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Elementary particle physics and cosmology for engineers (and others)

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1.0 Introduction

- Engineers, only figuratively speaking.
- Purpose: motivational and non-encyclopedic.
- Language of Nature: not Greek or English, but MATHEMATICS.

$\Rightarrow$ each lecture centered on one $\boxed{\text{EQUATION}}$.

$\Rightarrow$ instead of a lot of words, we will try to do a few CALCULATIONS.

1.0 Introduction

Four lectures:

1. Special Relativity
2. Quantum Mechanics
3. Quantum Field Theory & Standard Model & Higgs
4. General Relativity & Friedmann–Robertson–Walker universe

And now the same with acronyms:

1. SR
2. QM
3. QFT & SM & Higgs
4. GR & FRW universe

Four crucial references collected at the end.

1.1 SR

Einstein 1905 [1] essentially uses the following **two postulates**:

P1 : All laws of physics and all experimental results (except those of gravity) are equivalent in different inertial reference frames having constant uniform relative velocities.

P2′ : Such different inertial reference frames K$(x,\, y,\, z,\, t)$ and K′$(x',\, y',\, z',\, t')$ are related by Lorentz transformations (LTs).

Einstein formulates it better, because he tells us how to get the LTs (see later). For now, we just give the LTs explicitly.

[1] A. Einstein, Ann. Phys. (Leipzig) **17**, 891 (1905).

1.1 SR

For constant and uniform velocity v along the x axis, the LTs are:

$$\boxed{\begin{aligned} x' &= \gamma\,(x - v\,t)\,, \\ y' &= y\,, \\ z' &= z\,, \\ t' &= \gamma\,(t - v\,x/c^2)\,, \\ \gamma &\equiv 1/\sqrt{1 - v^2/c^2}\,. \end{aligned}} \tag{1}$$

1.1 SR

The LTs for $v/c \to 0$ reduce to the three Galilei transformations (GTs):

$$
\begin{aligned}
x' &= x - vt\,, && \text{(2a)}\\
y' &= y\,, && \text{(2b)}\\
z' &= z\,, && \text{(2c)}
\end{aligned}
$$

together with a “trivial” transformation,

$$
t' = t\,. \qquad \text{(2d)}
$$

Now, precisely (2d) shows Einstein’s genius.

1.1 SR

A velocity is the relative change of position with time. From

$$u_x \equiv \frac{dx}{dt}\,, \qquad u_y \equiv \frac{dy}{dt}\,, \qquad u_z \equiv \frac{dz}{dt}\,, \tag{3a}$$

$$u'_{x'} \equiv \frac{dx'}{dt'}\,, \qquad u'_{y'} \equiv \frac{dy'}{dt'}\,, \qquad u'_{z'} \equiv \frac{dz'}{dt'}\,, \tag{3b}$$

a straightforward but tedious algebra gives:

$$u'_{x'} = \frac{u_x - v}{1 - u_x\,v/c^2}\,, \tag{4a}$$

$$u'_{y'} = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x\,v/c^2}\,u_y\,, \tag{4b}$$

$$u'_{z'} = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x\,v/c^2}\,u_z\,. \tag{4c}$$

Two remarks. First, these velocity transformations for $v/c \to 0$ reduce to the three Galilei transformations:

$$
\begin{aligned}
u'_{x'} &= u_x - v \,, && \text{(5a)}\\
u'_{y'} &= u_y \,, && \text{(5b)}\\
u'_{z'} &= u_y \,. && \text{(5c)}
\end{aligned}
$$

Second, if the velocity has magnitude c in reference frame K, then the same holds in reference frame K' :

$$
K : \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad K' : \begin{pmatrix} u'_{x'} \\ u'_{y'} \\ u'_{z'} \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} . \qquad (6)
$$

In fact, this is the form of the second postulate used by Einstein (P2: light velocity c independent of the motion of source or detector).

1.1 SR

Three famous experiments are now explained as purely kinematic effects from SR:

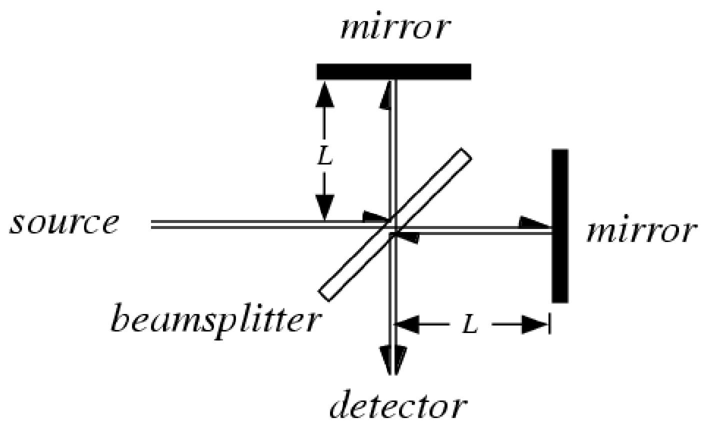
1. Stellar aberration [Bradley, 1727];
2. Fizeau experiment [1851];
3. Michelson–Morley experiment [1887].

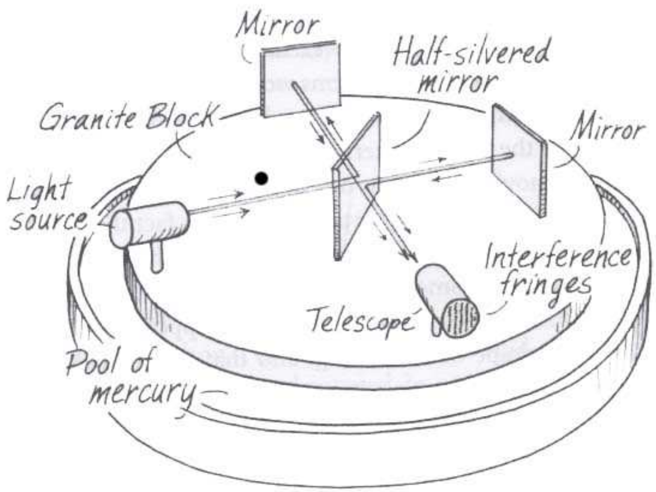
Time permitting, discuss experiments 1+2 on the blackboard.

