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Sphalerons and anomalies (an introduction)

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1. Overview

Context of this talk: **high-energy physics**.

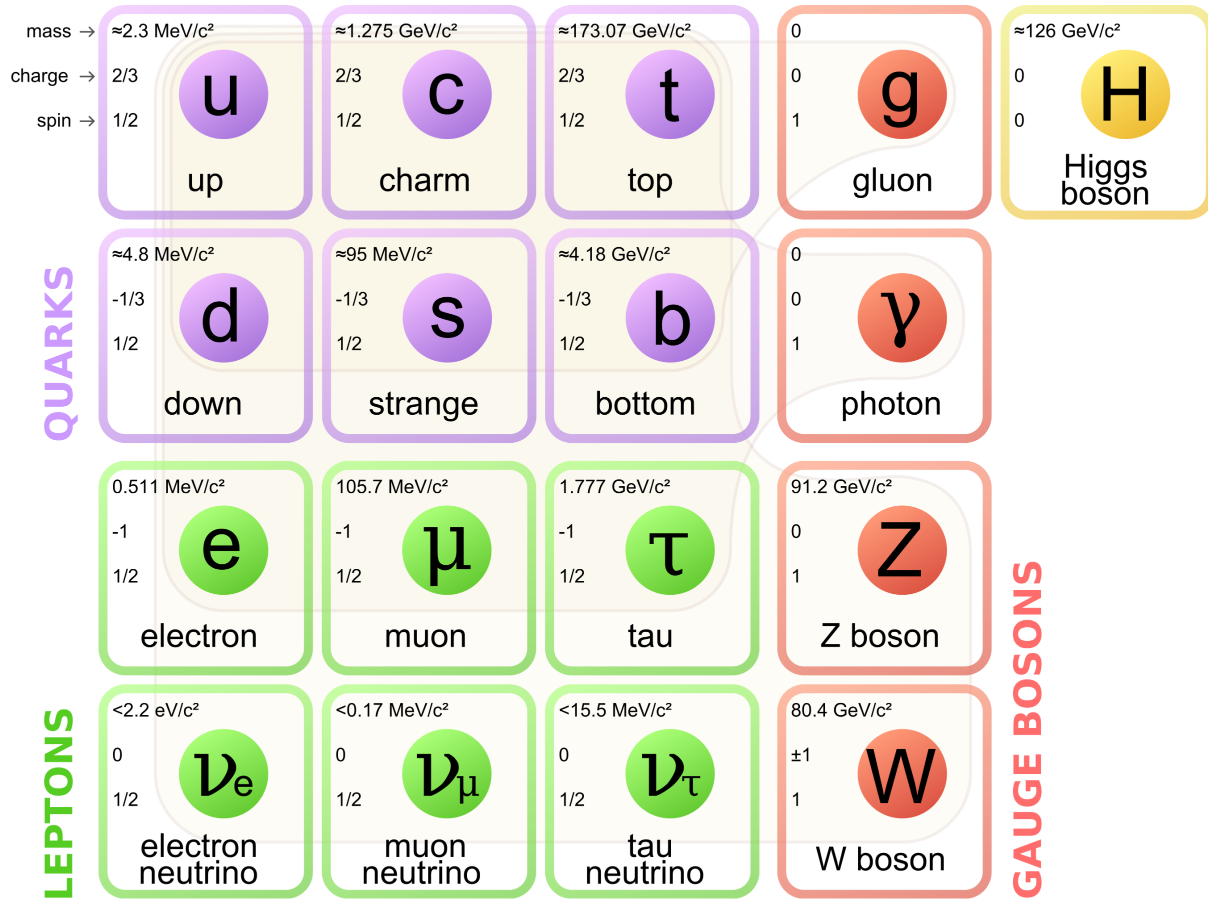
Now, **all** of our current knowledge of high-energy physics is contained in the so-called **Standard Model** (SM).

Incomplete list of SM founding fathers:

. . . , Yang and Mills, 1954; Glashow, 1961;
Englert and Brout, 1964; Higgs, 1964;
Fadde'ev and Popov, 1967;
Weinberg, 1967; Salam, 1968; Glashow, Iliopoulos, and Maiani, 1970;
't Hooft and Veltman, 1972; Lee and Zinn-Justin, 1972;
Weinberg, 1973; Fritzsch, Gell-Mann, and Leutwyler, 1973;
Gross and Wilczek, 1973; Politzer, 1973; . . .

1. Overview

SM:



Elementary particles of the SM [`https://commons.wikimedia.org/wiki`].

1. Overview

But there is more to the SM than particles and Feynman diagrams.

In the $SU(3)$ Yang–Mills theory of QCD:
the **instanton I** [Belavin, Polyakov, Schwartz, and Tyupkin, 1975].

