Cosmological constant problem: Revisiting the unimodular-gravity approach

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1. Introduction

The main **Cosmological Constant Problem** (CCP1) can be phrased as follows (Pauli, 1933; Bohr, 1948; Veltman, 1974; see [1, 2] for two reviews):

*why do the quantum fields in the vacuum not produce naturally a large cosmological constant $\Lambda$ in the Einstein gravitational field equation?*

The magnitude of the problem is enormous:

$$|\Lambda^{\text{theory}}|/|\Lambda^{\text{experiment}}| \geq 10^{54},$$

where the large number on the RHS will be explained on the next slide.

From now on, $c = 1$ and $\hbar = 1.$
1. Introduction

With the ATLAS and CMS results \cite{3,4} in support of the Higgs mechanism, it is clear that the EWSM in the laboratory involves a vacuum energy density of order

$$|\epsilon^{(EWSM)}_V| \sim (100 \text{ GeV})^4 \sim 10^{44} \text{ eV}^4.$$ 

Moreover, this energy density can be expected to change as the temperature $T$ of the Universe drops,

$$\epsilon^{(EWSM)}_V = \epsilon^{(EWSM)}_V(T).$$

How can the Universe then end up with a vacuum energy density

$$|\Lambda^{(obs)}| < 10^{-28} \text{ g cm}^{-3} \sim 10^{-10} \text{ eV}^4?$$

Here, there are 54 orders of magnitude to explain:

$$|\Lambda^{(obs)}/\epsilon^{(EWSM)}_V| \leq 0.000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 000 \ 001.$$
1. Introduction

In short, the main cosmological constant problem is

**CCP1** – why $|\Lambda| \ll (E_{\text{QCD}})^4 \ll (E_{\text{electroweak}})^4 \ll (E_{\text{Planck}})^4$ ?

Still more CCPs after the discovery of the “accelerating Universe”:

**CCP2a** – why $\Lambda \neq 0$ ?

**CCP2b** – why $\Lambda \approx \rho_{\text{matter}}|_{\text{present}} \sim +10^{-11} \text{ eV}^4$ ?

Hundreds of papers have been published on CCP2. But, most likely:

**CCP1 needs to be solved first, before CCP2 can even be addressed.**
1. Introduction

Here, a discussion of one particular approach to CCP1 by Volovik and the speaker, which goes under the name of $q$–theory [5, 6, 7, 8, 9, 10] (a brief review appears in App. A of [11]).

Originally, we considered four explicit realizations of $q$–theory using

1. a three-form gauge field [12, 13, 14, 15],

2. a massless vector-field [16, 17],

3. a spacetime 4D-brane [18],

4. an elasticity tetrad from a spacetime crystal [19].

The present talk will, however, focus on an entirely new and attractive realization.
1. Introduction

OUTLINE:

1. Introduction
2. Basics of $q$–theory ← original idea
3. Postulated three-form gauge field
4. Metric determinant ← new approach
5. Metric determinant: Cosmology
6. Conclusion
7. References
2. Basics of $q$–theory

Crucial insight [5]: there is vacuum energy and vacuum energy.

More specifically and introducing an appropriate notation:

the vacuum energy density $\epsilon$ appearing in the action

need not be the same as

the vacuum energy density $\rho_V$ in the Einstein field equation.

How could this happen concretely . . .
2. Basics of $q$–theory

Assume the full quantum vacuum to be a **self-sustained medium** (as is a droplet of water in free fall).

That medium would be characterized by some **conserved charge**.

Study, then, the **macroscopic** equations of this conserved **microscopic** variable (later called $q$), whose precise nature need not be known.

An analogy:

- Take the mass density $\rho$ of a liquid, for example, liquid Argon.
- This $\rho$ describes microscopic quantities ($\rho = m_{Ar} \, n_{Ar}$ with number density $n_{Ar}$ and mass $m_{Ar}$ of the atoms).
- Still, $\rho$ obeys the macroscopic equations of hydrodynamics, because of particle-number conservation and mass conservation.
2. Basics of $q$–theory

However, is the quantum vacuum similar to a “normal” liquid?

