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Construction of a new constrained instanton in Yang-Mills-Higgs theory

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Abstract

A self-consistent ansatz is presented for a four-dimensional euclidean solution (instanton) in the vacuum sector of constrained $SU(2)$ Yang-Mills-Higgs theory.

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1 Introduction

A topological argument has been given [1] for the existence of a new instanton* in the vacuum sector of euclidean $SU(2)$ Yang-Mills-Higgs (YMH) theory, with a constraint added to fix the scale of the solution. The existence argument was based on the construction of a suitable non-contractible loop (NCL) of 4-dimensional field configurations. The basic idea now is that by making appropriate modifications of the configuration at the top of this particular NCL one arrives at an exact solution of the field equations. The present paper gives the resulting self-consistent ansatz for this new constrained instanton $\mathrm{I}^\star$. Technically the construction of $\mathrm{I}^\star$ is more complicated than for the related 3-dimensional sphaleron solution $\mathrm{S}^\star$ [4], but the method is essentially the same.

Most likely, $\mathrm{I}^\star$ is the lowest action constrained instanton solution in the vacuum sector of euclidean $SU(2)$ YMH theory. As such it is expected to be of fundamental importance for the quantum field theory of the electroweak interactions. In particular, $\mathrm{I}^\star$ is believed to play a role in the asymptotics of Feynman perturbation theory [5].

The outline of this paper is as follows. Section 2 describes briefly the theory considered. Section 3 presents the ansatz and Section 4 lists the symmetries, which are crucial for the derivation. Section 5 gives the action evaluated for the ansatz fields and Section 6 explains how the ansatz leads to having a non-trivial solution of the field equations. Section 7, finally, gives a brief discussion of the potential physics applications mentioned above. There are also three appendices. Appendix A puts the NCL of [1] in a form appropriate for the construction of the ansatz of Section 3. Appendix B gives the transformations of the ansatz functions under the residual gauge symmetry discussed in Section 4. Appendix C, finally, gives the ansatz actiondensity of Section 5 in a compact notation.

2 Theory

Consider a classical Yang-Mills theory, with non-Abelian gauge group $SU(2)$ and gauge coupling constant g, coupled to a Higgs scalar field in the fundamental representation, with vacuum expectation value v and quartic coupling constant λ. The two mass scales of the theory are $M_\mathrm{W} \equiv \frac{1}{2} g\, v$ for the three W bosons and $M_\mathrm{H} \equiv \sqrt{2\lambda}\, v$ for the single Higgs scalar. The total euclidean action is

$$A = A_\mathrm{YMH} + A_\mathrm{C}\,, \tag{1}$$

*Here, and in the following, the term "instanton" refers to *any* localized, finite action solution of the general field equations belonging to the euclidean action of the theory considered, not exclusively to solutions of certain first order self-duality conditions [2, 3].

with the Yang-Mills-Higgs action

$$A_{\rm YMH} = \int\limits_{\mathbb{R}^4} d^4x \left[-\frac{1}{2\,g^2}\,{\rm Tr}\,W_{\mu\nu}^2 + |D_\mu \Phi|^2 + \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 \right], \tag{2}$$

where $W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]$, $D_\mu \Phi \equiv (\partial_\mu + W_\mu)\Phi$, $W_\mu \equiv W_\mu^a \sigma^a/(2i)$ and σ^a are the standard Pauli matrices, and a constraint term [6]

$$A_{\rm C} = \frac{\kappa}{g^2} \left(\frac{1}{M_W^4} \int\limits_{\mathbb{R}^4} d^4x\, O_8 - \frac{8\pi^2 c}{(M_{\rm W}\,\varrho)^4} \right), \tag{3}$$

with κ an arbitrary positive constant (Lagrange multiplier) and ϱ the "size" of the field configuration. Specifically, we choose for the constraint operator

$$O_8 = q_{\rm P}^2\,, \tag{4}$$

where $q_{\rm P}$ is the Pontryagin density

$$q_{\rm P} \equiv -\frac{1}{4}\,\epsilon_{\kappa\lambda\mu\nu}\,{\rm Tr}\,W_{\kappa\lambda} W_{\mu\nu}\,, \tag{5}$$

and for the numerical constant c we take the value

$$c = \frac{288}{21}\,. \tag{6}$$

This particular choice for c reproduces the usual scale parameter of a BPST [2] instanton–anti-instanton pair at infinite separation. Of course, there are many other constraint operators O_d possible, as long as they have canonical mass dimension $d > 4$. Also, there might, in principle, exist solutions for which the single constraint operator (4) does not suffice and further terms need to be added.