In the $SU(2) \times U(1)$ Yang–Mills–Higgs theory of the EWSM:
the **sphaleron S** [Klinkhamer and Manton, 1984].

The terminology will be explained on the next slide, but here we note already that these **nonperturbative** results complete the perturbative construction of the SM outlined previously.

1. Overview

Terminology:

an “instanton” is a localized, **finite-action** solution of the classical field equations for imaginary time τ ($\tau^2 \leq 0$);

a “topological soliton” is a static, stable, **finite-energy** solution of the classical field equations for real time t ($t^2 \geq 0$);

a “sphaleron” is a static, unstable, **finite-energy** solution of the classical field equations for real time t.

1. Overview

Generally speaking, instantons (and topological solitons) are relevant to equilibrium properties of the theory, whereas sphalerons are relevant to the dynamics.

Specifically, the two types of nonperturbative solutions of the SM are relevant to the following physical effects:

instantons for the gluon condensate and the η' mass,

sphalerons for the origin of the cosmic matter–antimatter asymmetry.

1. Overview

OUTLINE:

1. Overview
2. SU(2) x U(1) sphaleron S and EWBNV *
3. Spectral flow and anomalies
4. SU(3) sphaleron $\widehat{S}$
5. Conclusion
6. References

* EWBNV = electroweak baryon number violation

1. Overview

Admittedly, a difficult talk for a physics colloquium !!!!!!!!!!!!!

Goal is to explain to a general audience that "high-energy physics" is more than "elementary particle physics."

Primarily an introduction, but we can already emphasize three points:

1. The Higgs field is of importance for the sphaleron S.
2. EWBNV is not fully understood.
3. Sphalerons S, S^*, and $\widehat{S}$ are related to chiral anomalies.

2.0 General remarks

How to discover nonperturbative solutions, such as the instanton I or the sphaleron S?

Well, just follow this recipe:

1. make an appropriate *Ansatz* for the fields;
2. solve the resulting reduced field equations.

Of course, the subtlety in getting the “appropriate” *Ansatz* of step 1.

Here, topological insights have played a role.

2.1 SU(2) x U(1) sphaleron S

The electroweak Standard Model (EWSM), with $\sin^2\theta_w \approx 0.23$ and $m_H \approx 125$ GeV, has, most likely, no topological solitons but does have two sphalerons, S [1] and S* [2]. The extended $SU(3)$ theory also has a third sphaleron, $\widehat{\text{S}}$ [3].

The solution S is the best known [1, 4] and its energy is numerically equal to

$$E_S \sim 10 \text{ TeV},$$

and parametrically equal to

$$E_S \sim v/g \sim M_W/\alpha\,,$$

with the Higgs vacuum expectation value v, the $SU(2)$ coupling constant g, the mass $M_W = \frac{1}{2}\,g\,v$ of the charged vector bosons $W^\pm$, and the fine-structure constant $\alpha = e^2/(4\pi) = g^2\,\sin^2\theta_w/(4\pi)$.

2.1 SU(2) x U(1) sphaleron S

In simple terms, the sphaleron solution S of the EWSM

- is a slightly elongated blob of field energy with size of order $1/M_W \sim 10^{-2}$ fm and energy density of order $(1/\alpha)\, M_W^4$;
- has “tangled” fields (hence, the existence of fermion zero modes; see the discussion on spectral flow below) ;
- corresponds to an unstable configuration of fields, which, after a small perturbation, decays to the vacuum by emission of many particles (number of order $1/\alpha \sim 100$) .

2.1 SU(2) x U(1) sphaleron S

But how does S fit in configuration space?

A simple sketch is as follows (more details later):