MM experiment next slide.

SR not only made post-dictions but also pre-dictions, for example, the transverse Doppler effect [Ives and Stilwell, 1938].

1.1 SR





Experimental setup moves along with the Earth (velocity v_{Earth} around the Sun) and, therefore, through the "aether." Hence, the Galilean expectation is to have different velocities along the arms ($c_{\text{GT}} \sim c \pm v_{\text{Earth}}$), which must give a change in the interference pattern at the detector. But no such change was observed in 1887 and later.

As discussed before, SR gives immediately $c'_{\text{LT}} = c$.

1.1 SR

SR also provides us with a new **“Weltbild”** [Minkowski, 1908]:

space (3-dimensional) and time (1-dimensional)
are merged into **spacetime** (4-dimensional).

Trick is to use coordinates with the same dimension (here, length):

$$(x^1,\, x^2,\, x^3,\, x^0) \equiv (x,\, y,\, z,\, ct)\,, \tag{7}$$

which may be considered as components of a 4-vector x^μ with indices $\mu = 1, 2, 3, 0$.

This provides new insight into the issue of **causality**, with so-called past and future light-cones ($\rightarrow$ blackboard).

The Minkowski-spacetime point of view was also key towards the discovery of GR, where spacetime becomes dynamic (see later).

2.1 QM

Main postulate to describe the atom and three further assumptions:

1. quantum state is determined completely by the **wave function** ψ ;
2. wave equation is **linear** and **homogenous** [$\rightarrow$ superposition];
3. wave equation is a differential equation of **first order in time** [$\rightarrow \psi(t_0)$ determines $\psi(t)$ for $t \geq t_0$)];
4. **correspondence principle**, the predictions from the quantum theory must, in certain limits, correspond to those of classical mechanics.

These and other considerations led Schrödinger to his famous equation [article received on 21 June, 1926].

We will discuss this equation in its simplest possible form.

2.1 QM

Schrödinger equation for a particle in 1 space dimension (coordinate x):

$$\boxed{i\,\hbar\,\frac{d}{dt}\,\psi(x,t) = \left(-\,\frac{\hbar^2}{2m}\,\frac{d^2}{dx^2} + V(x,t)\right)\psi(x,t)\,,} \qquad (8)$$

where the mass m and the potential V are considered to be given (particle is spinless and has nonrelativistic kinetic energy).

Three obvious questions,

Q1: what is $\hbar$?

Q2: what does $i \equiv \sqrt{-1}$ do here?

Q3: what is the *physical* meaning of $\psi(x,t) \in \mathbb{C}$?

2.1 QM

A1: With dimension of V equal to energy, the dimension of $\hbar$ is: $[\hbar] = [\text{energy} \times \text{time}]$.

Now, $\hbar$ is a **new fundamental constant of Nature**, the Planck constant, with the experimental value

$$\hbar \equiv h/(2\pi) = 1.054571628(53) \times 10^{-34} \text{ J s}\,. \tag{9}$$

Its small value in "human" units explains that quantum effects are tiny for us, but not for an atom.

A2: All classical laws of physics (e.g., Newton and Maxwell) are real. **All observables are real too**. So, the presence of i in the Schrödinger equation (8) makes clear that something unusual must go on, $\rightarrow$ Q3.

A3: Surprising explanation given by Born [June 25, 1926]: $\psi(x,t) \in \mathbb{C}$ determines the **state** of the atom and $|\psi(x,t)|^2 \geq 0$ gives the **probability** of observing the particle at position x for time t.

2.1 QM

A very strange situation:

The quantum state of the system evolves **deterministically**, as given by the Schrödinger equation (8).

But the physics results predicted are **probabilistic** in nature.

The birth of QM was long and painful, which makes the original papers (e.g., the one of Heisenberg from July 29, 1925) difficult to read.

Still, a beautiful synthesis can already be found in the so-called “Dreimännerarbeit,” dated November 16, 1925 [2].

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[2] M. Born, W. Heisenberg, and P. Jordan, Z. Phys. **35**, 557 (1925).

2.1 QM

Remains to actually **solve** the Schrödinger equation (8), which is a partial differential equation (PDE).

Let’s do this for a relatively simple case, ‘simple’ mathematically but not physically.

Consider a static potential

$$V(x,t) \quad = \quad W(x)\,, \tag{10}$$

and look for a so-called stationary state,

$$\psi(x,t) \quad = \quad \phi(x)\, e^{-i\,E\,t/\hbar}\,, \tag{11a}$$

$$\phi(x) \quad \in \quad \mathbb{C}\,, \qquad E \in \mathbb{R}\,. \tag{11b}$$

2.1 QM

The original time-dependent Schrödinger equation (8) is reduced to an ordinary differential equation (ODE):

$$\phi''(x) + [\epsilon - U(x)]\,\phi(x) = 0\,, \tag{12a}$$

$$U(x) \equiv [2m/\hbar^2]\,W(x)\,, \tag{12b}$$

$$\epsilon \equiv [2m/\hbar^2]\,E\,, \tag{12c}$$

where the prime stands for a derivative with respect to x.