No, the quantum vacuum behaves like a liquid but not like a “normal” liquid.
2. Basics of $q$–theory

In fact, the quantum vacuum is known to be Lorentz invariant (cf. experimental limits at the $10^{-15}$ level in the photon sector [20]).

The Lorentz invariance of the vacuum rules out the standard type of charge density, which arises from the time component $j_0$ of a conserved vector current $j_\mu$.

Needed is a new type of relativistic conserved charge, called the vacuum variable $q$.

In other words, look for a relativistic generalization ($q$) of the number density ($n$) which characterizes the known material liquids.
2. Basics of $q$–theory

With such a variable $q(x)$, the vacuum energy density of the effective action can be a generic function

$$
\epsilon = \epsilon(q) = \Lambda_{\text{bare}} + \epsilon_{\text{nonconstant}}(q),
$$

(1)

including a possible constant term $\Lambda_{\text{bare}}$ from the zero-point energies of the fields of the Standard Model (SM).

From ① thermodynamics and ② Lorentz invariance follows that [5]

$$
P_V \overset{①}{=} - \left( \epsilon - q \frac{d\epsilon}{dq} \right) \overset{②}{=} -\rho_V,
$$

(2)

where the first equality corresponds to an integrated form of the Gibbs–Duhem equation for chemical potential $\mu \equiv d\epsilon/dq$.

Recall GD eq: $N \, d\mu = V \, dP - S \, dT \Rightarrow dP = (N/V) \, d\mu$ for $dT = 0$. 
2. Basics of $q$–theory

Both terms entering $\rho_V$ from (2) can be of order $(E_{\text{Planck}})^4$, but they cancel exactly for an appropriate value $q_0$ of the vacuum variable $q$.

Hence, for a generic function $\epsilon(q)$,

$$\exists q_0 = \text{const} : \Lambda \equiv \rho_V = \left[ \epsilon(q) - q \frac{d\epsilon(q)}{dq} \right]_{q=q_0} = 0 , \quad (3)$$

with constant vacuum variable $q_0$ [a similar constant variable is known to play a role for the Larkin–Pikin effect (1969) in solid-state physics].

Great, CCP1 solved, in principle . . .

**But, is such a relativistic vacuum variable $q$ possible at all?**

**Yes**, there exist several theories which contain such a $q$ variable and one example will be given in Sec. 3.
3 Postulated three-form gauge field

Vacuum variable $q$ may arise from a 3–form gauge field $A$ \cite{12, 13}.

Start from the effective action of GR+SM,

$$S^{\text{eff}}[g, \psi] = \int_{\mathbb{R}^4} d^4 x \sqrt{-\det g} \left( K_N R[g] + \Lambda_{\text{SM}} + L_{\text{SM}}^{\text{eff}}[\psi, g] \right),$$  \hspace{1cm} (4)

with gravitational coupling constant $K_N \equiv 1/(16\pi G_N)$ and $c = 1 = \hbar$.

Add a 3–form gauge field $A$ and get \cite{6, 7}:

$$\tilde{S}^{\text{eff}}[A, g, \psi] = \int_{\mathbb{R}^4} d^4 x \sqrt{-\det g} \left( K(q) R[g] + \epsilon(q) + L_{\text{SM}}^{\text{eff}}[\psi, g] \right),$$  \hspace{1cm} (5a)

$$q \equiv -\frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} \nabla_\alpha A_{\beta\gamma\delta} / \sqrt{-g},$$  \hspace{1cm} (5b)

where $\epsilon(q)$ is a generic function of $q$, which arises from the 4-form field strength $F = dA$. The gravitational coupling $K(q)$ is a positive function.