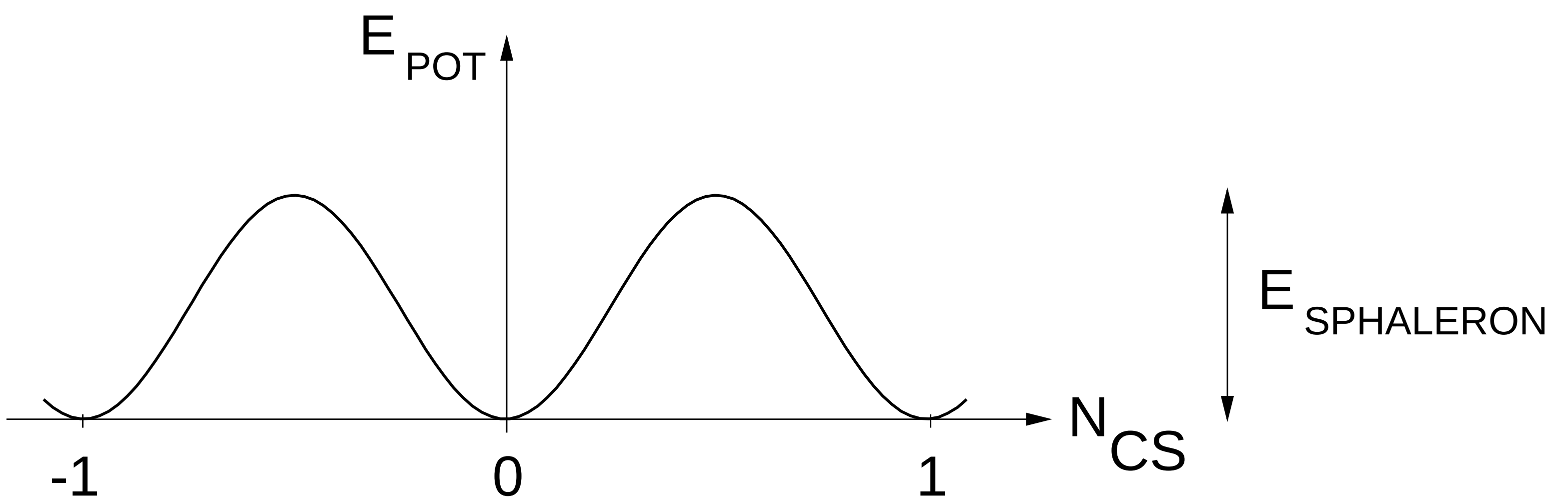


Figure 1: Potential energy over a slice of configuration space.

Side remark: small oscillations near $N_{\text{CS}} = 0$ (or any other integer) correspond to the SM elementary particles W, Z, γ, etc.

2.1 SU(2) x U(1) sphaleron S

Now a technical remark, triggered by the ATLAS and CMS discovery [5, 6] :

the energy density of S is only finite because of the Higgs **field**.

Setting $\theta_w = 0$, for simplicity, and using the dimensionless radial coordinate $\xi \equiv g\, v\, r$ (a prime indicating the derivative w.r.t. ξ), the energy is:

$$
\begin{aligned}
E_S \;\; &= \;\; \frac{v}{g} \int_0^\infty d\xi \; 4\pi\xi^2 \left[\frac{4}{\xi^2}\,(f')^2 + \frac{8}{\xi^4}\,[f(1-f)]^2 \right. && \longleftarrow \;\; (F_{mn})^2 \;\; \text{term} \\
&+ \frac{1}{2}\,(h')^2 + \frac{1}{\xi^2}\,[h(1-f)]^2 && \longleftarrow \;\; (D_m\Phi)^2 \;\; \text{term} \\
&\left. + \frac{1}{4}\,\frac{\lambda}{g^2}\,(1-h^2)^2 \right], && \longleftarrow \;\; V(\Phi) \;\; \text{term}
\end{aligned}
$$

with the following boundary conditions on the radial functions $f(r)$ and $h(r)$:

$$
f(0) = h(0) = 0\,, \quad f(\infty) = h(\infty) = 1\,.
$$

2.1 SU(2) x U(1) sphaleron S

The energy density from the above E_S integral shows that having $h(r) \equiv 1$ (i.e., absence of a dynamical Higgs field) would make the YM-mass-term contribution $\propto r^{-2}\,(1-f)^2$ diverge at $r=0$.

Physically, it is doubtful that such a divergent classical field configuration would play a role. At the very least, the EWSM would need to be modified if there were no Higgs field.*

The Higgs field is also important for another reason (spectral flow), as will be discussed in Sec. 3.

Having the nonperturbative classical solution S, the question remains what it does for physics?

The answer is cosmic baryon number violation, so let us discuss that first.

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* A similar conclusion follows from the well-known $W^+W^+ \to W^+W^+$ unitarity argument.

2.2 Electroweak baryon number violation

Conditions for cosmological baryogenesis [Sakharov, 1967]:

1. C and CP violation	Yes	(SM)
2. Thermal nonequilibrium	Yes	(FRW)
3. Baryon number (B) violation	?	

Strictly speaking, we know of only one physical theory that is expected to have B violation:

the electroweak Standard Model (EWSM).

[Side remark: the *ultimate* fate of black holes is uncertain and, hence, it is not known if black-hole physics violates baryon number conservation or not.]

2.2 Electroweak baryon number violation

But the relevant physical processes of the EWSM at

$$T \ll M_W \approx 10^2 \text{ GeV} ,$$

have a rate (tunneling through the barrier of Fig. 1) which is negligible [7],

$$\Gamma^{\text{(tunneling)}} \propto \exp[\,-2\,\mathcal{S}_{\text{BPST}}\,/\,\hbar\,] = \exp[\,-4\,\pi\,\sin^2\theta_w\,/\,\alpha\,] \approx 0\ ,$$

with an exponent given by twice the action of the BPST instanton.

