod}} = -\frac{\varepsilon}{2}\,\frac{m}{\omega^2}\,(\ddot{x})^2 + \frac{m}{2}\,(\dot{x})^2\,. \tag{5.13}$$

This modification is motivated by the fact that A_{123} never appears without a time derivative in the action (5.1), the reason being gauge invariance. Hence, we have, for the A_{123} gauge, the following arguments of the Lagrangian:

$$L\left[A,\,\partial_0 A,\,\partial_0^2 A\right]\Big|^{(A_{123}\text{ gauge})} = L\left[\partial_0 A_{123},\,\partial_0^2 A_{123}\right]\,. \tag{5.14}$$

Switching from $\overline{L}$ to $\overline{L}_{\text{mod}}$ has no effect on the canonical variables, since the canonical variables are completely determined by the terms with time derivatives. As a result, the canonical Hamiltonian $\overline{H}_{\text{mod}}$ is still linear in the canonical momentum $\overline{P}_1$. However, the equation of motion derived from $\overline{L}_{\text{mod}}$ [given by (5.3) without the $m\omega^2 x$ term] now implies that $\overline{P}_1$ is constant, $\partial_0\,\overline{P}_1 = 0$. It follows that $\overline{Q}_2$ and $\overline{P}_2$ cannot grow arbitrarily large in time, since $\overline{H}_{\text{mod}}$ [given by (5.5) without the $m\,\omega^2\,(\overline{Q}_1)^2/2$ term] is conserved and, for a fixed constant value of $\overline{P}_1$, bounded from above. The canonical coordinate $\overline{Q}_1$ can, in principle, grow arbitrarily large, as $\overline{Q}_1$ without derivatives does not appear in $\overline{H}_{\text{mod}}$. The Lagrangian $\overline{L}_{\text{mod}}$ possesses indeed a shift symmetry [$x(t) \to x'(t) = x(t) + b$ for an arbitrary constant b], which prevents $\overline{Q}_1 = x$ from appearing in any physical quantity. Therefore, the same argument as given above shows that the modified higher-derivative harmonic oscillator with Lagrangian (5.13) is not affected by the Ostrogradsky instability, as long as the shift symmetry is imposed.

5.2.2. Gauge choice with vanishing A_{123}

In order to see what happens if the component A_{123} associated with the higher-order time derivatives is gauged away, we will now consider an alternative gauge choice. Specifically, take the A_{023} gauge from Eq. (2.8), which leaves A_{023} as the only nonvanishing component of A. For flat spacetime, Eq. (2.9) then gives

$$q = 3!\,\partial_1 A_{023}\,. \tag{5.15}$$

The possibility to gauge away all higher-order time derivatives of the Lagrangian suggests that the theory does not suffer from the Ostrogradsky instability. We will now explicitly show that this is indeed the case.

As no higher-order time derivatives are associated with A_{023}, there is only one canonical coordinate $\widetilde{Q}$ with associated momentum $\widetilde{P}$. Introducing a factor of 3! for convenience, we find

$$\widetilde{Q} = 3!\,A_{023}\,, \tag{5.16a}$$

$$\widetilde{P} = \frac{\partial L}{\partial(3!\,\partial_0 A_{023})} = \partial_1\left[\,C(q)\,\nabla^0 q\right]\,. \tag{5.16b}$$

The Lagrangian L depends at least quadratically on both $\widetilde{Q}$ and $\widetilde{P}$. Therefore, the canonical Hamiltonian $\widetilde{H}$, obtained by a Legendre transformation with respect to $\widetilde{Q}$ and $\widetilde{P}$, will not be linear in any canonical variable. At this point, we might be tempted to conclude that the theory is not affected by the Ostrogradsky instability, but there is a subtlety.

For flat Minkowski spacetime, we write $\widetilde{H}$ in terms of q and perform an integration by parts,

$$\begin{aligned}\widetilde{H} &= \int d^3x\, \widetilde{P}\, \partial_0 \widetilde{Q} - L\left[\widetilde{Q}, \widetilde{P}\right] \\ &= \int d^3x\, \left(\epsilon(q) + \frac{1}{2}\, C(q)\, (\partial_0 q)^2 + \frac{1}{2}\, C(q)\, (\partial_a q)^2\right) . \end{aligned} \tag{5.17}$$

In comparison with the Hamiltonian H from (5.11c), the term $-\mu q$ of the integrand is missing. For typical nonequilibrium solutions of q-theory, this $-\mu q$ term generates a nonconstant contribution to H implying that $\widetilde{H}$ is not conserved [19, 31]. As in Sec. 5.2.1, μ refers to the quantity defined in Eq. (3.3), which is constant by the A field equations. In principle, the instability could hide in this nonconservation of $\widetilde{H}$. Furthermore, it is rather unusual that the Hamiltonian depends on the gauge choice. In electrodynamics, for example, the Hamiltonians derived in different gauges differ merely by a total derivative (cf. Sec. 3.5.3 of Ref. [43]).

Regarding the difference between H and $\widetilde{H}$, we note that the $-\mu q$ term is indeed a total derivative in the A_{023} gauge. The reason is that μ is constant and $q \propto \partial_1 A_{023} \propto \partial_1 \widetilde{Q}$. However, as already mentioned in the previous paragraph, typical solutions of q-theory are such that the spatial integral over $q \propto \partial_1 A_{023}$ does not vanish. Consequently, H and $\widetilde{H}$ do not usually coincide in q-theory. In particular, this implies that $\widetilde{H}$ differs from the energy derived from the gravitational energy-momentum tensor, while it does generate the correct time evolution for $\widetilde{Q}$ and $\widetilde{P}$. The discrepancy between $\widetilde{H}$ and the conserved energy is not really problematic, as the defining property of the canonical Hamiltonian is that it generates the time evolution of the canonical variables and not that it is conserved. It does, however, prevent us from deciding whether or not the theory is unstable.

We can now solve this problem of the nonconservation of $\widetilde{H}$ by simply adding the $-\mu q$ term,

$$\widetilde{H}_{\text{conserved}} = \widetilde{H} - \mu \int d^3x\, q\,, \tag{5.18}$$

with the constant μ as discussed a few lines below (5.17). This addition is allowed since the $-\mu q$ term is a total derivative according to (5.15) and total derivatives do not affect the time evolution of the canonical variables. In this way, we arrive at a Hamiltonian which is conserved in flat Minkowski spacetime and which contains a term linear in the canonical variable $\widetilde{Q}$, namely the $-\mu q$ term. But the linear term $-\mu q$ does not lead to an instability, if we consider this linear term together with the potential term $\epsilon(q)$ and recall the assumptions on $\epsilon(q)$ as stated at the beginning of this chapter. (In a general spacetime, $-\mu q$ must be replaced by $-\sqrt{-g}\,\mu q$, where μ as given by Eq. (3.3) is constant by the A field equations also in a curved spacetime. Adding the spatial integral of $-\sqrt{-g}\,\mu q$ to $\widetilde{H}$ still amounts to adding the integral of a total derivative to $\widetilde{H}$, as q is proportional to $1/\sqrt{-g}$, and the same conclusion holds for general spacetimes as for flat spacetime.) Hence, also in the A_{023} gauge, we conclude that the Ostrogradsky instability is absent.

5.3. Discussion

In this chapter, we have shown that the four-form realization of q-theory with a kinetic term for q is free from the Ostrogradsky instability. We have derived this result in two different gauges. The fact that the chemical potential μ (using the terminology discussed in Sec. 3.1) is constant has been crucial for both derivations. With the A_{123} gauge, a constant μ implies that the canonical momentum appearing linearly in the Hamiltonian is constant. With the A_{023} gauge, the required $-\mu q$ term for a conserved Hamiltonian is a total derivative only if μ is constant.

Note that the treatment up till now has been completely classical. At the quantum level, the Ostrogradsky instability typically leads to an additional propagating degree of freedom which carries negative energy, as discussed in Sec. 5.1. At this point, we cannot decide whether or not a quantized higher-derivative q-theory exhibits the corresponding vacuum-instability problem, because we do not yet have a quantized q-theory at our disposal. In Chapter 6, a possible quantized q-theory is proposed. However, there are still points that require further study regarding this quantum theory from Chapter 6, such that no definite conclusions can yet be drawn. It is also possible that the correct quantized theory requires a detailed knowledge of all microscopic degrees of freedom.

Still, we expect the quantized theory corresponding to the action (5.1) not to have negative-energy propagating degrees of freedom. The reason is the following. Consider flat Minkowski spacetime with the constant equilibrium solution q_0 for the q-field and the corresponding fixed value $\mu_0 = d\epsilon/dq|_{q=q_0}$ for μ. The discussion of Sec. 4.2 then implies that a linear perturbation $\varphi(x)$ of this constant solution q_0,

$$q(x) \equiv q_0 + \varphi(x)/\sqrt{C(q_0)}\,, \tag{5.19}$$

satisfies the Klein–Gordon equation Eq. (4.12a),

$$0 = \Box\,\varphi - m^2(\mu_0)\,\varphi\,. \tag{5.20}$$

The corresponding conserved Hamiltonian is given by the spatial integral of Eq. (4.13),

$$H = \int d^3x \left(\frac{1}{2}\,(\partial_0\varphi)^2 + \frac{1}{2}\,(\partial_a\varphi)^2 + \frac{1}{2}\,m^2(\mu_0)\,\varphi^2\right)\,. \tag{5.21}$$

Both the linearized equation of motion Eq. (5.20) and the corresponding Hamiltonian (5.21) have precisely the same formal structure as in the case of a fundamental scalar field.

The last observation suggests that quantizing the four-form realization of q-theory with a kinetic term for q leads to one propagating degree of freedom only, in contrast to what we expect from a typical higher-derivative theory. If this is indeed correct, and if μ remains constant in the quantized theory, we arrive at the following scenario. Theories suffering from the Ostrogradsky instability typically contain two propagating degrees of freedom with opposite energies, so that the vacuum can decay into pairs of positive-energy and negative-energy degrees of freedom. In our case, however, there is only one propagating degree of freedom. The additional degree of freedom represented by μ is nonpropagating and does not lead to a dynamical instability.

6. Quantization

In the previous chapters, we have considered the four-form realization of q-theory as a classical theory only. The aim of this chapter is to find a corresponding quantum theory. We are particularly interested in understanding the quantity μ, classically defined by Eq. (3.3), but now at the quantum level. This is because, classically, the equations of motion for A require μ to be constant. Yet, as discussed in Sec. 3.1, μ transforms nontrivially under parity, such that the operator corresponding to μ at the quantum level cannot simply be proportional to the identity operator.