Variational principle gives generalized Einstein and Maxwell equations:
3 Postulated three-form gauge field

\[ 2K(q) \left( R_{\alpha\beta} - g_{\alpha\beta} \frac{R}{2} \right) = -2 \left( \nabla_\alpha \nabla_\beta - g_{\alpha\beta} \Box \right) K(q) \]
\[ + \rho_V(q) g_{\alpha\beta} - T^M_{\alpha\beta}, \quad (6a) \]
\[ \frac{d\rho_V(q)}{dq} + R \frac{dK(q)}{dq} = 0, \quad (6b) \]

with a vacuum energy density,

\[ \rho_V = \epsilon - q \left( \frac{d\epsilon}{dq} + R \frac{dK}{dq} \right) = \epsilon - q \mu, \quad (7) \]

for integration constant (chemical potential) \( \mu \). Eq. (7) is precisely of the Gibbs–Duhem form (2) in Minkowski spacetime \( (R = 0) \). Technically, the extra \( g_{\alpha\beta} \) term on the RHS of (6a) appears because \( q = q(A, g) \).

The expression (5b) shows that \( q \) is a non-fundamental scalar field, which invalidates Weinberg’s no-go theorem (see [7] for details).
4. Metric determinant – Preliminaries

Preliminaries:

We have several examples of potential $q$-fields (e.g., 4-form field strength and 4D-brane), but all were added by hand.

The idea, here, is to use only the known fields from GR+SM, but perhaps to reinterpret them differently.

In fact, we propose to use the metric determinant

$$g(x) \equiv \det \left( g_{\alpha\beta}(x) \right). \quad (8)$$

Yet, $g(x)$ is not a scalar but only a scalar density. Still, it is a scalar if coordinate transformations are restricted to those of unit Jacobian:

$$\det \left( \frac{\partial x'\alpha}{\partial x^\beta} \right) = 1. \quad (9)$$
This reminds us of the so-called unimodular-gravity approach to the CCP [21, 22, 2], which uses restricted coordinate invariance and eliminates $g$ as a dynamical variable ($\Lambda$ then arises as a constant of integration).

For us, $g$ is not eliminated from the dynamics, but plays an essential role in the cancellation of the cosmological constant.

In short, the metric determinant is a dynamical variable.
4. Metric determinant – Motivation

Motivation (cond-mat inspired, courtesy of G.E. Volovik):

It is possible that the metric field $g_{\alpha\beta}(x)$ arises from a spacetime crystal with elasticity tetrads [19]. Then, the density of lattice points $n(x)$ [with dimension of $1/\text{length}^4$] would be proportional to the metric determinant,

$$M^{-4} n(x) = \sqrt{-g(x)},$$

where the crystal has a fundamental length scale $\ell \equiv 1/M$.

The total number of lattice points is given by

$$N = \int d^4 x \, n(x),$$

and it is natural to assume that this number is conserved.
Then there is a Lagrange multiplier in the action:

$$S_N = -\mu N = -\mu \int d^4 x \ n(x),$$

(12)

where $\mu$ is the corresponding (dimensionless) chemical potential.

The crucial observation is that $n$ can enter the matter action, provided coordinate invariance is restricted by (9). (The possibility of adding extra $\sqrt{-g}$ terms in the matter action was already noted in, e.g., Ref. [22], but was not pursued further.)

We, next, simplify the theory to the bare minimum. In fact, we only need a standard real scalar $X(x)$ for the appropriate expansion of the FRW-type model later on.
4. Metric determinant – Action

Action [23]:

\[ S = S_G + S_M + S_{\Lambda-\text{plus}} + S_N, \]  \hspace{1cm} (13a)

\[ S_G = \int d^4x \sqrt{-g} \frac{R}{16\pi G_N}, \]  \hspace{1cm} (13b)

\[ S_M = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha X \partial_\beta X + \frac{1}{2} g_2 M^2 X^2 \right], \]  \hspace{1cm} (13c)

\[ S_{\Lambda-\text{plus}} = \int d^4x \sqrt{-g} \epsilon(\Lambda, n) = \int d^4x \sqrt{-g} \left[ \Lambda + \zeta n \right], \]  \hspace{1cm} (13d)

\[ S_N = -\mu \int d^4x \ n(x), \]  \hspace{1cm} (13e)

\[ n(x) = \sqrt{-g(x)} \ M^4, \]  \hspace{1cm} (13f)

where we have used the simplest possible \textit{Ansätze} in (13c) and (13d), with real parameters \( \zeta > 0 \) and \( g_2 \geq 0 \).
Strictly speaking, the only new input is the single term \( n \propto \sqrt{-g} \) in the potential \( (13d) \), consistent with having coordinate invariance restricted by \( (9) \).