For $T \sim 10^2$ GeV, the rate (thermal excitation over the barrier of Fig. 1) contains a Boltzmann factor [1],

$$\Gamma^{\text{(thermal)}} \propto \exp[\,-E_S\,/\,(k\,T)\,]\ ,$$

in terms of the barrier height, the sphaleron energy E_S.

Note the respective factors of $\hbar$ and k in the two rates Γ: different physics!

2.2 Electroweak baryon number violation

Clearly, we should study electroweak baryon number violation for the conditions of the early universe,

$$T \gtrsim 10^2 \text{ GeV} .$$

This is a difficult problem, but entirely well-posed.

In this talk, we focus on the fundamental physics, i.e., the microscopic process, and we must really deal with the fermions [7, 8, 9, 10, 11, 12, 13].

2.3 EWBNV – Classic results

Consider $SU(2)$ Yang–Mills–Higgs theory with vanishing Yukawa couplings. Actually, forget about the Higgs, which may be reasonable above the EW phase transition.

Triangle anomaly in the AAA-diagram, provided the VVV-diagram is anomaly-free [14, 15]. [Side remark: this is Feynman *perturbation* theory.]

The gauge vertices of the EWSM are V–A and must be nonanomalous (gauge invariance is needed for unitarity). Then, the $B + L$ current becomes anomalous [7]:

$$\Delta(B - L) = 0 \,,$$

$$\underbrace{\Delta(B + L)}_{\text{change of fermion number}} = \underbrace{2\,N_{\text{fam}}}_{\text{integer}} \times \underbrace{\Delta N_{\text{CS}}}_{\text{gauge field characteristic}} \,.$$

2.3 EWBNV – Classic results

In the $A_0 = 0$ gauge, one has the Chern–Simons number

$$N_{\mathsf{CS}}(t) = N_{\mathsf{CS}}[\vec{A}(\vec{x}, t)]$$

and

$$\Delta N_{\mathsf{CS}} \equiv N_{\mathsf{CS}}(t_{\mathsf{out}}) - N_{\mathsf{CS}}(t_{\mathsf{in}}) \ .$$

For the record (using differential forms and the Yang–Mills field strength 2-form $F \equiv dA + A^2$), we have

$$N_{\mathsf{CS}}[A] \equiv \frac{1}{8\pi^2} \int_{M_3} \left(AdA + \frac{2}{3} A^3 \right) = \frac{1}{8\pi^2} \int_{M_3} \left(AF - \frac{1}{3} A^3 \right) \ .$$

2.3 EWBNV – Classic results

't Hooft [7] calculated the tunneling amplitude using the BPST instanton. This BPST instanton, which is a finite action solution over Euclidean spacetime (imaginary-time theory), gives

$$\Delta N_{\text{CS}} = Q[A_{\text{finite action}}] \in \mathbb{Z} \,,$$

where the topological charge Q is the winding number of the map

$$S^3\big|_{|x|=\infty} \to SU(2) \sim S^3 \,.$$

This holds only for transitions from near-vacuum to near-vacuum, i.e., at very low temperatures or energies. As mentioned above, the rate is then effectively zero, but, at least, $\Delta(B+L)$ is an integer, namely $2\, N_{\text{fam}} \times \Delta N_{\text{CS}}$.

2.4 EWBNV – Open question

For real-time processes at nonzero energies or temperatures, the topological charge Q is, in general, noninteger.

Hence, the question [12]

$$\Delta(B+L) \propto \text{which gauge field characteristic ?}$$

In the following, we consider pure $SU(2)$ Yang–Mills theory with a single isodoublet of left-handed fermions.

(The fermion number $B+L$ of the EWSM follows by multiplying with $2\,N_{\text{fam}}$. Recall that $B-L$ remains conserved in the EWSM.)

Furthermore, the gauge fields will be called dissipative if their energy density approaches zero uniformly as $t \to \pm\infty$.

2.5 EWBNV – Partial answer

Start from the eigenvalue equation of the time-dependent Dirac Hamiltonian:

$$H(\vec{x},t)\,\Psi(\vec{x},t) = E(t)\,\Psi(\vec{x},t)\ ,$$

where H is a functional of the background gauge field $\vec{A}(\vec{x},t)$.

Then, fermion number violation is related to the so-called spectral flow $\mathcal{F}$. See, e.g., Refs. [8, 13].

Definition:

$\mathcal{F}[\,t_f, t_i\,]$ is the number of eigenvalues of the Dirac Hamiltonian that cross zero from below minus the number of eigenvalues that cross zero from above, for the time interval $[\,t_i, t_f\,]$ with $t_i < t_f$.