Following the idea of q-theory as an emergent theory of the quantum vacuum [12], it can be argued that it is unnecessary to quantize q-theory [44]. This is because q-theory gives an effective, macroscopic description of an underlying microscopic theory and does, therefore, not need to be quantized for the same reason we do not need to quantize, for example, hydrodynamics. Still, suppose we allow for some quanta (e.g., quanta corresponding to quantized matter fields) to interact with the classical q-field of q-theory. Then, strictly speaking, we arrive at an inconsistent theory (cf. Sec. 1.1(b) of Ref. [45]). From this point of view, the quantization procedure discussed in this chapter may be understood as an exercise in finding a consistent quantum theory involving the effective vacuum variable q of q-theory.

Below, we will consider the action S_{kin} from Eq. (1.31) with a nonvanishing kinetic term for q,

$$C(q) \equiv C = \text{const} > 0\,, \tag{6.1}$$

and flat Minkowski spacetime,

$$g_{\alpha\beta} \equiv [\text{diag}(-1, 1, 1, 1)]_{\alpha\beta}\,. \tag{6.2}$$

In Sec. 6.1, we will propose a quantization procedure for this four-form realization of q-theory in the case of a quadratic potential $\epsilon(q)$. Then, we will generalize this quantization procedure to the case of a quartic potential in Sec. 6.2 In Sec. 6.3, we will discuss some physical consequences of the constructed quantum theory. Finally, we will summarize our findings and discuss some open questions in Sec. 6.4.

6.1. Quantization of the free theory

Let us consider the following potential $\epsilon(q)$:

$$\epsilon(q) = \Lambda + a\,q^2\,, \tag{6.3}$$

where $a > 0$ and Λ are constants. In Sec. 6.1.1, we will construct a quantum theory for the case of the potential (6.3), which we will refer to as the free theory. This construction will not directly follow the canonical formalism, since it is not obvious how to deal with the integration constant μ in the canonical formalism. We will, however, check whether or not the constructed theory agrees with what we expect from the canonical formalism in Sec. 6.1.2. In order to simplify generalizing to a different potential in Sec. 6.2, we will not explicitly write $\Lambda + aq^2$ for $\epsilon(q)$ in Sec. 6.1.1.

6.1.1. Construction

The quantity μ from Eq. (3.3) corresponds to an integration constant according to Sec. 3.1. Here, we start by considering a fixed constant value of this quantity μ and recall the definition of the effective potential $\epsilon_{\text{eff},\mu}$ from Eq. (3.10),

$$\epsilon_{\text{eff},\mu}(q) = \epsilon(q) - \mu\, q\,. \tag{6.4}$$

Following Sec. 3.1, the classical equations of motion for A and the Hamiltonian as derived from the gravitational energy-momentum tensor can then be written as

$$0 = \epsilon'_{\text{eff},\mu}(q) - C\,\Box\, q\,, \tag{6.5a}$$

$$H = \int d^3\vec{x}\, \left(\epsilon_{\text{eff},\mu}(q) + \frac{1}{2}C(\partial_0 q)^2 + \frac{1}{2}C(\partial_a q)^2\right)\,, \tag{6.5b}$$

where a prime denotes differentiation with respect to q.

Now consider a fundamental scalar ϕ_μ with an action given by S_{kin} with parameters as discussed above, but with ϵ replaced by $\epsilon_{\text{eff},\mu}$ and q replaced by ϕ_μ. Then, the associated equation of motion and the associated Hamiltonian have precisely the same formal structure as Eqs. (6.5a) and (6.5b) according to the discussion in Sec. 3.1. This theory of ϕ_μ can be quantized by the usual canonical quantization procedure for a free scalar field yielding a quantum field $\hat{\phi}_\mu$. Note that, similar to the scalar field ϕ from Sec. 3.1, ϕ_μ has mass dimension 2, which is rather unusual for a fundamental scalar field. However, first rescaling ϕ_μ with an appropriate dimensionful constant, then quantizing the theory, and finally going back to the unrescaled field leads to the same quantum theory as directly quantizing ϕ_μ. Therefore, we will keep ϕ_μ as it is for now.

In the quantum theory of $\hat{\phi}_\mu$, the equation of motion and the Hamiltonian $\hat{H}_\mu$ are respectively given by Eqs. (6.5a) and (6.5b) with q replaced by $\hat{\phi}_\mu$ and H replaced by $\hat{H}_\mu$. Also, $\hat{\phi}_\mu$ satisfies the canonical commutation relation

$$[\, \hat{\phi}_\mu(t,\vec{x}),\, C\,\partial_0\hat{\phi}_\mu(t,\vec{y})\,] = i\delta(\vec{x}-\vec{y})\,. \tag{6.6}$$

We denote the associated Hilbert space by $\mathcal{H}_\mu$ and the elements of a basis of $\mathcal{H}_\mu$ by $|\mu, n_\mu\rangle$ with some label n_μ. Strictly speaking, the index μ of n_μ is not needed for the potential (6.3). Still, we choose to keep it in our notation, as it is needed for more general cases discussed further below. Since $\hat{\phi}_\mu$ is a scalar, the parity operator $\mathcal{P}_\mu$ on $\mathcal{H}_\mu$ satisfies

$$\mathcal{P}_\mu\, \hat{\phi}_\mu(t,\vec{x}) = +\hat{\phi}_\mu(t,-\vec{x})\,\mathcal{P}_\mu\,. \tag{6.7}$$

The above discussion now allows us to construct a quantized four-form realization of q-theory as follows. We start by defining the Hilbert space $\mathcal{H}$ of this quantized four-form realization of q-theory as the direct sum of all $\mathcal{H}_\mu$. This means that any element $|\psi\rangle_{\mathcal{H}}$ in $\mathcal{H}$ can be written as

$$|\psi\rangle_{\mathcal{H}} = \int d\mu \sum_{n_\mu} c_{n_\mu}(\mu)\, |\mu, n_\mu\rangle_{\mathcal{H}} \ , \tag{6.8}$$

where a discrete notation for the n_μ is used for simplicity and the $c_{n_\mu}(\mu)$ are complex numbers. We also take the inner product on $\mathcal{H}$ to be defined by

$$_{\mathcal{H}}\langle \mu, n_\mu | \overline{\mu}, m_{\overline{\mu}}\rangle_{\mathcal{H}} \equiv \delta(\mu - \overline{\mu}) \cdot \begin{cases} _{\mathcal{H}_\mu}\langle \mu, n_\mu | \mu, m_\mu\rangle_{\mathcal{H}_\mu} & \mu = \overline{\mu} \\ 0 & \mu \neq \overline{\mu} \end{cases} . \tag{6.9}$$

In Eqs. (6.8) and (6.9), the indices of the vectors indicate that these are to be taken as elements of $\mathcal{H}$ and $\mathcal{H}_\mu$, respectively. In the following, we will mostly drop this explicit notation for simplicity.

Our next step is to define the operator $\hat{\mu}$ on $\mathcal{H}$ as the linear operator with

$$\hat{\mu}\, |\mu, n_\mu\rangle \equiv \mu\, |\mu, n_\mu\rangle \ . \tag{6.10}$$

Similarly, we define the operator $\hat{q}$ on $\mathcal{H}$ as the linear operator with

$$\hat{q}\, |\mu, n_\mu\rangle \equiv \hat{\phi}_\mu\, |\mu, n_\mu\rangle \ . \tag{6.11}$$

The equation of motion of $\hat{\phi}_\mu$, together with the linearity of $\hat{q}$ and $\hat{\mu}$, then implies

$$\hat{\mu} = \epsilon'(\hat{q}) - C\,\Box\,\hat{q}\,, \tag{6.12}$$

which corresponds to a quantized version of Eq. (6.5a). We obtain a quantized Hamiltonian corresponding to Eq. (6.5b) by defining $\hat{H}$ as the linear operator with

$$\hat{H}\, |\mu, n_\mu\rangle \equiv \hat{H}_\mu\, |\mu, n_\mu\rangle \ , \tag{6.13}$$

which implies that $\hat{H}$ is given by the right-hand side of Eq. (6.5b) with μ and q replaced with $\hat{\mu}$ and $\hat{q}$, respectively.

Defining the parity operator on $\mathcal{H}$ requires more care, since, classically, μ changes its sign under parity, as can be seen from Eq. (3.3) in flat spacetime. Therefore, we tentatively define the parity operator $\mathcal{P}$ on $\mathcal{H}$ as the linear operator with

$$\mathcal{P}\, |\mu, n_\mu\rangle \equiv \mathcal{P}_{-\mu}\, |-\mu, n_{-\mu}\rangle \ . \tag{6.14}$$

This is well-defined for the following reason: First, note that $\epsilon(q)$ is an even function of q such that Eq. (6.5) implies $\phi_{-\mu}(x) = -\phi_\mu(x)$ and $H_\mu = H_{-\mu}$. Further, since the same quantization procedure is employed for μ and $-\mu$, we also have $\hat{\phi}_{-\mu}(x) = -\hat{\phi}_\mu(x)$, $\hat{H}_\mu = \hat{H}_{-\mu}$, and $\mathcal{H}_\mu = \mathcal{H}_{-\mu}$. Therefore, the state $|-\mu, n_{-\mu}\rangle$ exists and Eq. (6.14) is well-defined. Note that we take the same labeling to be employed for the states in $\mathcal{H}_\mu$ and $\mathcal{H}_{-\mu}$.

As a consequence of Eq. (6.14), $\hat{\mu}$ is parity-odd,

$$\mathcal{P}\,\hat{\mu} = -\hat{\mu}\,\mathcal{P}\,. \tag{6.15}$$

In order to agree with the classical theory, $\hat{q}$ must be parity-odd as well. A parity-odd $\hat{q}$ is also needed in order to obtain $[\hat{H}, \mathcal{P}] = 0$. Indeed, Appendix C.1 shows that

$$\mathcal{P}\,\hat{q}(t,\vec{x}) = -\hat{q}(t,-\vec{x})\,\mathcal{P}\,. \tag{6.16}$$

Up to this point, we have not discussed the role of the three-form gauge field A. In the above construction, the fact that $\hat{q}$ is not a fundamental field is taken into account by the presence of the additional degree of freedom represented by $\hat{\mu}$. We will now also define the quantized three-form gauge field $\hat{A}$ for completeness, although no physical quantities will depend on the specific form of $\hat{A}$. To this end, we first choose the A_{023} gauge (2.8) at the classical level. We then define the quantum operators $\hat{A}_{\alpha\beta\gamma}$ by replacing each classical field in Eq. (2.8) by its respective corresponding quantum field,

$$\hat{A}_{1\beta\gamma} \equiv 0\,, \tag{6.17a}$$

$$3!\,\hat{A}_{023}(t,\vec{x}) \equiv \int_0^{x^1} d\tilde{x}^1\,\hat{q}(t,\tilde{x}^1,x^2,x^3)\,. \tag{6.17b}$$

Regarding the operator $\hat{\mu}$, we note that Eqs. (6.8), (6.9), and (6.10) imply an analogy between $\hat{\mu}$ and a quantum-mechanical degree of freedom such as the momentum operator $\hat{p}$ in ordinary quantum mechanics. But we also note that this analogy is not perfect, as no physical quantity is to $\hat{\mu}$ what the position operator $\hat{x}$ is to $\hat{p}$. Also, at least for more general potentials discussed in Sec. 6.2 below, the Hilbert space $\mathcal{H}$ does not factorize with one factor corresponding to eigenstates of $\hat{\mu}$.