Now, take a static rectangular barrier and an eigenvalue ϵ below the top of the barrier ($\rightarrow$ figure on the blackboard):

$$U(x) = \begin{cases} 0 & \text{in region I} \quad x > L\,, \\ U_2 & \text{in region II} \quad 0 < x < L\,, \\ 0 & \text{in region III} \quad x < 0\,, \end{cases} \tag{13a}$$

$$\epsilon \in (0,\, U_2)\,. \tag{13b}$$

2.1 QM

With boundary conditions corresponding to an outgoing wave in region III and with appropriate continuity conditions at $x = 0$ and $x = L$, the following solution is found:

$$\phi_\epsilon \quad \propto \quad \begin{cases} e^{-i\sqrt{\epsilon}x} + R\, e^{+i\sqrt{\epsilon}x} & \text{in region I} \quad x > L\,, \\ A\, e^{+\mu\, x} + B\, e^{-\mu\, x} & \text{in region II} \quad 0 < x < L\,, \\ S\, e^{-i\sqrt{\epsilon}x} & \text{in region III} \quad x < 0\,, \end{cases} \tag{14a}$$

$$\mu \quad \equiv \quad \sqrt{U_2 - \epsilon}\,, \tag{14b}$$

with explicit coefficients $A, B, R, S \in \mathbb{C}$. The real surprise is that $S \neq 0$:

$$T \equiv |S|^2 = \frac{4\,\epsilon\,\mu^2}{4\,\epsilon\,\mu^2 + (U_2)^2\ \sinh^2(\mu\, L)}\,. \tag{15}$$

2.1 QM

A **classical** particle coming in from the left (region I) with a kinetic energy less than the potential barrier will simply be reflected at $x = 0$.

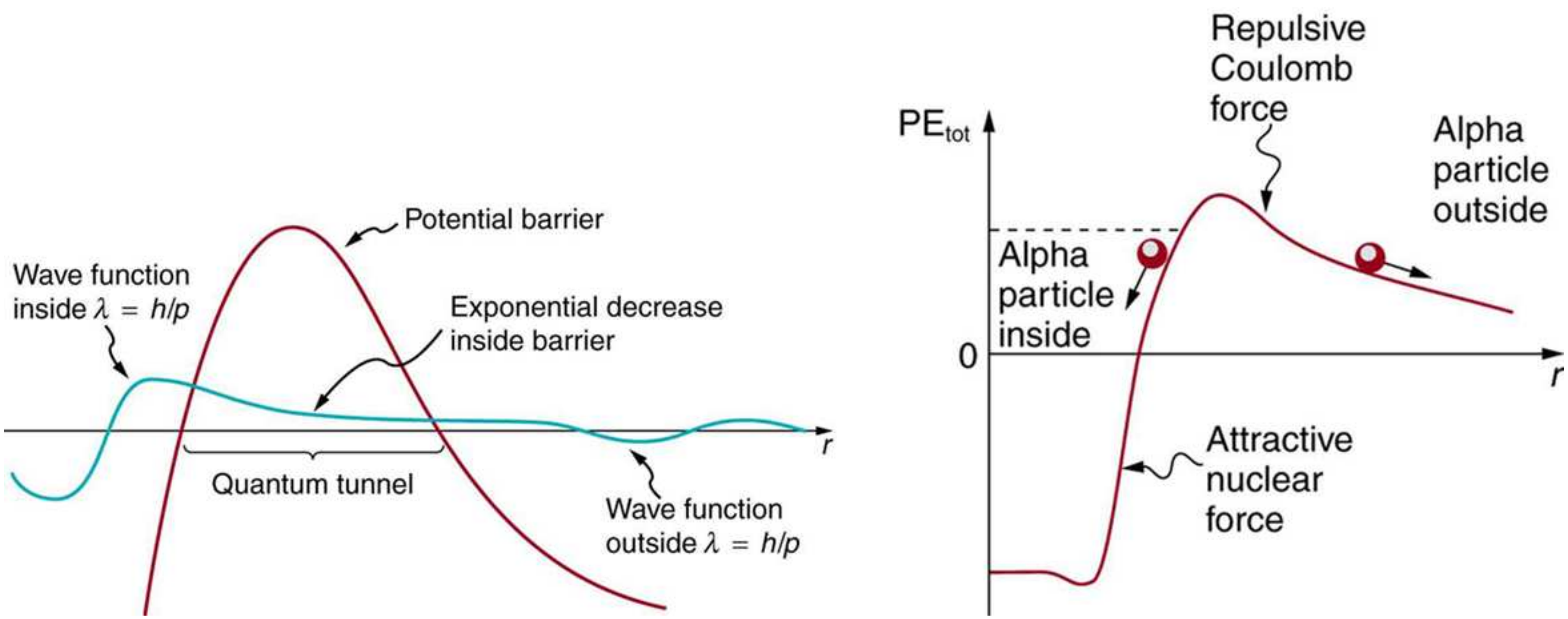
A **quantum** particle coming in from the left (region I) will be "partially" reflected at $x = 0$ and "partially" transmitted to $x = L$.

This is the so-called **tunneling effect**, where the transmission probability is controlled by the coefficient

$$T \equiv |S|^2 \,. \qquad (16)$$

Quantum tunneling explains radioactive decay [Gamov, August 1928] and, most importantly, the **concept** of half-life for identical particles (inexplainable within the realm of classical physics).

2.1 QM



Potential barrier
Wave function inside $\lambda = h/p$
Exponential decrease inside barrier
Quantum tunnel
Wave function outside $\lambda = h/p$
r
Repulsive Coulomb force
PE_{tot}
Alpha particle outside
Alpha particle inside
0
r
Attractive nuclear force

3.1 QFT – e^+e^-

In short, QM + SR = QFT .

First result in quantum field theory (QFT):

theoretical prediction of the existence of **antimatter** [Dirac, May 1931],
experimental discovery of antimatter [Anderson, December 1931].

More specifically, Dirac wrote about

a new kind of particle, unknown to experimental physics, having the same mass and opposite charge of the electron.

Title of Anderson's paper:

The apparent existence of easily deflectable positives,

which is precisely what was observed ($\rightarrow$ cloud-chamber picture)

The antiparticle corresponding to the electron (e^-) is called the positron (e^+).