Having defined the theory we proceed in three steps. First, we obtain for a given positive value of κ a solution $W^\star, \Phi^\star$ of the field equations resulting from variations δW, $\delta\Phi$ of the total action A. Second, we solve for the scale $\varrho^\star = \varrho^\star(\kappa, W^\star, \Phi^\star)$, so that $A_{\rm C}^\star = 0$ and $A^\star = A_{\rm YMH}^\star(\kappa, W^\star, \Phi^\star)$, and eliminate κ between $\varrho^\star$ and $A_{\rm YMH}^\star$. Third, with these values $\varrho^\star$ and $A_{\rm YMH}^\star(\varrho^\star, W^\star, \Phi^\star)$, we integrate over the collective coordinate ϱ in the path integral of the particular Green's function considered. Further details may be found in [1, 6].

Here, we focus on the first crucial step, namely to discover a non-trivial solution of the field equations belonging to the euclidean action (1). More specificially, we look for a finite action solution with Pontryagin index (topological charge)

$$Q_{\rm P} \equiv \frac{1}{8\pi^2} \int\limits_{\mathbb{R}^4} d^4x\; q_{\rm P} \tag{7}$$

vanishing, i. e. a new constrained instanton in the vacuum sector.

3 Ansatz

Define cylindrical coordinates ρ, φ, z and τ in terms of the cartesian coordinates

$$(x_1, x_2, x_3, x_4) \equiv (\rho\cos\varphi, \rho\sin\varphi, z, \tau)$$

and introduce a triad of matrices

$$\begin{aligned}
\mathbf{u} &\equiv \sin\varphi\, \tau^2 + \cos\varphi\, \tau^1 \\
\mathbf{v} &\equiv \cos\varphi\, \tau^2 - \sin\varphi\, \tau^1 \\
\mathbf{w} &\equiv \tau^3,
\end{aligned}$$

with $\tau^a \equiv \sigma^a/(2i)$ and σ^a the Pauli matrices

$$\sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The ansatz is then given by

$$\begin{aligned}
W_1 &= \frac{1}{\rho}\,\{\, (C_1\cos\varphi - C_3\sin\varphi)\,\mathbf{u} + (C_2\cos\varphi - C_4\sin\varphi)\,\mathbf{v} + (C_5\sin\varphi + C_6\cos\varphi)\,\mathbf{w} \,\} \\
W_2 &= \frac{1}{\rho}\,\{\, (C_1\sin\varphi + C_3\cos\varphi)\,\mathbf{u} + (C_2\sin\varphi + C_4\cos\varphi)\,\mathbf{v} - (C_5\cos\varphi - C_6\sin\varphi)\,\mathbf{w} \,\} \\
W_3 &= \frac{1}{z}\,\{\, C_7\,\mathbf{u} - C_8\,\mathbf{v} + C_9\,\mathbf{w} \,\} \\
W_4 &= \frac{1}{\tau}\,\{\, C_{10}\,\mathbf{u} - C_{11}\,\mathbf{v} + C_{12}\,\mathbf{w} \,\} \\
\Phi &= \frac{v}{\sqrt{2}} \begin{pmatrix} -(H_2 + iH_1)\,e^{-i\varphi} \\ H_0 + iH_3 \end{pmatrix},
\end{aligned} \tag{8}$$

with

$$\begin{gathered}
C_1 = 4\frac{\rho}{X}f_1 \qquad C_2 = 4\frac{\rho z}{X^2}f_2 \qquad C_3 = 4\frac{\rho z}{X^2}f_3 \qquad C_4 = 4\frac{\rho}{X}f_4 \\
C_5 = 4\frac{\rho^2}{X^2}f_5 \qquad C_6 = 4\frac{\rho^2 z}{X^3}f_6 \\
C_7 = 4\frac{\rho z^2}{X^3}f_7 \qquad C_8 = 4\frac{\rho z}{X^2}f_8 \qquad C_9 = 4\frac{z}{X}f_9 \\
C_{10} = 4\frac{\rho\tau}{X^2}f_{10} \qquad C_{11} = 4\frac{\rho z\tau}{X^3}f_{11} \qquad C_{12} = 4\frac{\tau z}{X^2}f_{12} \\
H_0 = \frac{z}{X}h_0 \qquad H_1 = 2\frac{\rho z}{X^2}h_1 \qquad H_2 = 2\frac{\rho}{X}h_2 \qquad H_3 = h_3
\end{gathered} \tag{9}$$