Furthermore, taking the direct sum instead of the direct product when defining $\mathcal{H}$ makes the constructed theory essentially different from a theory in which the fundamental scalar fields ϕ_μ are combined into a single theory in the standard way. More precisely, we expect that canonical quantization of the action

$$S_\otimes \equiv -\int d^4x \int \frac{d\mu}{E^2}\left(\epsilon(\phi_\mu) - \mu\,\phi_\mu + \frac{1}{2}C\,\partial_\beta\,\phi_\mu\,\partial^\beta\phi_\mu\right) \tag{6.18}$$

with some constant E with mass dimension 1, roughly[1], leads to the direct product $\mathcal{H}_\otimes$ instead of the direct sum $\mathcal{H}$ of the Hilbert spaces $\mathcal{H}_\mu$. We further expect that employing $\mathcal{H}_\otimes$ instead of $\mathcal{H}$ cannot lead to a quantum theory of the four-form realization of q-theory for the following reason. Having chosen all initial conditions, only one particular value of μ enters the calculation of physical quantities in the classical four-form realization of q-theory. Similarly, taking the Hilbert space $\mathcal{H}$ allows to select certain values of μ by choosing an appropriate superposition (6.8). In contrast, every state in $\mathcal{H}_\otimes$ contains *all* values of μ. This is because every element of $\mathcal{H}_\otimes$ is a superposition of states of the form $\prod_\mu |\psi_\mu\rangle$, where $|\psi_\mu\rangle$ is an element of $\mathcal{H}_\mu$. Hence, a quantum theory corresponding to the classical four-form realization of q-theory cannot be obtained using $\mathcal{H}_\otimes$.

[1] One needs to be careful about the continuum nature of μ.

6.1.2. Commutation relations and time evolution

We can infer the following commutation relations from the construction in Sec. 6.1.1:

$$[\,\hat{q}(t,\vec{x}),\, C\,\partial_0\hat{q}(t,\vec{y})\,] = i\delta(\vec{x}-\vec{y})\,, \tag{6.19a}$$

$$[\,\hat{\mu},\, \hat{q}\,] = [\,\hat{\mu}, C\,\partial_0\hat{q}\,] = [\,\hat{\mu},\, \hat{A}_{\alpha\beta\gamma}\,] = 0\,, \tag{6.19b}$$

$$[\,3!\,\hat{A}_{023}(t,\vec{x}),\, -C\,\partial_1\partial_0\hat{q}(t,\vec{y})\,] = i\delta(\vec{x}-\vec{y}) - i\delta(\vec{x}-\vec{y}|_{x^1=0})\,. \tag{6.19c}$$

Let us now compare these inferred commutation relations (6.19) to what we expect from the canonical formalism. The classical canonical variables in the A_{023} gauge are given by Eq. (5.16) in flat Minkowski spacetime,

$$\widetilde{Q} = 3!\,A_{023}\,, \tag{6.20a}$$

$$\widetilde{P} = -C\,\partial_1\partial_0 q\,. \tag{6.20b}$$

Eq. (6.19c) then implies noncanonical commutation relations between the quantum fields $\widetilde{Q}_{\text{quant}}$ and $\widetilde{P}_{\text{quant}}$, which correspond to the classical fields $\widetilde{Q}$ and $\widetilde{P}$ and are obtained from Eq. (6.20) by replacing q and A_{023} by $\hat{q}$ and $\hat{A}_{023}$, respectively. However, noncanonical commutation relations should be expected, since $\widetilde{Q} \propto A_{023}$ is not gauge-invariant and gauge-noninvariant fields such as the vector field in electrodynamics typically satisfy noncanonical commutation relations [46].

The main principle for finding suitable commutation relations for a gauge theory is then the requirement that the commutation relations must be compatible with the chosen gauge as well as the equations of motion [46]. In our case, it can be verified that the commutation relations (6.19) are compatible with both the chosen gauge and the equations of motion. In particular, the second, noncanonical term on the right-hand side of Eq. (6.19c) ensures compatibility with the gauge choice (6.17). We conclude that the canonical formalism is compatible with the commutation relations (6.19).

As discussed in Sec. 5.2.2 at the classical level, it is a peculiarity of the A_{023} gauge that the canonical Hamiltonian $\widetilde{H}$ obtained by a Legendre transformation with respect to the canonical variables (6.20) does not agree with the Hamiltonian (6.5b). Specifically, $\widetilde{H}$ is missing the term proportional to μq and is therefore not conserved. However, we may simply add the μq term to $\widetilde{H}$. The reason is that this μq term is a total derivative in the A_{023} gauge and does, therefore, not affect the time evolution, which the Hamiltonian generates for the canonical variables. This reasoning still holds in the quantum theory constructed above. Indeed, using the commutation relations (6.19), it can be explicitly checked that the time evolution of $\widetilde{Q}_{\text{quant}}$ and $\widetilde{P}_{\text{quant}}$ is given by their respective commutator with $\hat{H}$. Hence, the canonical formalism agrees with the construction in Sec. 6.1.1 also regarding the Hamiltonian.

We note that, in a gauge choice similar to the A_{123} gauge (2.6), there is a subtlety regarding the time evolution of a gauge-noninvariant canonical coordinate, which requires further study, as discussed in Appendix C.2. Still, this alternative gauge choice strengthens the analogy of $\hat{\mu}$ to a quantum-mechanical degree of freedom, as also discussed in Appendix C.2.

6.2. Quantization with interactions

In Sec. 6.1.1, the quantum fields $\hat{\phi}_\mu$ turned out to be free fields such that we did not have to deal with the issue of renormalization, which arises in an interacting quantum field theory. We will now revisit the construction of Sec. 6.1.1 in the case of an interacting theory with a potential $\epsilon(q)$ containing higher than quadratic powers of q. Specifically, we consider the following potential:

$$\epsilon(q) = \Lambda - a\,q^2 + b\,q^4\,, \tag{6.21}$$

where $a > 0$, $b > 0$, and Λ are constants. The reason for this particular choice will be explained below.

As in Sec. 6.1.1, we start by fixing a value of μ and consider the classical theory of the fundamental scalar field ϕ_μ obtained by replacing ϵ by $\epsilon_{\text{eff},\mu}$ and q by ϕ_μ in the action S_{kin} from Eq. (1.31). Again, the corresponding equation of motion as well as the Hamiltonian derived from the gravitational energy-momentum tensor have the same form as in Eq. (6.5) but with q replaced by ϕ_μ. We then take this theory of ϕ_μ to be quantized using canonical quantization and some renormalization scheme with associated renormalization scale M. Further, we use the same notation as in Sec. 6.1.1 for the quantized field $\hat{\phi}_\mu$, the Hilbert space $\mathcal{H}_\mu$, the basis elements of $\mathcal{H}_\mu$, and the Hamiltonian $\hat{H}_\mu$, where μ now denotes a renormalized quantity.

As in the case of the free theory, we define the Hilbert space $\mathcal{H}$ of the quantized four-form realization of q-theory as the direct sum of all $\mathcal{H}_\mu$. Similarly, we follow Sec. 6.1.1 in defining the inner product on $\mathcal{H}$ as well as the quantum operators $\hat{q}$, $\hat{\mu}$, $\hat{H}$, and $\hat{A}_{\alpha\beta\gamma}$.

Note that renormalization did not play a significant role up till now. However, we have not yet discussed if the quantized version (6.12) of the equation of motion (6.5a) and parity-conservation generalize to the potential (6.21). This will be done next and renormalization will be crucial there.

To this end, let us first consider the renormalization of the theory of $\hat{\phi}_\mu$. We define

$$\varphi_\mu \equiv \phi_\mu / E\,, \tag{6.22}$$

where E is some constant with mass dimension 1 such that φ_μ also has mass dimension 1. The theory of ϕ_μ is then equivalent to that of φ_μ with the following action, as noted in Sec. 6.1.1:

$$S_{\varphi_\mu} = -\int d^4x \left(\Lambda - m^2\,\varphi_\mu^2 + \lambda\,\varphi_\mu^4 - \tilde{\mu}\,\varphi_\mu + \frac{Z}{2}\,\partial_\beta\varphi_\mu\,\partial^\beta\varphi_\mu\right)\,, \tag{6.23}$$

where $Z \equiv CE^2$, $m^2 \equiv aE^2$, $\lambda \equiv bE^4$ and $\tilde{\mu} \equiv \mu E$. We can now see that the quantum theory corresponding to the action (6.23) is renormalizable, such that the quantum theory of ϕ_μ is renormalizable as well. But we also see that it may not be consistent to leave out a possible cubic term proportional to φ^3 in (6.23), since there is no symmetry protecting such a term from arising due to quantum corrections. If, in fact, such a cubic term were to arise due to quantum corrections, it would not be clear, if the corresponding quantum q-theory defined above would be parity-conserving.

Another potential issue is that there can be no simple renormalized version of Eq. (6.12), if beta-functions of parameters other than $\tilde{\mu}$ itself depend on $\tilde{\mu}$. This is because Eq. (6.12) contains a finite number of parameters, all of which are independent of μ. It is only the operator $\hat{\mu}$ that appears in Eq. (6.12). But this cannot hold independently of the renormalization scale M, if beta-functions of parameters other than μ itself depend on μ. Instead, there would only be the respective equation of motion of $\hat{\phi}_\mu$ for each value of μ in this case.

However, there is a renormalization scheme in which, at least perturbatively, both of the above-mentioned complications are absent. In order to see this, we first note that the mass dimensions of the coefficient λ_3 of a potential cubic term and the other parameters in the action (6.23) are given by

$$[Z] = [\lambda] = 0\,, \quad [\lambda_3] = 1\,, \quad [m^2] = 2\,, \quad [\tilde{\mu}] = 3\,, \quad [\Lambda] = 4\,. \tag{6.24}$$

Therefore, the action (6.23) contains a discrete symmetry $\varphi_\mu \to -\varphi_\mu$, which is *softly* broken by $\tilde{\mu}$. Since the action (6.23) also leads to a renormalizable theory, it follows that there is indeed a renormalization scheme, in which the presence of $\tilde{\mu}$ does not affect the beta-functions of the other parameters, at least perturbatively (cf. Sec. 10.1 of Ref. [47][2]). An example of such a renormalization scheme is the $\overline{MS}$ scheme [47].