3.1 QFT – e^+e^-

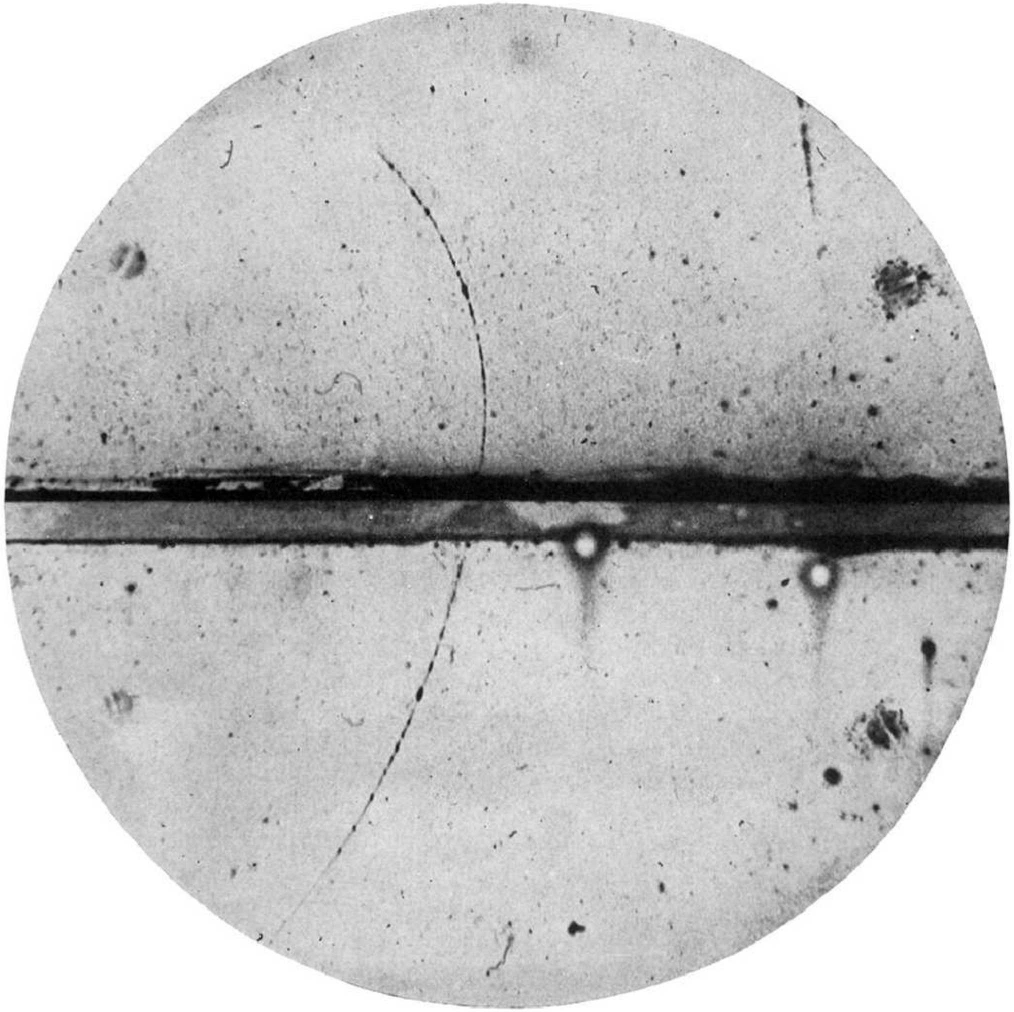


FIG. 1. A 63 million volt positron ($H\rho = 2.1 \times 10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5 \times 10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

3.1 QFT – e^+e^-

Start with the relativistic relation between energy E and 3-momentum $\vec{p}$ of a particle:

$$\boxed{E^2 = c^2\,|\vec{p}\,|^2 + m^2\,c^4\,.} \qquad (17)$$

Now, mathematically there are positive and negative roots,

$$E = \pm\sqrt{c^2\,|\vec{p}\,|^2 + m^2\,c^4}\,. \qquad (18)$$

But, physically, what do the negative roots mean?
For fermions, one possible explanation uses the so-called *Dirac sea*, but this explanation does not work for bosons.
Feynman has given a remarkable interpretation of antiparticles: *they are negative-energy particles that run backward in time!* Symbolically,

$$E\times t = (+t)\times\left(+\sqrt{c^2\,|\vec{p}\,|^2 + m^2\,c^4}\right) = (-t)\times\left(-\sqrt{c^2\,|\vec{p}\,|^2 + m^2\,c^4}\right). \qquad (19)$$

3.2 QFT – FDs

Turn now to **interactions** between particles and antiparticles.

Perturbative QFT is about calculating transition amplitudes $\mathcal{M} \in \mathbb{C}$, with the probability of the process being proportional to $|\mathcal{M}|^2 \in \mathbb{R}^+ \cup \{0\}$.

For example, the cross section for elastic electron-electron scattering is obtained from the amplitude $\mathcal{M}(e^- e^- \to e^- e^-) \in \mathbb{C}$ by taking the absolute square $|\mathcal{M}|^2$ (and doing some further integrals).

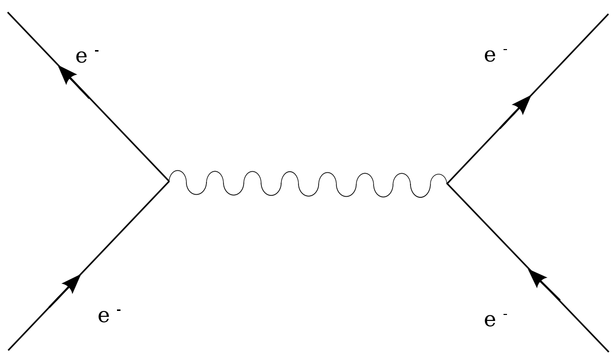
The calculation of these scattering amplitudes can be organized by use of so-called **Feynman diagrams** (FDs). FDs provide a kind of bookkeeping for complicated integrals [3].