and

$$X^2 \equiv x^2 + x_a^2 \equiv \rho^2 + z^2 + \tau^2 + x_a^2\,,$$

for arbitrary parameter x_a. The axial functions $f_i = f_i(\rho, z, \tau)$ and $h_j = h_j(\rho, z, \tau)$, with $i = 1, \ldots, 12$ and $j = 0, \ldots, 3$, are non-singular and have reflection symmetry

$$\begin{aligned} f_i(\rho, -z, \tau) &= f_i(\rho, z, \tau) \\ h_j(\rho, -z, \tau) &= h_j(\rho, z, \tau) \,. \end{aligned} \tag{10}$$

Continuity of the gauge fields at $\rho = 0$ demands

$$\begin{aligned} f_1(0, z, \tau) &= f_4(0, z, \tau) \\ -f_2(0, z, \tau) &= f_3(0, z, \tau) \,. \end{aligned} \tag{11}$$

All axial functions have furthermore Neumann boundary conditions at $\rho = 0$

$$\begin{aligned} \partial_\rho f_i(0, z, \tau) = 0 \\ \partial_\rho h_j(0, z, \tau) = 0 \end{aligned} \tag{12}$$

and Dirichlet-like boundary conditions at infinity

$$\lim_{|x|\to\infty} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \\ f_{12} \\ h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} \tau/x \\ -1 \\ (z^2 - \rho^2 + \tau^2)/x^2 \\ \tau(z^2 - \rho^2 + \tau^2)/x^3 \\ 2(\tau^2 + z^2)/x^2 \\ 0 \\ -2\tau/x \\ (\tau^2 - \rho^2 - z^2)/x^2 \\ 2\tau\rho^2/x^3 \\ (z^2 - \rho^2 - \tau^2)/x^2 \\ -2\tau/x \\ -2\rho^2/x^2 \\ 0 \\ -1 \\ -\tau/x \\ (\rho^2 - z^2 - \tau^2)/x^2 \end{bmatrix} . \tag{13}$$

The ansatz (8) has, as will be explained in the next Section, a residual gauge symmetry, which allows for the elimination of essentially three functions. One possible gauge choice is the well-known radial gauge $\hat{x}_\mu W_\mu = 0$:

$$C_1 + C_7 + C_{10} = C_2 - C_8 - C_{11} = C_6 + C_9 + C_{12} = 0 \,. \tag{14}$$

This completes the description of the ansatz*. Next, we study its symmetries and explain the logic behind the construction.

The $U(1)$ hypercharge gauge field of the electroweak standard model can easily be included, just as for the sphaleron S [4].

4 Symmetries

The ansatz of the previous Section has, by construction, several symmetries, which we will now discuss.

4.1 Axial symmetry

Generally, scalar and vector fields transform under infinitesimal coordinate transformations

$$x^\mu \to x^\mu + \delta\alpha\, \xi^\mu(x) \tag{15}$$

as follows:

$$\begin{aligned} \Phi(x) &\to \Phi(x) + \delta\alpha\, \mathbb{L}_\xi \Phi(x) && \text{(16a)} \\ W_\mu(x) &\to W_\mu(x) + \delta\alpha\, \mathbb{L}_\xi W_\mu(x)\,, && \text{(16b)} \end{aligned}$$

with Lie derivatives

$$\begin{aligned} \mathbb{L}_\xi \Phi &\equiv \xi^\nu \partial_\nu \Phi && \text{(17a)} \\ \mathbb{L}_\xi W_\mu &\equiv \xi^\nu \partial_\nu W_\mu + (\partial_\mu \xi^\nu)\, W_\nu\,. && \text{(17b)} \end{aligned}$$

In the present paper we will use a generalized form of invariance of the fields, namely by allowing for a compensating internal symmetry transformation. In short, invariance need only hold up to a gauge transformation (or any other internal symmetry transformation for that matter).