The resulting gravitational field equation reads:

\[
\frac{1}{8\pi G_N} \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) = \rho_{\text{vac}} g_{\alpha\beta} + T^M_{\alpha\beta}, \tag{14a}
\]

\[
\rho_{\text{vac}} = \Lambda + 2 \zeta n - \mu M^4, \tag{14b}
\]

\[
n = \sqrt{-g} M^4, \tag{14c}
\]

\[
\Lambda = \lambda M^4, \tag{14d}
\]

where the chemical potential \( \mu \) traces back to the action term \( (13e) \).
4. Metric determinant – Action

Taking the covariant divergence of (14a) and using the contracted Bianchi identities, the following combined energy-momentum conservation relation is obtained:

\[
\left( \rho_{\text{vac}} g_{\alpha\beta} + T_{\alpha\beta}^M \right) ;^\beta = 0 ,
\]

(15)

where the semicolon stands for a covariant partial derivative (the colon stands for a standard partial derivative).

If the matter component is separately conserved, \( (T_{\alpha\beta}^M) ;^\beta = 0 \), then equally so for the vacuum component, so that \( \rho_{\text{vac}} ;^\beta = 0 \).
5. Metric determinant: Cosmology

With diffeomorphisms restricted to those of unit Jacobian, the appropriate spatially-flat Robertson–Walker (RW) metric is given by [22]

\[ ds^2 = g_{\alpha\beta}(x) \, dx^\alpha \, dx^\beta = -\tilde{A}(t) \, dt^2 + \tilde{R}^2(t) \delta_{ij} \, dx^i \, dx^j, \quad (16) \]

where \( t \) is the cosmic time coordinate from \( x^0 = c t = t \) and \( \tilde{A}(t) > 0 \) an additional Ansatz function.

For \( \tilde{A}(t) = \text{const} > 0 \), we recover the standard spatially-flat RW metric.

Remark that the extended RW metric (16) gives the vacuum variable

\[ n \propto \sqrt{-g} = (\tilde{A})^{1/2} \, |\tilde{R}|^3. \quad (17) \]
Henceforth, we set

\[ E_{\text{Planck}} \equiv 1/\sqrt{G_N} = M \],

and introduce the following dimensionless quantities (the chemical potential \( \mu \) is already dimensionless):

\[ t \rightarrow \tau, \quad \rho_X(t) \rightarrow r_X(\tau), \quad \tilde{A}(t) \rightarrow a(\tau), \]  

\[ X(t) \rightarrow \chi(\tau), \quad P_X(t) \rightarrow p_X(\tau), \quad \tilde{R}(t) \rightarrow r(\tau), \]  

\[ n(t) \rightarrow n(\tau), \quad \Lambda \rightarrow \lambda, \]  

where \( n(\tau) \) is dimensionless and equal to \( \sqrt{a(\tau)}|r(\tau)|^3 \).
5. Metric determinant: Cosmology

From the field equations of the action (13) for the RW metric (16) and using the homogeneous perfect fluid from the $\chi$ scalar, we obtain the following ODEs:

\[
\dot{r}_\chi + 3 \left(1 + w_M\right) \left(\frac{\dot{r}}{r}\right) r_\chi = 0 , \quad (20a)
\]

\[
3 \left(\frac{\dot{r}}{r}\right)^2 = 8 \pi a \left(r_\chi + r_{\text{vac}}\right) , \quad (20b)
\]

\[
\frac{2 \ddot{r}}{r} + \left(\frac{\dot{r}}{r}\right)^2 - \left(\frac{\dot{a}}{a}\right) \left(\frac{\dot{r}}{r}\right) = -8 \pi a \left(w_M r_\chi - r_{\text{vac}}\right) , \quad (20c)
\]

\[
r_{\text{vac}} = \lambda + 2 \zeta \sqrt{a} |r|^3 - \mu , \quad (20d)
\]

where the overdot stands for differentiation with respect to $\tau$. These ODEs have three real parameters: the matter equation-of-state parameter $w_M \equiv p_\chi / r_\chi > -1$ and two parameters, $\zeta > 0$ and $\mu \neq 0$, entering the vacuum energy density $r_{\text{vac}}$. 
5. Metric determinant: Cosmology