2.5 EWBNV – Partial answer

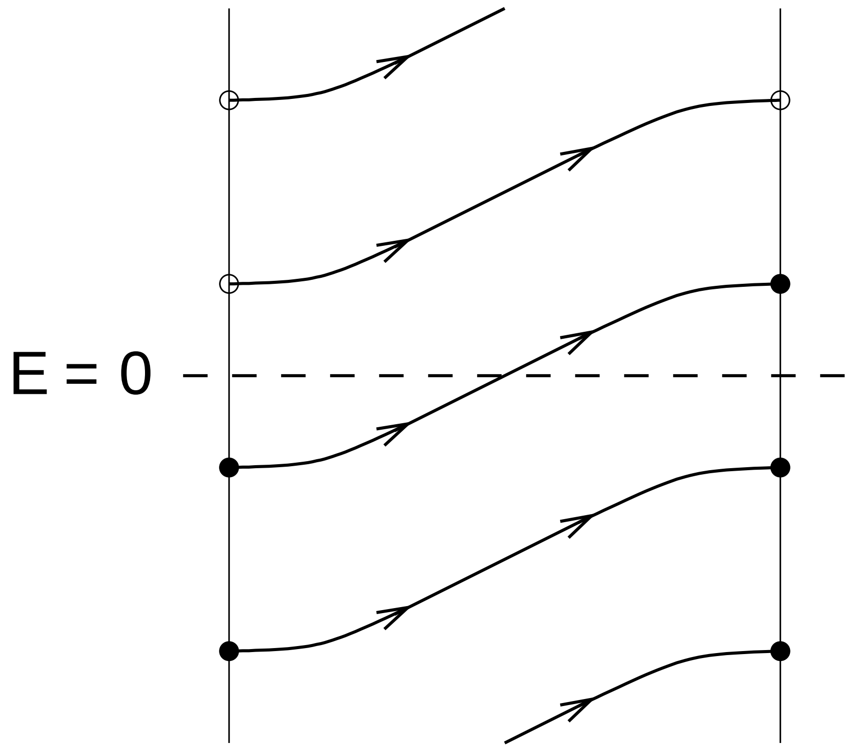

Figure 2: Spectral flow with $\mathcal{F}[t_f, t_i] = +1 - 0 = +1$. Filling the (infinite) Dirac sea at the initial time t_i results in one extra fermion at the final time t_f.

2.5 EWBNV – Partial answer

Strongly-dissipative* SU(2) gauge fields at finite energy have [8, 9, 10]:

$$\mathcal{F} \;=\; \Delta N_{\text{CS}}[A_{\text{associated vacuum}}] \equiv \Delta N_{\text{winding}} \in \mathbb{Z}\,.$$

Now, there exist three weakly-dissipative,* spherically symmetric gauge field solutions [Lüscher & Schechter, 1977] with

1. (low energy) $\Delta N_{\text{winding}} = 0$ and $\mathcal{F} = 0\,,$
2. (moderate energy) $\Delta N_{\text{winding}} = 1$ and $\mathcal{F} = 1\,,$
3. (high energy) $\Delta N_{\text{winding}} = 1$ and $\mathcal{F} = -1\,.$

$$\Rightarrow\; \left[\,\mathcal{F} \neq \Delta N_{\text{winding}}\,\right]_{\text{spherically symmetric fields}}\,.$$

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* For the precise definition of strongly/weakly-dissipative, see [11].

2.5 EWBNV – Partial answer

In fact, there is another gauge field characteristic [11]:

$$\Delta N_{\text{twist}} = 0 \qquad \text{for case 1 and 2}\,,$$
$$\Delta N_{\text{twist}} = -2 \qquad \text{for case 3}\,.$$

$$\Rightarrow\ \big[\,\mathcal{F} = \Delta N_{\text{winding}} + \Delta N_{\text{twist}}\,\big]_{\text{spherically symmetric fields}}\,.$$

For weakly dissipative or nondissipative gauge fields, one has thus

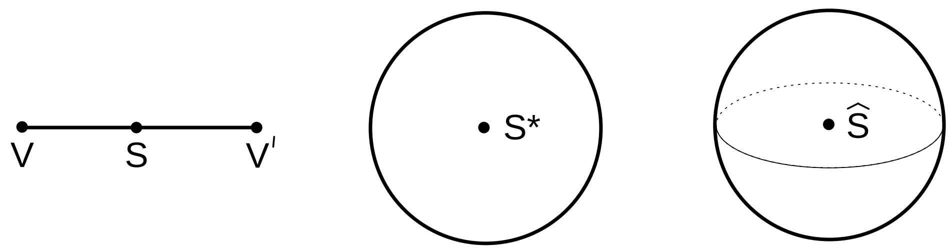
$$\Delta(B+L) \;=\; 2\,N_{\text{fam}} \times \Big(\Delta N_{\text{CS}}\,[A_{\,\text{associated vacuum}}] + \underline{\text{extra terms}}\Big).$$

But the “extra terms” are not known in general [12].

In short, the microphysics of EWBNV is not fully understood.