The coordinate transformation relevant here is the rotation

$$x^\mu \to x^\mu + \delta\varphi\, \xi^\mu_{\mathrm{R}}\,, \tag{18}$$

generated by the vector field

$$\xi^\mu_{\mathrm{R}} = (-x_2, x_1, 0, 0) \tag{19}$$

and infinitesimal parameter $\delta\varphi$. It is then straightforward to verify the invariance of the ansatz gauge fields (8) under the transformation (16b), with $\delta\alpha$ replaced by $\delta\varphi$ and ξ by ξ_{R}, combined with the global gauge transformation

$$W_\mu \to W_\mu - [W_\mu, \tau^3]\, \delta\varphi\,. \tag{20}$$

For the scalar field (8) there is in addition to the transformation (16a), again with $\delta\alpha$ replaced by $\delta\varphi$ and ξ by ξ_{R}, a compensating transformation

$$\Phi \to \Phi - i\,\delta\varphi \left(\frac{\mathbb{1} + \sigma^3}{2}\right) \Phi\,, \tag{21}$$

which combines a global $SU(2)$ gauge transformation with a $U(1)$ phase transformation. The ansatz (8) thus has $U(1)$ invariance under rotations in the x_1,x_2 plane.

4.2 Discrete symmetries

The ansatz (8–10) is invariant under the combined discrete transformation

$$P_{xz} \otimes C \otimes G_c \,, \tag{22}$$

with P_{xz} the following parity-like transformation of the classical fields (cartesian coordinates x_1, x_2, x_3, x_4 being written as x, y, z, τ) :

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ \Phi \end{bmatrix} (x,y,z,\tau) \rightarrow \begin{bmatrix} -W_1 \\ W_2 \\ -W_3 \\ W_4 \\ \Phi \end{bmatrix} (-x,y,-z,\tau) \,, \tag{23}$$

C the charge conjugation transformation

$$\begin{bmatrix} W_\mu \\ \Phi \end{bmatrix} \rightarrow \begin{bmatrix} W_\mu^* \\ \Phi^* \end{bmatrix} \tag{24}$$

and G_c the global $SU(2)$ gauge transformation with gauge parameter function $\Gamma = -1\!\!1_2$ in the center of the group

$$\begin{bmatrix} W_\mu \\ \Phi \end{bmatrix} \rightarrow \begin{bmatrix} W_\mu \\ -\Phi \end{bmatrix} \,. \tag{25}$$

The second discrete symmetry of the ansatz acts only on the fields in the $z = 0$ plane and corresponds to the usual parity transformation P

$$\begin{bmatrix} W_1 \\ W_2 \\ W_4 \\ \Phi \end{bmatrix} (x,y,0,\tau) \rightarrow \begin{bmatrix} -W_1 \\ -W_2 \\ -W_4 \\ \Phi \end{bmatrix} (-x,-y,0,-\tau) \,, \tag{26}$$

provided the ansatz functions (9) obey the following conditions :

$$\begin{aligned} -f_1(\rho,0,-\tau) &= f_1(\rho,0,\tau) \\ -f_4(\rho,0,-\tau) &= f_4(\rho,0,\tau) \\ f_5(\rho,0,-\tau) &= f_5(\rho,0,\tau) \\ f_{10}(\rho,0,-\tau) &= f_{10}(\rho,0,\tau) \\ -h_2(\rho,0,-\tau) &= h_2(\rho,0,\tau) \\ h_3(\rho,0,-\tau) &= h_3(\rho,0,\tau) \,. \end{aligned} \tag{27}$$

Note that the same parity reflection symmetry holds for the sphaleron S* [4], which corresponds in fact to the $z = 0$ slice of I*.

Both discrete symmetries also distinguish the configuration at the "top" of the NCL constructed previously [1]. The crucial idea behind the ansatz of Section 3 was to generalize that particular NCL configuration, while respecting the axial and discrete symmetries present. Further details can be found in Appendix A.

4.3 Residual gauge symmetry

The axisymmetric ansatz (8) is form invariant under gauge transformations

$$\begin{aligned} W_\mu &\rightarrow \Gamma\,(W_\mu + \partial_\mu)\,\Gamma^{-1} \\ \Phi &\rightarrow \Gamma\,\Phi\,, \end{aligned} \tag{28}$$

with gauge parameter function

$$\Gamma = \exp\left(\omega_1\,\mathbf{u} + \omega_2\,\mathbf{v} + \omega_3\,\mathbf{w}\right), \tag{29}$$

where $\omega_a = \omega_a(\rho, z, \tau)$. In order to maintain the discrete symmetries of the ansatz, there are the following conditions :

$$\begin{aligned} \omega_1(\rho,-z,\tau) &= \omega_1(\rho,z,\tau) \\ -\omega_2(\rho,-z,\tau) &= \omega_2(\rho,z,\tau) \\ -\omega_3(\rho,-z,\tau) &= \omega_3(\rho,z,\tau) \\ -\omega_1(\rho,0,-\tau) &= \omega_1(\rho,0,\tau)\,. \end{aligned} \tag{30}$$

The explicit transformations of the coefficient functions C_i and H_j are somewhat involved and are given in Appendix B. As mentioned above, these residual gauge transformations eliminate essentially 3 functions from the ansatz.