We can get analytic Friedmann-type and deSitter-type solutions from the following Ansatz functions:

\[
\begin{align*}
    a(\tau) &= \alpha \tau^{-2p}, \\
    r(\tau) &= \alpha^{-1/6} \hat{r} \tau^{p/3}, \\
    r_{\chi}(\tau) &= \alpha^{-1} \hat{\chi} \tau^{-m},
\end{align*}
\]

with positive parameters \(\alpha, p, \hat{r}, \hat{\chi},\) and \(m\).

The corresponding dimensionless Ricci and Kretschmann curvature scalars read:

\[
\begin{align*}
    \mathcal{R} &= \frac{2}{3} p \left(5p - 3\right) \frac{1}{\alpha} \tau^{-2(1-p)}, \\
    \mathcal{K} &= \frac{4}{27} p^2 \left(9 - 24p + 17p^2\right) \frac{1}{\alpha^2} \tau^{-4(1-p)}.
\end{align*}
\]
5. Metric determinant: Cosmology

Assuming $\mu > 0$ and $\lambda < \mu$, the analytic Friedmann-type solution with $r_{\text{vac}} = 0$ has parameters:

\[
\begin{align*}
\alpha_{\text{F-sol}} &> 0, \quad p_{\text{F-sol}} = \frac{2}{3 + w_M}, \quad \hat{r}_{\text{F-sol}} = \left[ \frac{1}{2 \zeta} (\mu - \lambda) \right]^{1/3}, \quad (23a) \\
m_{\text{F-sol}} &= \frac{2 \left( 1 + w_M \right)}{3 + w_M}, \quad \hat{\chi}_{\text{F-sol}} = \frac{1}{6 \pi \left( 3 + w_M \right)^2}, \quad (23b)
\end{align*}
\]

so that the Ricci and Kretschmann scalars (22) drop to 0 asymptotically.

Assuming $\mu > 0$ and $0 < \lambda < \mu$, a particular analytic deSitter-type solution with $r_{\text{vac}} = \lambda$ has parameters:

\[
\begin{align*}
\alpha_{\text{deS-sol}} &= \frac{1}{24 \pi \lambda}, \quad p_{\text{deS-sol}} = 1, \quad \hat{r}_{\text{deS-sol}} = \sqrt[3]{\frac{\mu}{2 \zeta}}, \quad (24a) \\
\hat{\chi}_{\text{deS-sol}} &= 0, \quad (24b)
\end{align*}
\]

so that the Ricci and Kretschmann curvature scalars (22) are constant.
The ODEs (20) give a constant vacuum energy density, $\dot{r}_{\text{vac}} = 0$. But, with $r_{\text{vac}} > 0$ initially, particle creation by the spacetime curvature will result in a decrease of $r_{\text{vac}}$ and an increase of $r_{\chi}$.

The modified ODEs with vacuum-matter energy exchange are given by

\begin{align*}
\dot{r}_{\chi} + 4 \left( \frac{\dot{r}}{r} \right) r_{\chi} &= \Gamma, \quad (25a) \\
\dot{r}_{\text{vac}} &= -\Gamma, \quad (25b) \\
3 \left( \frac{\dot{r}}{r} \right)^2 &= 8 \pi a \left( r_{\chi} + r_{\text{vac}} \right), \quad (25c) \\
\frac{1}{8 \pi a} \left[ \frac{2 \ddot{r}}{r} + 2 \left( \frac{\dot{r}}{r} \right)^2 - \left( \frac{\dot{r}}{r} \right) \left( \frac{\dot{a}}{a} \right) \right] &= \frac{4}{3} r_{\text{vac}}, \quad (25d) \\
r_{\text{vac}} &= \lambda + 2 \zeta \sqrt{a} |r|^3 - \mu. \quad (25e)
\end{align*}
5. Metric determinant: Cosmology