3. Spectral flow and anomalies

Three sphalerons are relevant to the SM, each related to having a **nontrivial vacuum structure**:



NCL of vacua

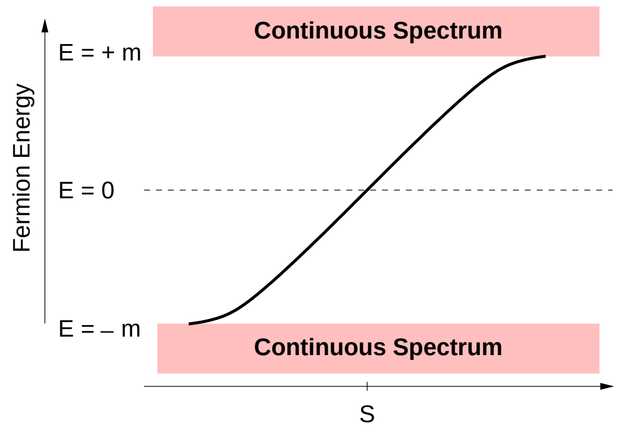
NCS of vacua

where NCL/S stands for noncontractible loop/sphere.

Note that the V–S–V′ sketch above corresponds to Fig. 1.

3. Spectral flow and anomalies

These sphalerons are, in fact, relevant to **spectral flow** (with fermion masses from the Higgs field). The picture for S is well known (cone-like for S^* and $\widehat{\mathrm{S}}$):



In turn, these sphalerons are related to **anomalies**:

S to the chiral $U(1)$ anomaly [Adler–Bell–Jackiw, 1969],

S^* to the chiral nonperturbative $SU(2)$ anomaly [Witten, 1982],

$\widehat{\mathrm{S}}$ to the chiral non-Abelian anomaly [Bardeen, 1969].

3. Spectral flow and anomalies

These sphalerons are then relevant to the following physical processes:

S to B+L violation for the matter-antimatter asymmetry in the early universe,

S^* to multiparticle production in high-energy scattering with $\sqrt{s} \geq E_{S^*}$,

$\widehat{\mathrm{S}}$ to nonperturbative dynamics of QCD.

The physics application of S is well known, even though far from being understood completely (as discussed before).

For the rest of the talk, let me focus on $\widehat{\mathrm{S}}$, which has an interesting mathematical structure but a less clear physics application.

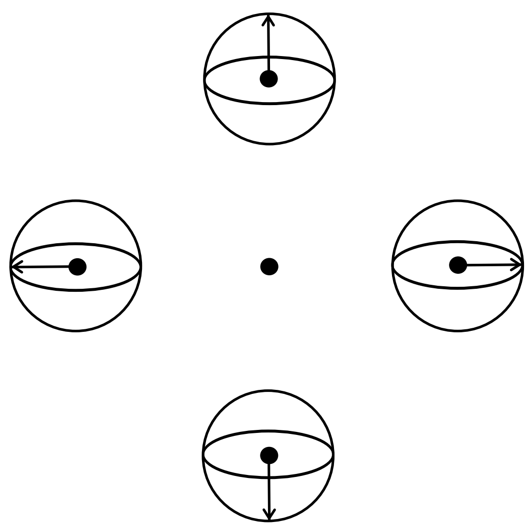
4.0 Preliminary remarks

Before discussing the $SU(3)$ sphaleron $\widehat{\mathrm{S}}$, recall three basic facts of S.

First, the $SU(2)$ sphaleron S can be embedded in $SU(3)$ YMH theory [strictly speaking, the embedded solution is the $SU(2) \times U(1)$ sphaleron].

Second, the $SU(2)$ gauge and Higgs fields of S are determined by two radial functions $f(r)$ and $h(r)$, as discussed before.

Third, the $SU(2)$ sphaleron S has a so-called **hedgehog structure**, i.e., a topologically nontrivial map $S_3^{(\text{space})} \to SU_2^{(\text{internal})} = S_3^{(\text{internal})}$.
Here a sketch of $S_2^{(\text{space})} \to S_2^{(\text{internal})}$:



4.1 SU(3) sphaleron S-hat

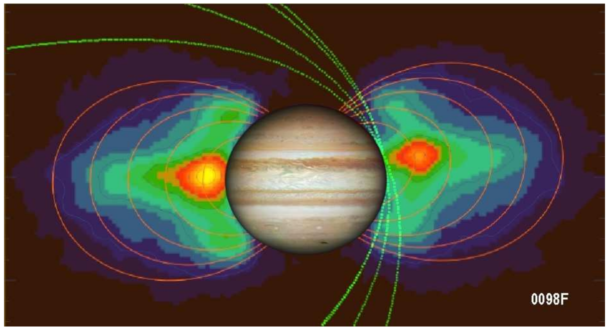
Now turn to $\widehat{\mathrm{S}}$, which is very different.

First, $\widehat{\mathrm{S}}$ exists in $SU(3)$ YMH but not in $SU(2)$ YMH theory.