Consequently, if we take the quantized q-theory to be defined using the $\overline{MS}$ scheme for the theories of $\hat{\phi}_\mu$, a renormalized version of Eq. (6.12) holds. Moreover, the parity operator $\mathcal{P}$ can then be defined as in Sec. 6.1.1 such that the constructed theory is manifestly parity-conserving. We would like to stress that choosing a different renormalization scheme does not lead to different physical predictions. Rather, with the $\overline{MS}$ scheme, the connection to classical q-theory is easier to see compared to other renormalization schemes.

We conclude that it is reasonable to regard the theory constructed above as a quantum theory corresponding to the action S_{kin} with potential (6.21). Although the potential (6.21) contains a quartic coupling constant with negative mass dimension, this quantized theory is renormalizable in the sense that the theory can be described by a finite number of parameters. The reason is that contact with the standard literature on renormalization can be made only after the rescaling (6.22), which leads to a dimensionless quartic coupling constant in the action (6.23). In contrast, if we had chosen to include higher than quartic powers of q in S_{kin}, we would not have ended up with a renormalizable theory and the dimensional analysis from Ref. [47] would not have been applicable. Therefore, it is not clear if it would have been sensible to call the resulting theory a quantized q-theory. This is the reason for the particular choice (6.21). More general choices require further study.

6.3. Physical consequences

If we consider an eigenstate of $\hat{\mu}$, the theory of $\hat{q}$ constructed above looks quite similar to the theory of the fundamental field $\hat{\phi}_\mu$, at least up to the effects of parity which takes μ

[2] Strictly speaking, the arguments from Ref. [47] do not exclude the possibility that $\tilde{\mu}$ affects the beta-function of Λ. However, it can be shown that, perturbatively and in the $\overline{MS}$ scheme, this is not the case for dimensional reasons.

to $-\mu$. However, an eigenstate of $\hat{\mu}$ is not a physical state, as such a state is necessarily nonnormalizable due to (6.9). Therefore, any physical state has to be a superposition of different values of μ.

With this in mind, we consider the equilibrium of q-theory, which, classically, entails

$$\epsilon(q_0) - q_0\,\mu_0 = 0\,, \tag{6.25}$$

with $\mu_0 = d\epsilon/dq\big|_{q=q_0}$ according to Eqs. (4.8) and (4.9). In the quantum theory constructed above, we will now tentatively take the equilibrium with an associated value μ_0 of μ to be given by the condition that the quantity ρ_μ,

$$\rho_\mu \equiv {}_{\mathcal{H}_\mu}\langle \mu, 0_\mu | \hat{\rho}_\mu | \mu, 0_\mu \rangle_{\mathcal{H}_\mu}\,, \tag{6.26}$$

vanishes. Here, $|\mu, 0_\mu\rangle$ denotes the vacuum state in $\mathcal{H}_\mu$ and $\hat{\rho}_\mu$ is the Hamiltonian density associated with $\hat{H}_\mu$. For the free theory discussed in Sec. 6.1, this definition coincides with Eq. (6.25), if the zero-point energies are absorbed into the parameter Λ of the potential.

As already mentioned, an eigenstate of $\hat{\mu}$ with eigenvalue μ_0 is not a physical state. Therefore, we consider instead a Gaussian superposition of different values of μ centered around μ_0. More specifically, we consider the superposition

$$|\psi\rangle \equiv \frac{1}{(2\pi)^{1/4}\sqrt{\Delta\mu}} \int d\mu\, e^{-\frac{(\mu-\mu_0)^2}{4(\Delta\mu)^2}}\, |\mu, 0_\mu\rangle\,, \tag{6.27}$$

where $\Delta\mu$ quantifies the uncertainty with respect to $\hat{\mu}$ and is taken to be much smaller than the Planck scale E_P,

$$\Delta\mu = \sqrt{\langle\hat{\mu}^2\rangle - \langle\hat{\mu}\rangle^2} \ll (E_\text{P})^2\,. \tag{6.28}$$

For the free theory, we can then calculate the expectation value of the Hamiltonian density $\hat{\rho}$ associated with $\hat{H}$, which enters the right-hand side of the semi-classical Einstein equations,

$$\langle\psi\,|\,\hat{\rho}\,|\,\psi\rangle = \frac{1}{\sqrt{2\pi}\,\Delta\mu} \int d\mu\, e^{-\frac{(\mu-\mu_0)^2}{2(\Delta\mu)^2}}\, [\epsilon(\phi_0(\mu)) - \mu\,\phi_0(\mu)] = -\frac{(\Delta\mu)^2}{2\,\epsilon''(q_0)}\,. \tag{6.29}$$

Here, the zero-point energies are absorbed into Λ and $\phi_0(\mu)$ denotes the vacuum expectation value of $\hat{\phi}_\mu$ in $\mathcal{H}_\mu$. The nonzero value of (6.29) now suggests that an uncertainty $\Delta\mu \ll (E_\text{P})^2$ with respect to $\hat{\mu}$ leads to a contribution to the energy density of the order of $(\Delta\mu)^2$. In order to obtain an equilibrium state corresponding to Minkowski spacetime, we might therefore argue that the equilibrium of q-theory in semi-classical gravity is described by a superposition centered around a value $\mu_0 + O(\Delta\mu)$ of μ, which is shifted relative to the classical equilibrium value μ_0.

Strictly speaking, however, no superposition of different values of μ can describe the perfect equilibrium of q-theory, since no such superposition is Lorentz-invariant due to not being an eigenstate of $\hat{H}$. Therefore, the perfect equilibrium of q-theory might have to be interpreted as the idealization of the nonnormalizable state $|\mu_0, 0_{\mu_0}\rangle$.

In any case, we can also consider, in flat spacetime and on the background of a superposition as in Eq. (6.27), the quantum analog of the propagating mode discussed in Sec. 4.2. To this end, we define

$$\hat{q} \equiv \hat{q}_{\text{vev}} + \delta\hat{q} \,/\, \sqrt{C}\,, \tag{6.30a}$$

$$\hat{q}_{\text{vev}}\, |\mu, n_\mu\rangle \equiv \phi_0(\mu)\, |\mu, n_\mu\rangle\ . \tag{6.30b}$$

Expanding in $\delta\hat{q}$, we obtain for the operators $\hat{\phi}_\mu$

$$0 = \Box\, \delta\hat{\varphi}_\mu - m^2(\mu)\, \delta\hat{\varphi}_\mu\,, \tag{6.31a}$$

$$\hat{H}_\mu = \frac{1}{2}\int d^3\vec{x}\ \Big((\partial_0\, \delta\hat{\varphi}_\mu)^2 + (\partial_a\, \delta\hat{\varphi}_\mu)^2 + m^2(\mu)\, (\delta\hat{\varphi}_\mu)^2 + 2c(\mu) \Big)\ , \tag{6.31b}$$

where $\delta\hat{\varphi}_\mu$, $m^2(\mu)$, and $c(\mu)$ are defined by

$$\hat{\phi}_\mu \equiv \phi_0(\mu) + \delta\hat{\varphi}_\mu \,/\, \sqrt{C}\,, \tag{6.32a}$$

$$m^2(\mu) \equiv \epsilon''(\phi_0(\mu)) \,/\, C\,, \tag{6.32b}$$

$$c(\mu) \equiv \epsilon(\phi_0(\mu)) - \mu\, \phi_0(\mu)\,. \tag{6.32c}$$

This implies that the theory contains a continuum of particles with masses depending on μ. However, due to the fact that $\mathcal{H}$ is constructed as the direct sum and not as the direct product of all $\mathcal{H}_\mu$, these particles exhibit some unusual properties. For example, one-particle states corresponding to different values of μ cannot be combined into a multi-particle state. Moreover, all of these particles are allowed to self-interact, while particles corresponding to different values of μ cannot interact with each other since $\hat{\mu}$ is conserved.

Above, we have noted that the state $|\mu_0, 0_{\mu_0}\rangle$, which may or may not represent the perfect equilibrium of q-theory, is not normalizable due to (6.9). A related observation is that there can be no normalizable and Lorentz-invariant vacuum state. In fact, there is not even a lowest-energy state, as μ appears linearly in the Hamiltonian (6.5b). Still, the theory is not necessarily unstable, since states corresponding to different values of μ do not interact with each other.

This last remark is related to the question, whether or not the action S_{kin} leads to the Ostrogradsky instability, as discussed in Chapter 5 at the classical level. Here, we will not thoroughly study the Ostrogradsky instability at the quantum level. We do, however, note that the quantum theory constructed in this chapter does not contain any negative-energy propagating degrees of freedom in its perturbative particle spectrum according to Eq. (6.31). This was already suggested in Sec. 5.3 and is in contrast to the quantum version of the higher-derivative harmonic oscillator discussed in Sec. 5.1.[3] Therefore, it seems indeed plausible that the quantum theory constructed in this chapter is not affected by the Ostrogradsky instability. Even if some of the particles associated with Eq. (6.31)

[3] We also stated in Sec. 5.3 that we expect a quantized theory corresponding to S_{kin} to have one propagating degree of freedom only. In contrast, the quantum theory constructed in this chapter, in a sense, contains an infinite number of propagating degrees of freedom. However, for a definite value of μ as assumed in Sec. 5.3, the theory constructed in this chapter does contain only one propagating degree of freedom.

did have opposite energy, this would not immediately correspond to an instability, since particles corresponding to different values of μ do not interact with each other, as noted above. However, a more definite conclusion regarding the Ostrogradsky instability at the quantum level requires further study.

6.4. Summary and outlook

In this chapter, we have proposed a quantization procedure for the four-form realization of q-theory with a kinetic term for q. The quantum operator corresponding to the dynamically emerging integration constant μ is given by $\hat{\mu}$ in the resulting quantum theory. As suggested at the beginning of this chapter, this quantum operator $\hat{\mu}$ is constant but not simply proportional to the identity operator. In particular, $\hat{\mu}$ is formally analogous to a quantum-mechanical degree of freedom. Note that this formal analogy is not perfect. For example, the Hilbert space $\mathcal{H}$ does not factorize in general. Instead, $\mathcal{H}$ is the direct sum of the Hilbert spaces $\mathcal{H}_\mu$. Also, no physical quantity is to $\hat{\mu}$ what, for example, the position operator $\hat{x}$ is to the momentum operator $\hat{p}$ in ordinary quantum mechanics. Nevertheless, this analogy provides an interpretation of μ at the quantum level, which is in addition to the interpretations of μ as a Lagrange multiplier and a chemical potential at the classical level (cf. Sec. 1.2.1 and Refs. [12, 19]).