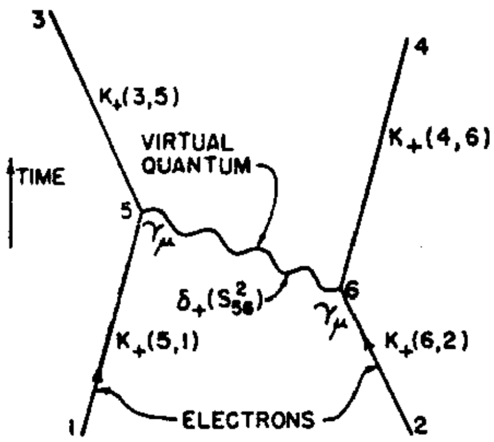
Now a few FDs, where the arrow keeps track of the flow of charge.

[3] R.P. Feynman, Phys. Rev. **76**, 769 (1949).

3.2 QFT – FDs

The basic FD for elastic electron-electron scattering ($e^- e^- \to e^- e^-$) is:

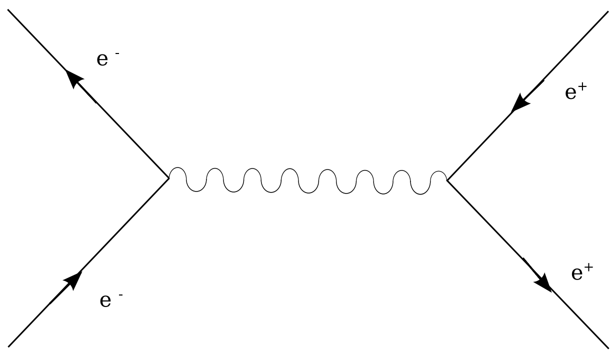


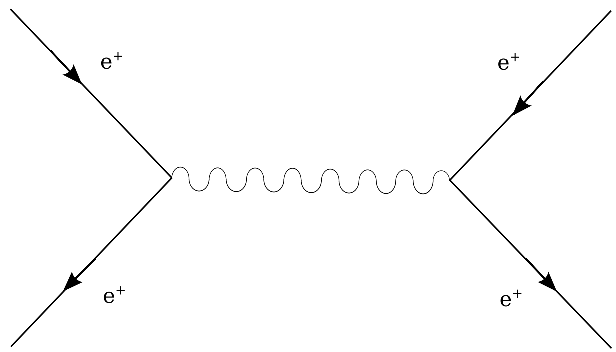


where the diagram on the left is the modern minimalist one and the diagram on the right the original from Feynman's 1949 paper [3].

3.2 QFT – FDs

Below, on the left, is the FD for elastic electron-positron scattering ($e^-e^+ \to e^-e^+$) and, on the right, the FD for elastic positron-positron scattering ($e^-e^+ \to e^-e^+$):





Obviously, these three amplitudes are related.
Very beautiful indeed.

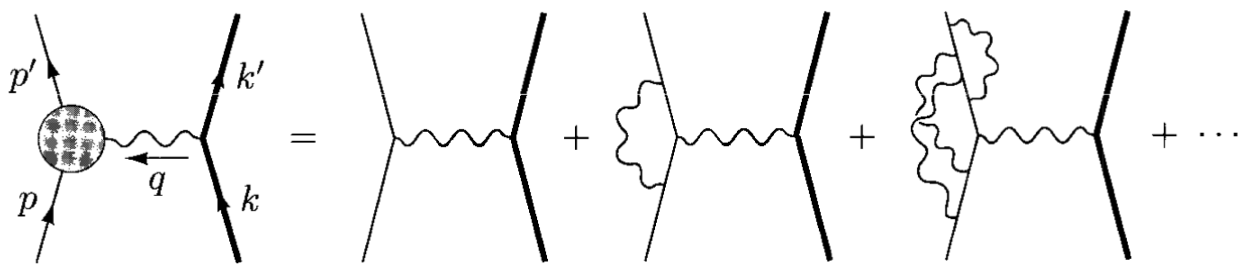
3.2 QFT – FDs

Up till now, we have only shown **tree diagrams**.

Quantum corrections have **loops** and, in principle, all diagrams for a particular process need to be summed:

$$\text{complete amplitude} = \text{tree} + \text{1-loop} + \text{2-loops} + \text{etc.} \qquad (20)$$

For example, for the dressed vertex on the left:



and similarly for the vertex of the heavy line on the right and the photon line connecting the two dressed vertices.

3.3 QFT – SM

A brief (one-slide) history of elementary-particle physics:

Quantum electrodynamics (QED) describes the interaction between electrons and positrons. (The classical electrodynamics of Faraday and Maxwell appears in the limit of weak fields.)

The original QED theory was formulated by Dirac, Jordan, Pauli, and Heisenberg in the 1930's.

Prompted by the experimental discovery of the Lamb shift, the first **renormalization** calculations of QED were performed by Schwinger, Feynman, and Tomonoga in 1948.

Also in the 1940's, modern experimental particle physics took off, with the discovery of more and more 'elementary' particles, resonances, later understood as combinations of **quarks**.

The **Standard Model** (SM), describing the interactions of quarks and leptons, was constructed in the 1960's.

3.3 QFT – SM

The formal (one-slide) description of QED and SM is as follows.

QED is an **Abelian $U(1)$ gauge field theory** with particle content:

- 1 spin-one-half particle [electron or positron];
- 1 spin-one gauge boson [photon].

The SM is a **non-Abelian $SU(3) \times SU(2) \times U(1)$ gauge field theory** with particle content (quarks & leptons, and gauge bosons):

- 3 $\times$ 15 = 45 chiral spin-one-half particles in the representation $R_L = 3 \times [(3,\, 2)_{1/3} + (\overline{3},\, 1)_{-4/3} + (\overline{3},\, 1)_{2/3} + (1,\, 2)_{-1} + (1,\, 1)_{2}]$;
- 12 spin-one gauge bosons in the respective adjoint representations [8 gluons (g_{ab}), 3 weak vector bosons ($W^{\pm}$, Z^0) and 1 photon (γ)].