Second, the self-consistent *Ansatz* of $\widehat{\mathrm{S}}$ requires eight axial functions for the gauge field and three axial functions for the fundamental Higgs field.

Third, $\widehat{\mathrm{S}}$ does not have a hedgehog structure but a **Jupiter-like structure**:

for a given half-plane through the symmetry–axis with azimuthal angle ϕ, the parallel components A_r and A_θ involve only one particular $su(2)$ subalgebra of $su(3)$, whereas the orthogonal component A_ϕ excites precisely the other five generators of $su(3)$.



4.1 SU(3) sphaleron S-hat

As to the reduced field equations, they are very difficult to solve, even numerically.

Still it is possible to obtain an upper bound on the energy [3]:

$$E_{\widehat{\mathsf{S}}}\Big|_{\lambda/g^2=0} < 1.72 \times E_S \,, \tag{2}$$

with $E_S \equiv 1.52 \times 4\pi v/g$ and λ the quartic Higgs coupling constant.

After several years of work, the numerical solution of the reduced field equations has been obtained recently [K & Nagel, 2015] and the numerical value for the energy is:

$$E_{\widehat{\mathsf{S}}}\Big|_{\lambda/g^2=0} = (1.240 \pm 0.003) \times E_S \,. \tag{3}$$

4.1 SU(3) sphaleron S-hat

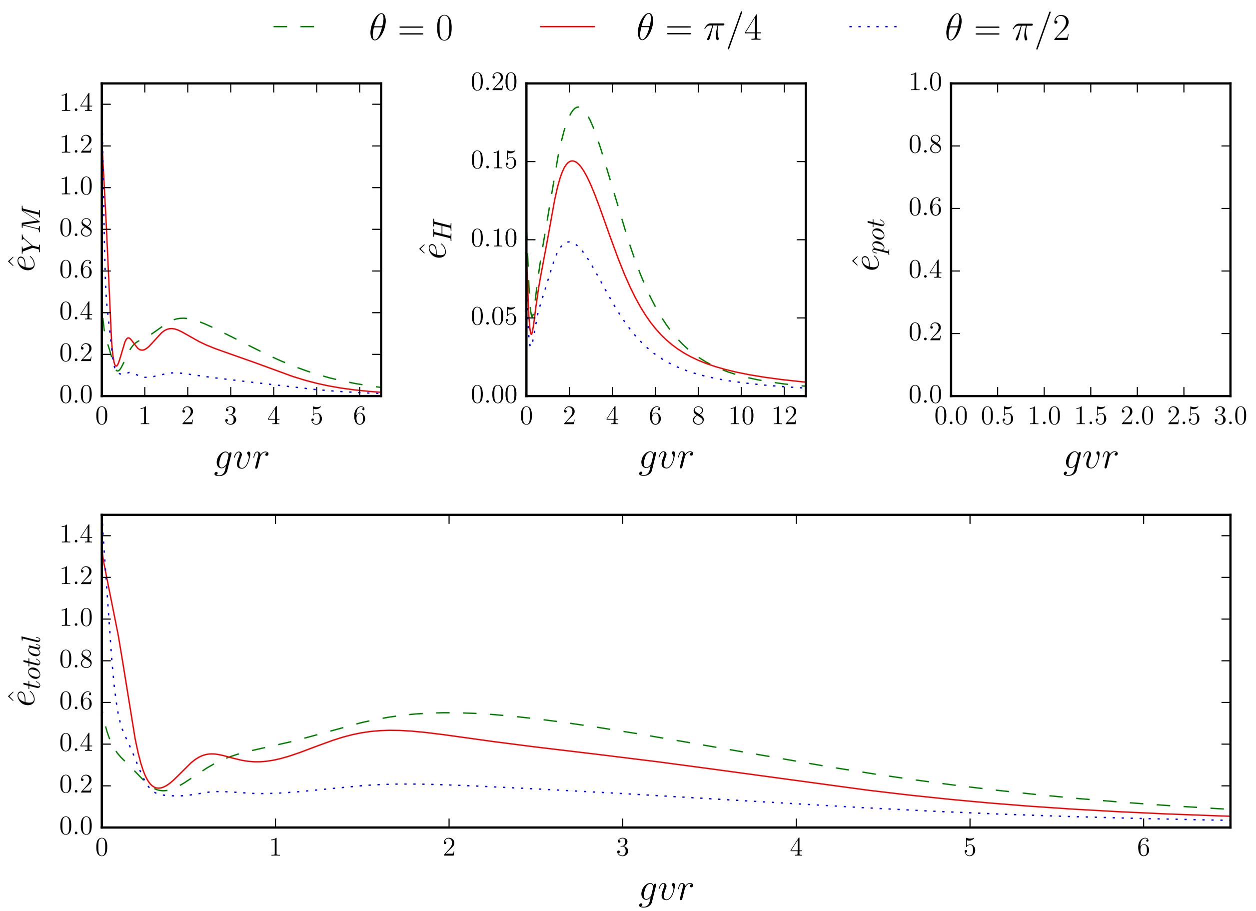


Figure 3: $\widehat{\mathrm{S}}$ energy densities for $\lambda/g^2 = 0$.