Further, the operator $\hat{\mu}$ as well as the quantum field $\hat{q}$ are parity-odd in agreement with the classical theory, leading to a parity-conserving quantized theory. The constructed theory is also manifestly gauge-invariant, since the operators corresponding to the three-form gauge field do not appear in any physical quantities. Still, the gauge choice discussed in Appendix C.2 leads to a subtlety regarding the time evolution of a gauge-noninvariant canonical coordinate, which requires further study.

Another point that requires further study is related to actions with non-constant $C(q)$ or potentials $\epsilon(q)$ with higher than quartic powers of q. This is because, in this chapter, we have employed arguments which can be applied only in the case of constant $C(q)$ and a quartic potential $\epsilon(q)$, namely the arguments showing that renormalization does not prevent us from regarding the constructed theory as a quantum theory corresponding to the action S_{kin}.

A physical consequence of the fact that $\hat{\mu}$ is formally analogous to a quantum-mechanical degree of freedom is that any physical state must be a superposition of a continuum of different values of μ. As a result, the equilibrium of q-theory in semi-classical gravity might be described by such a superposition centered around a value of μ, which is shifted relative to the classical equilibrium value μ_0. But taking such a superposition as the perfect equilibrium does break its Lorentz invariance, so that we might have to interpret the perfect equilibrium as the idealization of the nonnormalizable state $|\mu_0, 0_{\mu_0}\rangle$.

Furthermore, the constructed theory contains a particle with mass given by Eq. (6.32b) for each value of μ. But rather unusually, one-particle states corresponding to different values of μ cannot be combined into a multi-particle state. This is a consequence of taking the direct sum instead of the direct product when constructing $\mathcal{H}$. Also, particles corresponding to different values of μ do not interact with each other due to μ-conservation.

In Chapter 5, we have argued that the four-form realization of q-theory with a kinetic term for q is not affected by the Ostrogradsky instability at the classical level. Following the discussion in Sec. 5.3, we are optimistic that the corresponding quantum theory constructed in this chapter does not suffer from the Ostrogradsky instability as well. The reason is that there are no negative-energy particles associated with this quantized theory, which is in contrast to what we expect from a typical quantum theory suffering from the Ostrogradsky instability. However, a definite conclusion about the Ostrogradsky instability at the quantum level requires further study.

We would like to close this chapter by noting three more points that might be interesting to study further. First, one might ask, if and how the path integral approach to a quantum theory of the three-form gauge field A, as discussed in Refs. [22, 37], is related to the quantization procedure from this chapter. Second, one could study whether or not the construction from Secs. 6.1 and 6.2 can be generalized to curved spacetimes. This may be relevant for far-from-equilibrium processes in q-theory, where the background spacetime cannot be approximated by Minkowski spacetime. And third, it may be interesting to study a quantized q-theory in the presence of couplings between q and ordinary matter. This might provide further insights into the results of Ref. [20] regarding quantum-mechanical particle production and the associated backreaction.

7. Conclusion

In this master's thesis, we have discussed various aspects of the four-form realization of q-theory with a kinetic term for q. In particular, we have first considered gauge-fixing procedures for the three-form gauge field A. The main result is a complete gauge-fixing procedure which is independent of the chosen action as well as the behavior of the gauge field at the spacetime boundaries. Then, we have shown that the field equations associated with a general action of the four-form realization of q-theory have the same formal structure as the field equations associated with a certain corresponding action of a fundamental scalar field. Facilitated by these general results, we have rederived the results from Ref. [30] regarding a possible propagating mode as a consequence of a kinetic term for q. Additionally, we have argued that, in the "empty" vacuum with $q = 0$, this potential propagating mode may have a finite, negative mass-square, corresponding to an instability.

We have further shown that the four-form realization of q-theory with a kinetic term for q is free from the Ostrogradsky instability at the classical level. This is despite the fact that a kinetic term for q in the Lagrangian contains higher-order time derivatives, which typically signals the presence of the Ostrogradsky instability. The quantity μ from Eq. (3.3), which can be interpreted as a chemical potential or a Lagrange multiplier [12, 19], has played an important role in deriving the absence of the Ostrogradsky instability. In particular, it has been crucial that μ is required to be constant by the A field equations.

Finally, we have proposed a possible quantization procedure for the four-form realization of q-theory with a kinetic term for q. There are still some questions requiring further study regarding this quantization procedure. Nevertheless, the resulting quantum theory suggests that there is a formal analogy between the operator corresponding to μ at the quantum level and a quantum-mechanical degree of freedom such as the momentum operator $\hat{p}$ in ordinary quantum mechanics. This result provides an additional interpretation of the quantity μ at the quantum level.

Throughout this master's thesis, μ appears as an integration constant, which is determined by initial conditions. Therefore, taken at face value, the theories considered in this master's thesis cannot describe the equilibration of the chemical potential of q-theory in so far as this chemical potential is represented by the integration constant μ. This is why dissipation from irreversible processes has been considered in Ref. [21].

There are now at least two ways, in which the results of this master's thesis may nevertheless be helpful in better understanding the equilibration of the chemical potential of q-theory in the context of the four-form realization. First, as mentioned above, having a constant μ plays a role in avoiding the Ostrogradsky instability. Hence, this result might restrict the possible ways, in which μ can be promoted to a dynamical degree of freedom without introducing unwanted instabilities. Second, insights into quantum effects might be gained by further working out the quantization procedure for the four-form realization proposed in this master's thesis. An example of such quantum effects could be those discussed in Ref. [20], where a nonconstant chemical potential of q-theory follows from the backreaction due to quantum-mechanical particle production.

A. Generalized Lorenz gauge

A.1. Obtaining the generalized Lorenz gauge

In this appendix, we will show that the gauge functions $\lambda^{\beta\gamma}$ from Eq. (2.14), which are employed in order to obtain the generalized Lorenz gauge (2.10), satisfy not only Eq. (2.13a) but also Eq. (2.13b), $\partial_\beta \lambda^{\beta\gamma} = 0$. To this end, we define

$$t_{\text{ret}} \equiv t - |\vec{x} - \vec{x}'| \,, \tag{A.1}$$

which implies

$$\begin{aligned} \partial_0 \, t_{\text{ret}} &= 1 \,, \\ \partial_a \, t_{\text{ret}} &= -\partial'_a \, t_{\text{ret}} \,. \end{aligned} \tag{A.2}$$

Here, ∂_a denotes the partial derivative with respect to the components of $\vec{x}$, while ∂'_a denotes the partial derivative with respect to the components of $\vec{x}'$. Neglecting boundary terms, Eq. (2.13b) then follows from a direct calculation,

$$\begin{aligned}
\partial_\beta \lambda^{\beta\gamma}(t, \vec{x}) &\propto \int d^3\vec{x}' \, \partial_\beta \, \frac{\left(\partial_\rho A^{\rho\beta\gamma}\right)(t_{\text{ret}}, \vec{x}')}{|\vec{x} - \vec{x}'|} \\
&= \int d^3\vec{x}' \left(\partial_a \frac{1}{|\vec{x} - \vec{x}'|}\right) \left(\partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}') + \int d^3\vec{x}' \, \frac{\left(\partial_0 \partial_\rho A^{\rho\beta\gamma}\right)(t_{\text{ret}}, \vec{x}')}{|\vec{x} - \vec{x}'|} \, \partial_\beta t_{\text{ret}} \\
&= \int d^3\vec{x}' \left(-\partial'_a \frac{1}{|\vec{x} - \vec{x}'|}\right) \left(\partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}') + \int d^3\vec{x}' \, \frac{\left(\partial_0 \partial_\rho A^{\rho\beta\gamma}\right)(t_{\text{ret}}, \vec{x}')}{|\vec{x} - \vec{x}'|} \, \partial_\beta t_{\text{ret}} \\
&= \int d^3\vec{x}' \, \frac{1}{|\vec{x} - \vec{x}'|} \left[\partial'_a \left\{\left(\partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}')\right\} + \left(\partial_0 \partial_\rho A^{\rho\beta\gamma}\right)(t_{\text{ret}}, \vec{x}') \cdot \partial_\beta t_{\text{ret}}\right] \\
&= \int d^3\vec{x}' \frac{1}{|\vec{x} - \vec{x}'|} \Big[\left(\partial_a \partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}') + \left(\partial_0 \partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}') \cdot (\partial'_a t_{\text{ret}}) \\
&\qquad + \left(\partial_0 \partial_\rho A^{\rho\beta\gamma}\right)(t_{\text{ret}}, \vec{x}') \cdot \partial_\beta t_{\text{ret}}\Big] \\
&= \int d^3\vec{x}' \, \frac{1}{|\vec{x} - \vec{x}'|} \Big[\left(\partial_a \partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}') - \left(\partial_0 \partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}') \cdot (\partial_a t_{\text{ret}}) \\
&\qquad + \left(\partial_0 \partial_\rho A^{\rho 0\gamma}\right)(t_{\text{ret}}, \vec{x}') + \left(\partial_0 \partial_\rho A^{\rho a\gamma}\right)(t_{\text{ret}}, \vec{x}') \cdot (\partial_a t_{\text{ret}})\Big] \\
&= \int d^3\vec{x}' \frac{1}{|\vec{x} - \vec{x}'|} \left(\partial_\delta \partial_\rho A^{\rho\delta\gamma}\right)(t_{\text{ret}}, \vec{x}') = 0 \,.
\end{aligned} \tag{A.3}$$

A.2. Plane-wave solutions in the generalized Lorenz gauge

In flat spacetime and with a quadratic potential $\epsilon(q)$, the equations of motion for the three-form gauge field A take the form of massless Klein–Gordon equations, if the generalized Lorenz gauge condition (2.10) is imposed. Here, we will explicitly treat the associated plane-wave solutions.