But, up till now, all these particles are **massless**.

Higgs mechanism needed to provide for masses as observed.

3.4 QFT – Higgs

Higgs mechanism in words:

1. introduce a **fundamental scalar field**, $\mathbf{\Phi(x)}$;
2. arrange for self-interactions, so that the ground-state corresponds to a **condensate**, $\langle 0|\,\Phi\,|0\rangle \neq 0$;
3. particles moving through the condensate pick up **effective mass**;
4. identify this effective mass as “the” mass of the SM particles, *viz.*, the quarks (u, d, c, s, t, b), the leptons (e, ν_e, μ, ν_μ, τ, ν_τ), and the weak vector bosons ($W^\pm$ and Z^0).

3.4 QFT – Higgs

Now, the Higgs condensate is **constant** over spacetime, so how to detect it (apart from gravity; see remark below)?

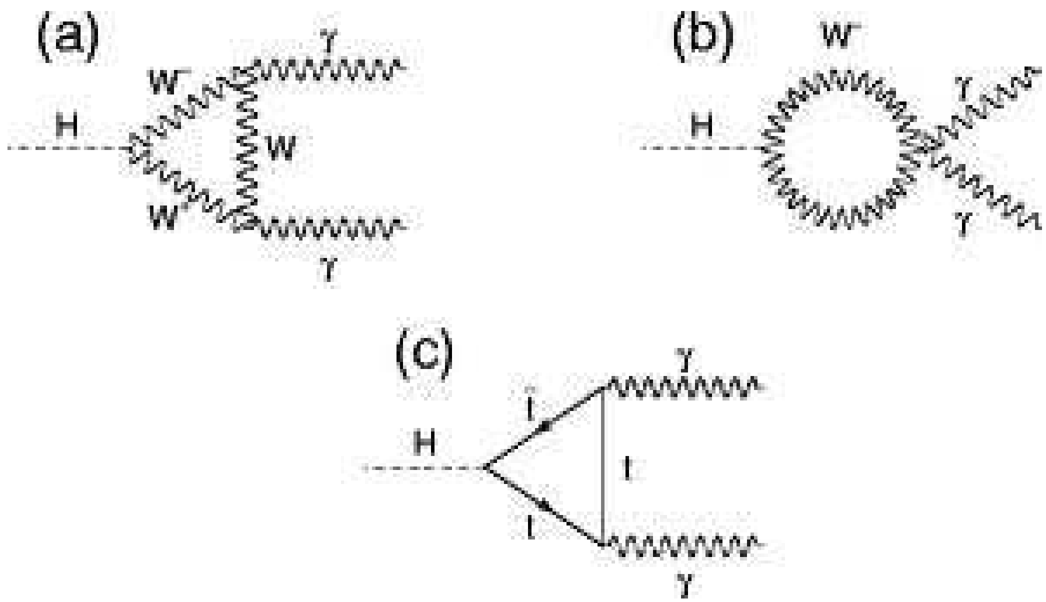
Idea is to look for small perturbations of the condensate (similar to waves at the top of the ocean), These small perturbations correspond to the **Higgs-boson particle**, $\boldsymbol{H}$.

But, as the particle H has the quantum numbers of the vacuum ($Q_H = 0$, $B_H = 0$, $L_H = 0$, $S_H = 0$, etc.), it will be difficult to detect $\ldots$

After many years of effort, the Higgs-boson particle H appears to have been **discovered** (announcement on the fourth of July, 2012) by the ATLAS and CMS experiments at the Large Hadron Collider, with a mass $\boxed{M_H \approx 126 \text{ GeV}}$.

One interesting decay channel is the one into two photons, which does not appear at tree level but only at 1-loop.