As the left-hand side of (25d) is proportional to the Ricci scalar, we have $\mathcal{R} \propto r_{\text{vac}}$. Therefore, we can write the Zeldovich–Starobinsky-type [24, 8] source term ($\Gamma \propto \mathcal{R}^2$) as the following simpler expression:

\[
\begin{align*}
\Gamma &= \tilde{\gamma} \left| \dot{r}/r \right| (r_{\text{vac}})^2, \\
\tilde{\gamma}(\tau) &= \gamma \left[ \frac{\tau^2 - \tau_{\text{bcs}}^2}{\tau^2 + 1} \right]^2, \\
\gamma &\geq 0,
\end{align*}
\]  

(26a, 26b, 26c)

where we have added a smooth switch-on function $\tilde{\gamma}(\tau)$ for initial boundary conditions at $\tau = \tau_{\text{bcs}}$, in order to ease the numerical evaluation of the ODEs.
We have obtained numerical solutions of the ODEs (25) with source term (26), for initial boundary conditions at or near the analytic Friedmann-type solution, which also holds for nonzero positive $\gamma$.

We have also obtained numerical solutions for initial boundary conditions from the analytic deSitter-type solution, which is a solution only for the $\gamma = 0$ case.

Full numerical results are given in Ref. [23]. Here, we only present some results with a start from the deSitter-type solution of the unmodified ODEs (20).
5. Metric determinant: Cosmology

Fig. 1: Numerical solution of the modified ODEs (25) with source term (26) and parameters $w_M = 1/3$, $\zeta = 1$, $\mu = 3$, $\lambda = 10^{-4}$, and $\gamma = 0$ (quantum-dissipative effects turned off). The initial boundary conditions are taken from the analytic de-Sitter-type solution (21) and (24), having $\bar{\alpha} \equiv \alpha_{\text{deS-sol}} = 132.629$ and $\bar{\tau} \equiv r_{\text{deS-sol}} = 1.14471$. The top row shows the three basic variables: the metric functions $r(\tau)$ and $a(\tau)$ and the dimensionless energy density $r_\chi$. The bottom row shows three derived quantities: the dimensionless Ricci curvature scalar $\mathcal{R}$, the dimensionless Kretschmann curvature scalar $\mathcal{K}$, the dimensionless gravitating vacuum energy density $r_{\text{vac}}$ from (25e).
5. Metric determinant: Cosmology

\[ \alpha^{1/6}/\tau \times \tau^{-1/3} \times r[\tau] \]
\[ 1/\alpha \times \tau^2 \times a[\tau] \]
\[ 10^4 \times r_j[\tau] \]

\[ 3/4 \times \alpha \times R[\tau] \]
\[ 27/8 \times \alpha^2 \times K[\tau] \]
\[ 10^4 \times r_{\text{vac}}[\tau] \]

\[ 3/4 \times \alpha \times R[\tau] \]
\[ 10^4 \times r_j[\tau] \]
\[ 10^4 \times r_{\text{vac}}[\tau] \]

\[ 1/2 \times \alpha^{1/6}/\tau \times \tau^{-1/5} \times r[\tau] \]
\[ 20/\alpha \times \tau^{6/5} \times a[\tau] \]
\[ 10^3 \times \tau^{4/5} \times r_j[\tau] \]

**Fig. 2:** Same as Fig. 1, but now with \( \gamma = 2 \times 10^{11} \) (quantum-dissipative effects turned on).
6 Conclusions

To summarize, the $q$–theory approach to the main Cosmological Constant Problem (CCP1) provides a solution. For the moment, this is only a possible solution, because it is not known for sure that the “beyond-the-Standard-Model” physics contains such a $q$–type variable.

GENERAL REMARK: it is clear that the SM harbors huge vacuum energy densities, which somehow need to be cancelled by new d.o.f., possibly related to the fundamental theory of spacetime and gravity.

BAD NEWS: nothing is known for sure about these fundamental d.o.f.

GOOD NEWS: even though the detailed (high-energy) microphysics is unknown, it may be possible to describe the macroscopic (low-energy) effects along the lines of $q$–theory, just as for the hydrodynamics of water.

GOOD NEWS (cont.): it is also possible that gravity is an emerging phenomenon and that the metric determinant can play the role of a $q$-type variable and cancel the cosmological constant.
8. References