More specifically, in flat spacetime and for a quadratic potential $\epsilon(q)$, the equations of motion for the three-form gauge field A together with the generalized Lorenz gauge condition (2.10) give

$$\Box A^{\alpha\beta\gamma} = 0\,, \tag{A.4a}$$

$$\partial_\alpha A^{\alpha\beta\gamma} = 0\,. \tag{A.4b}$$

For plane-wave solutions with wave vector k and antisymmetric polarization tensor ε,

$$A^{\alpha\beta\gamma}_{\text{pw}} = \varepsilon^{\alpha\beta\gamma} e^{ikx} + \text{cc}\,, \tag{A.5}$$

where *cc* means complex conjugation, Eq. (A.4) becomes

$$k^2 = 0\,, \tag{A.6a}$$

$$k_\alpha\, \varepsilon^{\alpha\beta\gamma} = 0\,. \tag{A.6b}$$

Like A itself, ε has four independent components, if no constraints except antisymmetry are imposed. These components can be taken to be, for example, ε^{012}, ε^{013}, ε^{023}, and ε^{123}. To see how many independent solutions Eq. (A.6) admits, we consider

$$k^\alpha = (\omega, 0, 0, \omega)^\alpha\,. \tag{A.7}$$

Due to Lorentz symmetry and Eq. (A.6a), there is always a reference frame, in which k takes the form (A.7).

For $\omega = 0$, all four components of ε can be chosen arbitrarily, since Eq. (A.6) does not impose any further conditions in this case. However, this mode is not very interesting, as it leads to $q = 0$ and is therefore gauge equivalent to $A = 0$.

For $\omega \neq 0$, Eq. (A.6b) implies

$$0 = \varepsilon^{013}\,, \tag{A.8a}$$

$$0 = \varepsilon^{023}\,, \tag{A.8b}$$

$$0 = \varepsilon^{012} - \varepsilon^{123}\,. \tag{A.8c}$$

As a result, ε has at most one independent component, $\varepsilon_0 \equiv \varepsilon^{012} = \varepsilon^{123}$.

As already noted in Sec. 2.2, the generalized Lorenz gauge (2.10) leaves a residual gauge freedom. In particular, any gauge transformation with gauge functions $\lambda^{\beta\gamma}$ satisfying the following conditions does not violate the generalized Lorenz gauge condition:

$$\Box \lambda^{\beta\gamma} = 0\,, \tag{A.9a}$$

$$\partial_\beta \lambda^{\beta\gamma} = 0\,. \tag{A.9b}$$

Here, we choose

$$\lambda^{\beta\gamma} \equiv \bar{\varepsilon}^{\beta\gamma} e^{ikx} + cc\,, \tag{A.10a}$$

$$k_\beta\, \bar{\varepsilon}^{\beta\gamma} = 0\,, \tag{A.10b}$$

with antisymmetric and constant $\bar{\varepsilon}$ and k as defined in Eq. (A.7). The constraint (A.10b) for the components of $\bar{\varepsilon}$ reads explicitly

$$0 = \bar{\varepsilon}^{01} - \bar{\varepsilon}^{31}\,, \tag{A.11a}$$

$$0 = \bar{\varepsilon}^{02} - \bar{\varepsilon}^{32}\,, \tag{A.11b}$$

$$0 = \bar{\varepsilon}^{03}\,. \tag{A.11c}$$

Note that the component $\bar{\varepsilon}^{12} = -\bar{\varepsilon}^{21}$ remains unconstrained.

A gauge transformation with gauge functions $\lambda^{\beta\gamma}$ as in Eq. (A.10) then gives

$$A^{\alpha\beta\gamma}_{\text{pw}} \to A^{\alpha\beta\gamma}_{\text{pw}} + \left(ik^{[\alpha}\, \bar{\varepsilon}^{\beta\gamma]} e^{ikx} + cc \right)\,, \tag{A.12}$$

which amounts to a shift in the polarization tensor ε,

$$\varepsilon^{\alpha\beta\gamma} \to \varepsilon^{\alpha\beta\gamma} + ik^{[\alpha}\, \bar{\varepsilon}^{\beta\gamma]}\,. \tag{A.13}$$

We will now choose specific values for the components of $\bar{\varepsilon}$. First, we take all the constrained components of $\bar{\varepsilon}$ to be zero, leaving $\bar{\varepsilon}^{12} = -\bar{\varepsilon}^{21}$ as the only nonvanishing components of $\bar{\varepsilon}$. As a result, Eq. (A.11) is trivially satisfied and only the components ε^{012} and ε^{123} of the polarization tensor ε are affected by (A.13),

$$\varepsilon^{012} \to \varepsilon^{012} + i\, k^{[0} \bar{\varepsilon}^{12]} = \varepsilon^{012} + 2i\, \omega\, \bar{\varepsilon}^{12} = \varepsilon_0 + 2i\, \omega\, \bar{\varepsilon}^{12}\,, \tag{A.14a}$$

$$\varepsilon^{123} \to \varepsilon^{123} + i\, k^{[1} \bar{\varepsilon}^{23]} = \varepsilon^{123} + 2i\, \omega\, \bar{\varepsilon}^{12} = \varepsilon_0 + 2i\, \omega\, \bar{\varepsilon}^{12}\,. \tag{A.14b}$$

The other components of ε remain zero. Eq. (A.14) then implies that we can get rid of both nonvanishing components ε^{012} and ε^{123} of ε at the same time by choosing $\bar{\varepsilon}^{12} = -\varepsilon_0/2i\omega$. Thus, we conclude that all plane-wave solutions can be gauged away,

$$A^{\alpha\beta\gamma}_{\text{pw}} = 0\,. \tag{A.15}$$

B. Propositions from Chapter 3

B.1. Proof of Proposition 1

The purpose of this appendix is to prove Proposition 1. To this end, we first define,

$$S_0[\phi, g] \equiv \int d^4x \sqrt{-g}\, \mathcal{L}[\phi, g] \,. \tag{B.1}$$

With this definition, we have

$$S_{\mathrm{f}}[\phi, g] = S_0[\phi, g] + \int d^4x \sqrt{-g}\, \mu_{\mathrm{f}}\, \phi \,, \tag{B.2a}$$

$$S_{\mathrm{nf}}[q(A, g), g] = S_0[q(A, g), g] \,. \tag{B.2b}$$

The equations of motion associated with the action S_{f} are then

$$0 = \frac{\delta S_{\mathrm{f}}[\phi, g]}{\delta \phi} = \frac{\delta S_0[\phi, g]}{\delta \phi} + \sqrt{-g}\, \mu_{\mathrm{f}} \,, \tag{B.3a}$$

$$0 = \frac{\delta S_{\mathrm{f}}[\phi, g]}{\delta g_{\alpha\beta}} = \frac{\delta S_0[\phi, g]}{\delta g_{\alpha\beta}} + \frac{1}{2} \sqrt{-g}\, \mu_{\mathrm{f}}\, \phi\, g^{\alpha\beta} \,, \tag{B.3b}$$

where we have used

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g}\, g^{\alpha\beta} \delta g_{\alpha\beta} \,. \tag{B.4}$$

Using the chain rule for functional derivatives and the expression (1.21) of q in terms of A, the equations of motion for A associated with S_{nf} can now be written as

$$\begin{aligned} 0 &= \frac{\delta S_{\mathrm{nf}}[q(A, g), g]}{\delta A_{\alpha\beta\gamma}(x)} = \int d^4y\, \frac{\delta S_{\mathrm{nf}}[q(A, g), g]}{\delta q(y)}\, \frac{\delta q(y)}{\delta A_{\alpha\beta\gamma}(x)} \\ &= \int d^4y\, \frac{\delta S_0[q, g]}{\delta q(y)} \left(-\frac{1}{\sqrt{-g}} \epsilon^{\delta\nu\rho\sigma} \right) \frac{\delta\, \partial_\delta A_{\nu\rho\sigma}(y)}{\delta A_{\alpha\beta\gamma}(x)} \\ &= \epsilon^{\delta\alpha\beta\gamma}\, \partial_\delta \left(\frac{1}{\sqrt{-g}}\, \frac{\delta S_0[q, g]}{\delta q(x)} \right) . \end{aligned} \tag{B.5}$$

In terms of the quantity μ from Eq. (3.3), Eq. (B.5) is the statement that μ is constant. This proves the first part of Proposition 1.

In order to obtain the equations of motion for the metric associated with S_{nf}, we will utilize the relation

$$\delta q = -\frac{1}{2}\, q\, g^{\alpha\beta}\, \delta g_{\alpha\beta} \,, \tag{B.6}$$

which immediately follows from Eqs. (1.21) and (B.4). The equations of motion for the metric associated with $S_{\rm nf}$ can then be obtained using Eq. (B.6) as well as the chain rule for functional derivatives,

$$\begin{aligned}0 = \frac{\delta S_{\rm nf}[q(A,g),g]}{\delta g_{\alpha\beta}(x)} &= \left.\frac{\delta S_{\rm nf}[\phi,g]}{\delta g_{\alpha\beta}(x)}\right|_{\phi=q} + \int d^4y\, \frac{\delta S_{\rm nf}[q(A,g),g]}{\delta q(y)} \frac{\delta q(y)}{\delta g_{\alpha\beta}(x)} \\ &= \left.\frac{\delta S_0[\phi,g]}{\delta g_{\alpha\beta}(x)}\right|_{\phi=q} + \frac{\delta S_0[q,g]}{\delta q(x)} \left(-\frac{1}{2}\, q\, g^{\alpha\beta}\right) .\end{aligned} \tag{B.7}$$

Using the definition of μ from Eq. (3.3), Eq. (B.7) can be simplified,

$$0 = \left.\frac{\delta S_0[\phi,g]}{\delta g_{\alpha\beta}}\right|_{\phi=q} + \frac{1}{2}\sqrt{-g}\,\mu\, q\, g^{\alpha\beta} . \tag{B.8}$$

It can now be seen that, for a fixed constant value of μ with $\mu = \mu_{\rm f}$, Eq. (B.8) has the same formal structure as Eq. (B.3b), which proves the second part of Proposition 1.

B.2. Proof of Proposition 2

In this appendix, we prove Proposition 2. The proof proceeds in five steps.