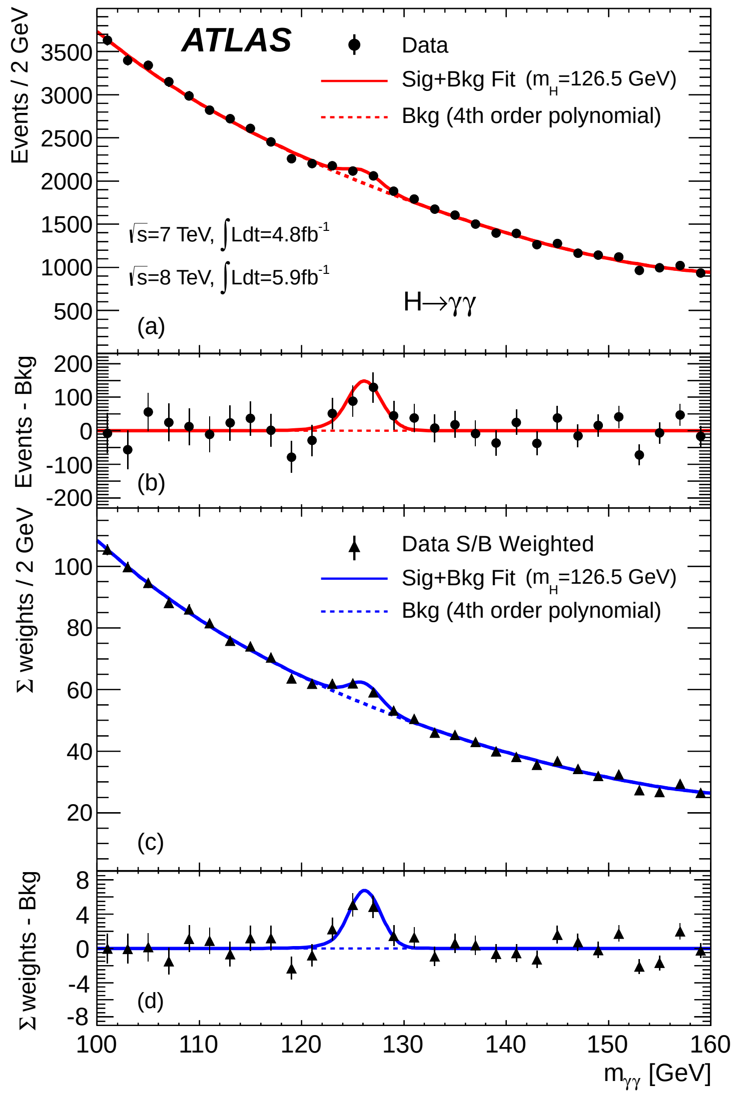
3.4 QFT – Higgs



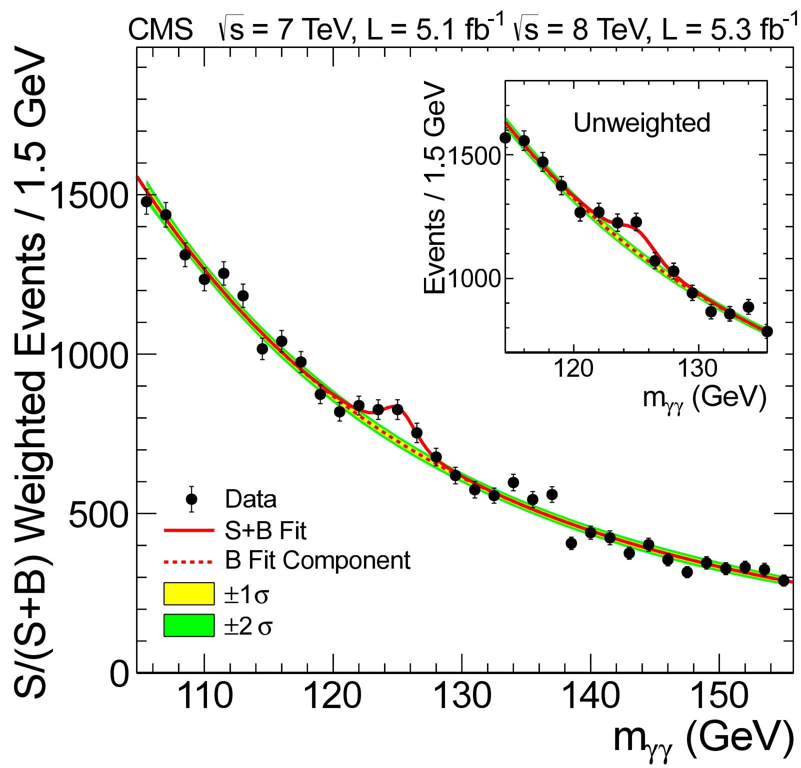
Decay rate: $\Gamma_{\text{1-loop}}(H \to \gamma\gamma) = \frac{\alpha\, M_H}{8\, \sin^2\theta_w} \left(\frac{M_H}{M_W}\right)^2 \frac{\alpha^2}{18\,\pi^2} \left| \sum_f Q_f^2\, N_c(f) - \frac{21}{4} \right|^2$

with Q_f the electric charge of fermion f and $N_c(f)$ the color degeneracy of fermion f, specifically, $N_c(\text{quark}) = 3$ and $N_c(\text{lepton}) = 1$ [Γ from the Peskin&Schroeder textbook].

3.4 QFT – Higgs



[ATLAS, arXiv:1207.7214; CMS, arXiv:1207.7235]



3.4 QFT – Higgs

Higgs boson, fantastic discovery!

Tip of the iceberg:

- possible gravity connection, $M_H \approx 126$ GeV $\stackrel{?}{\Rightarrow}$
  $E_{\text{Planck}} \equiv \sqrt{\frac{\hbar\, c^5}{G_N}} \approx 1.22 \times 10^{19}$ GeV $\approx 1\,220\,000\,000\,000\,000\,000\,0$ GeV.
- Cosmological Constant Problem (CCP),
  vacuum energy density from theory too large by 54 orders of magnitude:
  $\left|\epsilon_V^{\text{(exp)}} / \epsilon_V^{\text{(Higgs)}}\right| \leq$
  $0.000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,001\,.$

4.1 GR

Einstein gravitational field equation [November 25, 1915]:

$$\boxed{R_{\mu\nu}(x) - \tfrac{1}{2}\, g_{\mu\nu}(x)\, g^{\rho\sigma}(x)\, R_{\rho\sigma}(x) = -8\pi\, G_N\, T_{\mu\nu}(x)\,,} \qquad (21)$$

where (x) stands for $(x^0,\, x^1,\, x^2,\, x^3)$ and further conventions/notations are:

- spacetime indices $\mu,\, \nu,\, \rho,\, \sigma$ run over $0,\, 1,\, 2,\, 3$;
- Einstein summation convention, repeated up and down indices being summed over ;
- $g_{\mu\nu}$ is the metric with invariant line element $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$;
- $g^{\mu\nu}$ is the inverse metric with $g_{\mu\nu}\, g^{\nu\rho} = \delta_\mu{}^\rho$, where the RHS involves the Kronecker symbol δ ;
- $R_{\mu\nu}$ is the Ricci curvature tensor with 1st and 2nd order derivatives of the metric ;
- $T_{\mu\nu}$ is the energy-momentum tensor of the matter.