4.1 SU(3) sphaleron S-hat

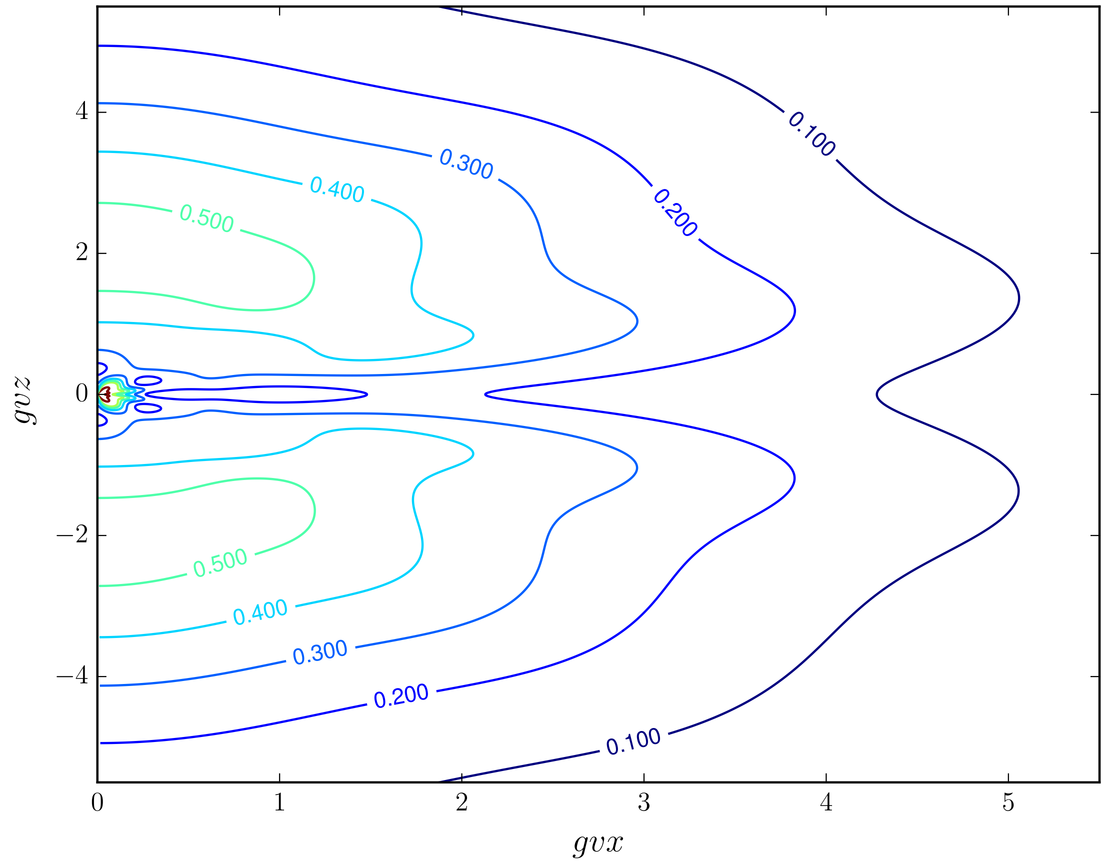


Figure 4: Contours of the total energy density of $\widehat{\text{S}}$ for $\lambda/g^2 = 0$.

4.1 SU(3) sphaleron S-hat

Mathematically, it is remarkable that the energy of $\widehat{\mathrm{S}}$ with eight gauge fields is close to that of S with only four gauge fields. Most likely, this is due to the highly-ordered (Jupiter-like) structure mentioned earlier.

Physically, it is important that the $\widehat{\mathrm{S}}$ barrier is low, as it implies that related processes are little suppressed at high energies/temperatures [the energy scale being set by QCD quantum effects, $\Lambda \sim 100$ MeV].

5. Conclusion

The mathematical physics of the sphaleron solutions is relatively straightforward. Really difficult are the physics applications.

Let us mention three **outstanding puzzles** related to the three sphalerons S, S^*, and $\widehat{\text{S}}$:

First, how does the B+L violation proceed microscopically at high energies or high temperatures (the scale being set by $E_S \sim 10$ TeV) and what is the proper selection rule?

Second, does EWSM multiparticle production in high-energy scattering with $\sqrt{s} \sim E_{S^*} \sim 20$ TeV reach the unitarity limit?

Third, does $\widehat{\text{S}}$ produce new physical effects in QCD?

6. References

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