Step 0

Similar to the proof of Proposition 1, we first define

$$\overline{S}_0[\phi,g] = \int d^4x \sqrt{-g}\mathcal{L}[\phi,g] . \tag{B.9}$$

With this definition, we have

$$\overline{S}_{\rm f}[\phi,g] = \overline{S}_0[\phi,g] + \int d^4x\, \sqrt{-g}\,\overline{\mu}_{\rm f}\,\phi , \tag{B.10a}$$

$$\overline{S}_{\rm nf}[\overline{q}(\overline{A},g),g] = \overline{S}_0[\overline{q}(\overline{A},g),g] . \tag{B.10b}$$

For $\overline{\mu}_{\rm f} = \mu_{\rm f}$, the action $\overline{S}_{\rm f}$ is actually the same action as the action $S_{\rm f}$ in the case of Proposition 1. Therefore, the equations of motion for ϕ and g in the case of $\overline{S}_{\rm f}$ are are also just those of Eq. (B.3) with appropriately adapted notation,

$$0 = \frac{\delta \overline{S}_{\rm f}[\phi,g]}{\delta\phi} = \frac{\delta \overline{S}_0[\phi,g]}{\delta\phi} + \sqrt{-g}\,\overline{\mu}_{\rm f} , \tag{B.11a}$$

$$0 = \frac{\delta \overline{S}_{\rm f}[\phi,g]}{\delta g_{\alpha\beta}} = \frac{\delta \overline{S}_0[\phi,g]}{\delta g_{\alpha\beta}} + \frac{1}{2}\sqrt{-g}\,\overline{\mu}_{\rm f}\,\phi\, g^{\alpha\beta} . \tag{B.11b}$$

Eq. (3.13) implies that the equations of motion for $\overline{A}$ are equivalent to having constant $\overline{\mu}$, with $\overline{\mu}$ as defined in Eq. (3.12). Therefore, Eq. (B.11a) already has the same formal structure as the $\overline{A}$ field equations for $\overline{\mu} = \overline{\mu}_{\mathrm{f}}$. Hence, it only remains to be shown that the equations of motion for the metric associated with $\overline{S}_{\mathrm{nf}}$ have the same formal structure as Eq. (B.11b).

It suffices to show that $\overline{q}$ is necessarily of the form

$$\overline{q} = \nabla_\alpha V^\alpha[g, \overline{A}]\,, \tag{B.12}$$

where $V^\alpha[g, \overline{A}]$ is a functional that transforms as a pseudovector or a vector depending on whether $\overline{q}$ is a scalar or a pseudoscalar. This is because Eq. (B.12) implies

$$\begin{aligned}
0 = \frac{\delta \overline{S}_{\mathrm{nf}}[\overline{q}(\overline{A}, g), g]}{\delta g_{\alpha\beta}(x)} &= \left.\frac{\delta \overline{S}_{\mathrm{nf}}[\phi, g]}{\delta g_{\alpha\beta}(x)}\right|_{\phi=\overline{q}} + \int d^4y\, \frac{\delta \overline{S}_{\mathrm{nf}}[\overline{q}(\overline{A}, g), g]}{\delta \overline{q}(y)} \frac{\delta \overline{q}(y)}{\delta g_{\alpha\beta}(x)} \\
&= \left.\frac{\delta \overline{S}_0[\phi, g]}{\delta g_{\alpha\beta}(x)}\right|_{\phi=\overline{q}} - \overline{\mu} \int d^4y\, \sqrt{-g}\, \frac{\delta \overline{q}(y)}{\delta g_{\alpha\beta}(x)} \\
&= \left.\frac{\delta \overline{S}_0[\phi, g]}{\delta g_{\alpha\beta}(x)}\right|_{\phi=\overline{q}} - \overline{\mu} \int d^4y\, \sqrt{-g}\, \frac{\delta}{\delta g_{\alpha\beta}(x)} \left(\frac{1}{\sqrt{-g}} \partial_\gamma \left(\sqrt{-g}\, V^\gamma\right)\right) \\
&= \left.\frac{\delta \overline{S}_0[\phi, g]}{\delta g_{\alpha\beta}(x)}\right|_{\phi=\overline{q}} + \frac{1}{2} \overline{\mu}\, \sqrt{-g}\, g^{\alpha\beta}\, \frac{1}{\sqrt{-g}} \partial_\gamma \left(\sqrt{-g}\, V^\gamma\right) \\
&= \left.\frac{\delta \overline{S}_0[\phi, g]}{\delta g_{\alpha\beta}(x)}\right|_{\phi=\overline{q}} + \frac{1}{2} \overline{\mu}\, \sqrt{-g}\, g^{\alpha\beta}\, \overline{q}\,,
\end{aligned} \tag{B.13}$$

where we have used Eqs. (B.4) and (3.13) as well as the relation

$$\nabla_\alpha V^\alpha = \frac{1}{\sqrt{-g}}\, \partial_\alpha \left(\sqrt{-g}\, V^\alpha\right)\,. \tag{B.14}$$

Eq. (B.13) has precisely the same formal structure as Eq. (B.11b) for $\overline{\mu} = \overline{\mu}_{\mathrm{f}}$, as was to be shown. Hence, the remaining four steps will consist in proving Eq. (B.12).

Step 1

Purely from the definition of $\overline{q}$ in terms of $\overline{A}$ and g, we define for arbitrary $\mathcal{F}$, g and $\overline{A}$,

$$E^{\alpha\ldots}[\mathcal{F}, g, \overline{A}] \equiv \int d^4y\, \sqrt{-g}\, \mathcal{F}(y)\, \frac{\delta \overline{q}(y)}{\delta \overline{A}_{\alpha\ldots}(x)}\,, \tag{B.15}$$

where $\mathcal{F}$ is a pseudoscalar or scalar quantity depending on whether $\overline{q}$ is a pseudoscalar or a scalar.

E has the following properties: It is linear in $\mathcal{F}$ (in particular, we have $E|_{\mathcal{F}=0} = 0$), it is a local functional, and it is independent of the chosen action $\overline{S}_{\mathrm{nf}}$. As a result, E can be written as a finite sum of the form

$$\begin{aligned}
E^{\alpha\ldots}[\mathcal{F}, g, \overline{A}] &= E_0^{\alpha\ldots}[g, \overline{A}]\, \mathcal{F} + E_1^{\rho\alpha\ldots}[g, \overline{A}]\, \nabla_\rho \mathcal{F} + E_2^{\sigma\rho\alpha\ldots}[g, \overline{A}]\, \nabla_\sigma \nabla_\rho \mathcal{F} + \ldots \\
&= E_0^{\alpha\ldots}[g, \overline{A}]\, \mathcal{F} + \left(E_1^{\rho\alpha\ldots}[g, \overline{A}] + E_2^{\sigma\rho\alpha\ldots}[g, \overline{A}]\, \nabla_\sigma + \ldots\right) \nabla_\rho \mathcal{F}\,,
\end{aligned} \tag{B.16}$$

where the E_i, $i = 0, 1, \ldots$ are independent of $\mathcal{F}$.

Step 2

For any given $\overline{S}_{\mathrm{nf}}$, the equations of motion for $\overline{A}$ can now be written as

$$0 = \int d^4y\, \frac{\delta \overline{S}_{\mathrm{nf}}}{\delta \overline{q}(y)} \frac{\delta \overline{q}(y)}{\delta \overline{A}_{\alpha\dots}(x)} = E^{\alpha\dots}\left[\frac{1}{\sqrt{-g}}\frac{\delta \overline{S}_{\mathrm{nf}}}{\delta \overline{q}}, g, \overline{A}\right] = E^{\alpha\dots}\left[-\overline{\mu}, g, \overline{A}\right] . \tag{B.17}$$

Thus, for any g_{s} and $\overline{A}_{\mathrm{s}}$ satisfying the equations of motion associated with $\overline{S}_{\mathrm{nf}}$, Eqs. (3.13) and (B.16) imply

$$0 = E_0^{\alpha\dots}[g_{\mathrm{s}}, \overline{A}_{\mathrm{s}}]\, \overline{\mu} . \tag{B.18}$$

Step 3

For a given value of $\overline{\mu} \neq 0$, Eq. (B.18) implies that E_0 vanishes for all $\overline{A}_{\mathrm{s}}$ and g_{s} solving of the equations of motion associated with $\overline{S}_{\mathrm{nf}}$,

$$E_0[g_{\mathrm{s}}, \overline{A}_{\mathrm{s}}] = 0 . \tag{B.19}$$

We are now interested in whether or not E_0 also vanishes for general g and $\overline{A}$,

$$E_0[g, \overline{A}] \stackrel{?}{=} 0 , \quad \text{for any } \ g, \overline{A} . \tag{B.20}$$

To this end, we can think of

$$E_0^{\alpha\dots}[g, \overline{A}] = 0 \tag{B.21}$$

as differential equations for g and $\overline{A}$ of a certain fixed order. These differential equations (B.21) are independent of the chosen $\overline{S}_{\mathrm{nf}}$ and have a certain set of solutions. Similarly, the equations which determine g_{s} and $\overline{A}_{\mathrm{s}}$ are also differential equations, namely

$$\overline{\mu} = \frac{1}{\sqrt{-g_{\mathrm{s}}}} \frac{\delta \overline{S}_{\mathrm{nf}}[\overline{q}(g_{\mathrm{s}}, \overline{A}_{\mathrm{s}}), g_{\mathrm{s}}]}{\delta \overline{q}} , \tag{B.22a}$$

$$0 = \frac{\delta \overline{S}_{\mathrm{nf}}[\overline{q}(g_{\mathrm{s}}, \overline{A}_{\mathrm{s}}), g_{\mathrm{s}}]}{\delta g_{\alpha\beta}} . \tag{B.22b}$$

In contrast to E_0, $\overline{S}_{\mathrm{nf}}$ and $\overline{\mu} = \text{const} \neq 0$ can be freely chosen. In particular, (B.22) can be differential equations of arbitrary order.

Hence, for some $\overline{S}_{\mathrm{nf}}$ and $\overline{\mu} \neq 0$, there certainly are solutions of Eq. (B.22) which are not solutions of Eq. (B.21) - except if $E_0 \equiv 0$. Therefore, we conclude that indeed $E_0 \equiv 0$, since $E_0^{\alpha\dots}[g_{\mathrm{s}}, \overline{A}_{\mathrm{s}}]$ has to vanish for arbitrary $\overline{S}_{\mathrm{nf}}$ and arbitrary $\overline{\mu} \neq 0$. As a result, Eq. (B.16) becomes

$$E^{\alpha\dots}[\mathcal{F}, g, \overline{A}] = \left(E_1^{\rho\alpha\dots}[g, \overline{A}] + E_2^{\sigma\rho\alpha\dots}[g, \overline{A}]\, \nabla_\sigma + \dots\right) \nabla_\rho \mathcal{F} . \tag{B.23}$$

Step 4

Eqs. (B.15) and (B.23) now imply that $\overline{q}$ must contain at least one derivative acting on $\overline{A}$, if E is not identically zero. Still, in principle $\overline{q}$ could additionally contain terms without derivatives acting on $\overline{A}$. In order to better understand the structure of $\overline{q}$, we symbolically write $\overline{q}$ as

$$\overline{q} = U_1[g,\overline{A}]\,\nabla\, V_1[g,\overline{A}] + U_2[g,\overline{A}]\,\nabla\, V_2[g,\overline{A}] + \cdots + C[g,\overline{A}]\,. \tag{B.24}$$

Here, U_i, V_i, $i = 1, 2, \ldots$ are local functionals of g and $\overline{A}$, which may carry tensor indices, and C is a (pseudo-)scalar local functional of g and $\overline{A}$. Further, we take C and the U_i to not contain derivatives acting on $\overline{A}$, while the V_i may include additional derivatives acting on $\overline{A}$. In other words, the ∇ written in Eq. (B.24) is the first derivative acting on $\overline{A}$ in each term.