4.1 GR

Three remarks:

1. The **basic structure** of (21) is to have spacetime structure (curvature) on the LHS and matter content (e.g., energy density) on the RHS.
2. **Einstein's theory reduces to Newton's** in the appropriate limit (e.g., small velocities and weak fields). This becomes particularly clear if Newtonian gravity is written as a **field** theory, given by the **Poisson equation** over 3-dimensional Euclidean space (coordinates y^1, y^2, y^3) with absolute time t:

$$\left(\frac{\partial}{\partial y^1}\frac{\partial}{\partial y^1}+\frac{\partial}{\partial y^2}\frac{\partial}{\partial y^2}+\frac{\partial}{\partial y^3}\frac{\partial}{\partial y^3}\right)\phi(y,t)=4\pi\, G_N\, \rho_{\text{mass}}(y,t)\,. \quad (22)$$

3. Careful discussion of (21) in a follow-up paper [4].

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[4] A. Einstein, Ann. Phys. (Leipzig) **49**, 769 (1916).

4.1 GR

Three classic tests of GR as given by the field equation (21):

1. **Perihelion precession** of Mercury [AE, November 18(!), 1915]:

$$\Delta\phi[\text{arcsec/century}]\Big|^{\text{(exp)}} = +43.11 \pm 0.45\,,$$

$$\Delta\phi[\text{arcsec/century}]\Big|^{\text{(GR)}} = +43.03\,,$$

2. **Deflection of light** by the Sun [Eddington et al, 1919];
3. **Gravitational redshift** of spectral lines [Pound and Rebka, 1960].

A further prediction is the existence of **gravitational waves**. Seen indirectly with the binary pulsar but not yet in a laboratory on Earth.

Also the **Lense-Thirring effect** [1918], which has been confirmed only recently by the Gravity-Probe-B satellite experiment [C.W.F. Everitt, et al., PRL 106, 221101 (2011), arXiv:1105.3456].

4.1 GR

One year later [February 8, 1917], Einstein generalized his field equation (21) by the introduction of the **cosmological constant $\boldsymbol{\lambda}$**:

$$\boxed{R_{\mu\nu}(x) - \tfrac{1}{2}\, g_{\mu\nu}(x)\, g^{\rho\sigma}(x)\, R_{\rho\sigma}(x) - \lambda\, g_{\mu\nu}(x) = -8\pi\, G_N\, T_{\mu\nu}(x)\,,} \qquad (23)$$

Remarks:

- λ has dimensions of inverse length squared;
- With $\lambda \equiv 1/L^2$ and $L \gg 1$ AU, all solar-system tests are satisfied.
- From astronomical observations, we now have $L \sim 10^{10}$ lyr and λ is indeed 'cosmological.'

4.2 FRW universe

The Universe over large scales is **homogeneous and isotropic**. Mathematically, it can then be shown that spacetime is described by the **Robertson–Walker metric**:

$$g_{\mu\nu}(x^1,\, x^2,\, x^3,\, t) \;=\; \begin{pmatrix} & -a(t)^2\,\widetilde{g}_{ij}(x) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{(24a)}$$

$$\widetilde{g}_{ij}(x^1,\, x^2,\, x^3) \;=\; \left(\frac{1}{1 + k\,|\vec{x}|^2/4}\right)^2 \delta_{ij}\,, \qquad \text{(24b)}$$

$$k \;\in\; \{-1,\, 0,\, +1\}\,. \qquad \text{(24c)}$$

Here, so-called comoving coordinates are used, with cosmic time coordinate t, and $a(t)$ is the scale factor of the constant-t hypersurface.

Constant-time slices correspond to the 3-sphere for $k = +1$, Euclidean 3-space for $k = 0$, and a constant hyperbolic 3-space for $k = -1$.

4.2 FRW universe

The matter content of the Universe (over length scales $\gg 10^8$ lyr) can be described by the energy-momentum tensor of a **perfect fluid** with energy density ρ and isotropic pressure P. With the RW metric and comoving coordinates, these quantities depend on cosmic time,

$$\rho = \rho(t)\,, \quad P = P(t)\,. \tag{25}$$

With this energy-momentum tensor,

$$T_{\mu\nu}(x) = T_{\mu\nu}^{\text{perfect-fluid}}[P(t),\, \rho(t)]\,, \tag{26}$$

and the RW metric (24), the Einstein equation (21) reduces to a system of ODEs, the **Friedmann equations**.

4.2 FRW universe

Three Friedmann equations for three variables $a(t)$, $\rho(t)$, and $P(t)$:

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\,\frac{k}{a^2} = 8\pi\,G_N\,\rho\,, \tag{27a}$$

$$\frac{d}{da}(\rho\,a^3) + 3Pa^2 = 0\,, \tag{27b}$$

$$\rho = \rho(P)\,, \tag{27c}$$

where the overdot stands for differentiation w.r.t. cosmic time t.

From these, a further 2nd-order ODE can be obtained:

$$3\,\frac{\ddot{a}}{a} = -4\pi\,G_N\,(\rho + 3\,P)\,. \tag{28}$$

4.2 FRW universe

Three phases of the FRW universe with Hubble expansion $\dot{a}/a > 0$:

1. **Early phase** dominated by **radiation** $[P = \frac{1}{3}\,\rho]$ with $a(t) \propto t^{1/2}$.
2. **Late phase** dominated by **matter** $[P = 0]$ with $a(t) \propto t^{2/3}$.
3. **Final phase** dominated by **vacuum energy** $[P_V = -\rho_V \equiv -\Lambda < 0]$ with a de-Sitter type expansion, $a(t) \propto \exp(\text{const} \times t)$.

From observations of very distant supernovae [released in 1999], it appears that our Universe ($t = t_0 \sim 10^{10}$ yr) has just entered this final phase. Hence, its description as the *"accelerating Universe,"* in contrast to the earlier *decelerated* phases from 'standard matter.'

According to (28), the **acceleration** comes really from the most unusual **equation of state**, $w_V \equiv P_V/\rho_V = -1$.

Fundamental-physics question (related to CCP mentioned earlier): *what sets the present value of the vacuum energy density,*

$$\rho_V(t_0) \sim 2 \times 10^{-11}\ \text{eV}^4 \,. \tag{29}$$

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