From Eq. (B.24), we can now calculate E using Eq. (B.15) and then compare to the supposed result from Eq. (B.23). The expressions for E as calculated form Eq. (B.24) and as given in Eq. (B.23) agree, if C does not depend on $\overline{A}$ and the U_i, $i = 1, 2, \ldots$ are constants. However, if the U_i, $i = 1, 2, \ldots$ are not constants or C depends on $\overline{A}$, additional terms may appear in E as calculated form Eq. (B.24). In particular, E as calculated from Eq. (B.24) may not result in a vanishing E_0. Explicitly, the terms that would contribute to E_0 are the following:

$$\int d^4y\sqrt{-g}\,\mathcal{F}\left(\frac{\delta C(y)}{\delta\overline{A}_{\alpha\ldots}(x)} + \sum_i \frac{\delta U_i(y)}{\delta\overline{A}_{\alpha\ldots}(x)}\,\nabla V_i(y) - \frac{\delta V_i(y)}{\delta\overline{A}_{\alpha\ldots}(x)}\,\nabla U_i(y)\right). \tag{B.25}$$

Therefore, in order to agree with Eq. (B.23), those terms have to vanish for arbitrary $\mathcal{F}$,

$$0 = \frac{\delta C(y)}{\delta\overline{A}_{\alpha\ldots}(x)} + \sum_i \frac{\delta U_i(y)}{\delta\overline{A}_{\alpha\ldots}(x)}\,\nabla V_i(y) - \frac{\delta V_i(y)}{\delta\overline{A}_{\alpha\ldots}(x)}\,\nabla U_i(y)\,. \tag{B.26}$$

Next, we write Eq. (B.24) as

$$\overline{q} = \nabla\sum_i U_i V_i - \sum_i V_i \nabla U_i + C\,. \tag{B.27}$$

Also, we note that Eq. (B.26) implies

$$\frac{\delta}{\delta\overline{A}(x)}\left(\sum_i V_i(y)\nabla U_i(y) - C(y)\right) = \sum_i \frac{\delta V_i(y)}{\delta\overline{A}(x)}\,\nabla U_i(y) - \frac{\delta U_i(y)}{\delta\overline{A}(x)}\,\nabla V_i(y) - \frac{\delta C(y)}{\delta\overline{A}(x)} = 0\,. \tag{B.28}$$

As a result, $\sum_i V_i\nabla U_i - C$ depends only on the metric such that $\overline{q}$ can be written as a total derivative plus terms independent of $\overline{A}$ according to Eq. (B.27). Since by assumption $\overline{q}|_{\overline{A}=0} = 0$, this means that $\overline{q}$ can indeed be written as in Eq. (B.12),

$$\overline{q} = \nabla_\alpha V^\alpha[g,\overline{A}]\,. \tag{B.29}$$

C. Quantization

C.1. Transformation of $\hat{q}$ under $\mathcal{P}$

In this appendix, we derive Eq. (6.16) from the definitions in Sec. 6.1.1. Note that it suffices to show that Eq. (6.16) holds with both sides multiplied by $|\mu, n_\mu\rangle$ since $\hat{q}$ and $\mathcal{P}$ are linear. We start with the following definitions:

$$\hat{\phi}_\mu(t,\vec{x})\,|\mu, n_\mu\rangle \equiv \sum_{\tilde{n}_\mu} c_{\tilde{n}_\mu}(t,\vec{x})\,|\mu, \tilde{n}_\mu\rangle\ , \tag{C.1a}$$

$$-\hat{\phi}_{-\mu}(t,\vec{x})\,|-\mu, n_{-\mu}\rangle \equiv \sum_{\tilde{n}_{-\mu}} d_{\tilde{n}_{-\mu}}(t,\vec{x})\,|-\mu, \tilde{n}_{-\mu}\rangle\ . \tag{C.1b}$$

As argued in Sec. 6.1.1, we have $\mathcal{H}_\mu = \mathcal{H}_{-\mu}$ and $\hat{\phi}_\mu(x) = -\hat{\phi}_{-\mu}(x)$. Hence, both expansion coefficients in (C.1) are the same, i.e., $c_{\tilde{n}_\mu} = d_{\tilde{n}_{-\mu}} \equiv c_{\tilde{n}}$. A direct calculation then yields

$$\begin{aligned}\mathcal{P}\,\hat{q}(t,\vec{x})\,|\mu, n_\mu\rangle &= \mathcal{P}\,\hat{\phi}_\mu(t,\vec{x})\,|\mu, n_\mu\rangle = \mathcal{P}\sum_{\tilde{n}} c_{\tilde{n}}(t,\vec{x})\,|\mu, \tilde{n}\rangle \\ &= \mathcal{P}_{-\mu}\sum_{\tilde{n}} c_{\tilde{n}}(t,\vec{x})\,|-\mu, \tilde{n}\rangle = \mathcal{P}_{-\mu}\left(-\hat{\phi}_{-\mu}(t,\vec{x})\right)|-\mu, n_{-\mu}\rangle \\ &= -\hat{\phi}_{-\mu}(t,-\vec{x})\,\mathcal{P}_{-\mu}\,|-\mu, n_{-\mu}\rangle = -\hat{q}(t,-\vec{x})\,\mathcal{P}_{-\mu}\,|-\mu, n_{-\mu}\rangle \\ &= -\hat{q}(t,-\vec{x})\,\mathcal{P}\,|\mu, n_\mu\rangle\ .\end{aligned} \tag{C.2}$$

C.2. Alternative gauge choice

In Sec. 6.1.1, the A_{023} gauge (2.6) was employed in order to construct quantum operators corresponding to the three-form gauge field A. Here, we consider a similar construction for an incomplete A_{123} gauge (2.6). By this we mean that we follow Sec. 2.1 in constructing the A_{123} gauge, but we do not remove the x^0-independent function $A_0(\vec{x})$ from A_{123},

$$A_{0\beta\gamma} = 0\,, \tag{C.3a}$$

$$-3!\,A_{123}(t,\vec{x}) = -A_0(\vec{x}) + \int_0^t d\tilde{t}\; q(\tilde{t},\vec{x})\,. \tag{C.3b}$$

Eq. (C.3) does not completely fix the gauge, since we do not specify $A_0(\vec{x})$ for the moment. The canonical variables in the gauge (C.3) are the same as those in the A_{123} gauge and are given in Eq. (5.10). Explicitly,

$$Q_1 = -3!\,A_{123}\,, \tag{C.4a}$$
$$P_1 = \delta S_{\text{kin}}/\delta q = -\mu\,, \tag{C.4b}$$
$$Q_2 = -3!\,\partial_0 A_{123} = q\,, \tag{C.4c}$$
$$P_2 = C\,\partial_0 q\,. \tag{C.4d}$$

Note that P_1 is proportional to the quantity μ from Eq. (3.3). As in Sec. 6.1.1, we define corresponding quantum operators by replacing each field in (C.3) and (C.4) by its respective quantum operator. We denote the resulting quantum operators by a hat, e.g. $\hat{Q}_1$, and leave the operator $\hat{A}_0(\vec{x})$ corresponding to $A_0(\vec{x})$ unspecified. Ignoring $\hat{A}_0$, we can then infer the following commutation relations:

$$[\,\hat{Q}_2(t,\vec{x}),\,\hat{P}_2(t,\vec{y})\,] = i\delta(\vec{x}-\vec{y})\,, \tag{C.5a}$$
$$[\,\hat{Q}_1,\,\hat{P}_1\,] = [\,\hat{Q}_2,\,\hat{P}_1\,] = [\,\hat{P}_2,\hat{P}_1\,] = 0\,. \tag{C.5b}$$

The commutators $[\hat{Q}_1,\hat{Q}_2]$ and $[\hat{Q}_1,\hat{P}_2]$ are not explicitly calculated here.

We see that $\hat{Q}_2$ and $\hat{P}_2$ satisfy canonical commutation relations, while at least some of the commutators involving $\hat{Q}_1$ are noncanonical. We can also see that a canonical commutator between $\hat{Q}_1$ and $\hat{P}_1$ is not possible, since $\hat{P}_1 \propto \hat{\mu}$ is required to be constant by the equations of motion for $\hat{A}$. As in Sec. 6.1.2, these noncanonical commutators are not problematic, since $\hat{Q}_1 \propto \hat{A}_{123}$ is gauge-noninvariant and Eq. (C.5) is consistent with the gauge choice as well as the equations of motion.

However, there is a subtlety regarding the time evolution of $\hat{Q}_1$. Unlike in the gauge (6.17), the quantized canonical Hamiltonian coincides with the Hamiltonian $\hat{H}$ from Eq. (6.13) in the gauge (C.3). But, ignoring $\hat{A}_0$, the time evolution of $\hat{Q}_1 \propto \hat{A}_{123}$ is not given by its commutator with $\hat{H}$,

$$[\,-3!\,\hat{A}_{123}(t,\vec{x}),\,\hat{H}\,] = i\,\hat{q}(t,\vec{x}) - i\,\hat{q}(0,\vec{x}) \neq i\,\hat{q}(t,\vec{x}) = i\,\partial_0\left(-3!\,\hat{A}_{123}(t,\vec{x})\right)\,. \tag{C.6}$$

It might be possible to ameliorate Eq. (C.6) by adjusting $\hat{H}$ or by imposing

$$[\,-\hat{A}_0(\vec{x}),\,\hat{H}\,] = i\,\hat{q}(0,\vec{x})\,. \tag{C.7}$$

However, we are currently able to explicitly fix Eq. (C.6) only for $C = 0$, $\epsilon \propto q^2$, and a finite spatial volume V. In this case, we can employ a similar construction as in Sec. 6.1.1, but with the operators $\hat{\phi}_\mu$ being trivial and $\hat{\mu} \propto \hat{q}$. We then obtain $[-3!\hat{A}_{123},\hat{H}] = [-\hat{A}_0,\hat{H}] = i\,\hat{q}$ by imposing

$$[\,\hat{A}_0,\,\hat{\mu}\,] = \frac{i}{V}\,. \tag{C.8}$$

Note that Eq. (C.8) is similar to a quantum-mechanical canonical commutation relation, strengthening the analogy of $\hat{\mu}$ to a quantum-mechanical degree of freedom. Also, imposing Eq. (C.8) is similar to what is done in Ref. [48] in order to quantize bosonic zero-frequency modes. Still, more general cases, especially with $C \neq 0$, require further study.

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