5 Action

The ansatz of section 3 gives for the Yang-Mills-Higgs action

$$A_{\text{YMH}} = 4\pi \int\limits_{-\infty}^{\infty} d\tau \int\limits_{0}^{\infty} dz \int\limits_{0}^{\infty} d\rho\,\rho\,a_{\text{YMH}}\,, \tag{31}$$

with actiondensity

$$a_{\text{YMH}} = a_{\text{WKIN}} + a_{\text{HKIN}} + a_{\text{HPOT}}\,, \tag{32}$$

where

$$
\begin{aligned}
& a_{\mathrm{WKIN}} = \\
& \frac{1}{2g^2}\Bigg\{ \left(\frac{\partial_z C_1}{\rho} - \frac{\partial_\rho C_7}{z} - \frac{C_2 C_9}{\rho z} - \frac{C_6 C_8}{\rho z}\right)^2 + \left(\frac{\partial_\tau C_1}{\rho} - \frac{\partial_\rho C_{10}}{\tau} - \frac{C_2 C_{12}}{\rho\tau} - \frac{C_6 C_{11}}{\rho\tau}\right)^2 \\
& + \left(\frac{\partial_z C_2}{\rho} + \frac{\partial_\rho C_8}{z} + \frac{C_1 C_9}{\rho z} - \frac{C_6 C_7}{\rho z}\right)^2 + \left(\frac{\partial_\tau C_2}{\rho} + \frac{\partial_\rho C_{11}}{\tau} + \frac{C_1 C_{12}}{\rho\tau} - \frac{C_6 C_{10}}{\rho\tau}\right)^2 \\
& + \left(\frac{\partial_z C_6}{\rho} - \frac{\partial_\rho C_9}{z} + \frac{C_1 C_8}{\rho z} + \frac{C_2 C_7}{\rho z}\right)^2 + \left(\frac{\partial_\tau C_6}{\rho} - \frac{\partial_\rho C_{12}}{\tau} + \frac{C_1 C_{11}}{\rho\tau} + \frac{C_2 C_{10}}{\rho\tau}\right)^2 \\
& + \left(\frac{\partial_\tau C_7}{z} - \frac{\partial_z C_{10}}{\tau} + \frac{C_8 C_{12}}{z\tau} - \frac{C_9 C_{11}}{z\tau}\right)^2 + \left(\frac{\partial_\tau C_8}{z} - \frac{\partial_z C_{11}}{\tau} - \frac{C_7 C_{12}}{z\tau} + \frac{C_9 C_{10}}{z\tau}\right)^2 \\
& + \left(\frac{\partial_\tau C_9}{z} - \frac{\partial_z C_{12}}{\tau} + \frac{C_7 C_{11}}{z\tau} - \frac{C_8 C_{10}}{z\tau}\right)^2 + \left(\frac{\partial_\rho C_3}{\rho} - \frac{C_4 C_6}{\rho^2} - \frac{C_2}{\rho^2}(C_5 - 1)\right)^2 \\
& + \left(\frac{\partial_z C_3}{\rho} - \frac{C_4 C_9}{\rho z} + \frac{C_8}{\rho z}(C_5 - 1)\right)^2 + \left(\frac{\partial_\tau C_3}{\rho} - \frac{C_4 C_{12}}{\rho\tau} + \frac{C_{11}}{\rho\tau}(C_5 - 1)\right)^2 \\
& + \left(\frac{\partial_\rho C_4}{\rho} + \frac{C_3 C_6}{\rho^2} + \frac{C_1}{\rho^2}(C_5 - 1)\right)^2 + \left(\frac{\partial_z C_4}{\rho} + \frac{C_3 C_9}{\rho z} + \frac{C_7}{\rho z}(C_5 - 1)\right)^2 \\
& + \left(\frac{\partial_\tau C_4}{\rho} + \frac{C_3 C_{12}}{\rho\tau} + \frac{C_{10}}{\rho\tau}(C_5 - 1)\right)^2 + \left(\frac{\partial_\rho C_5}{\rho} + \frac{C_2 C_3}{\rho^2} - \frac{C_1 C_4}{\rho^2}\right)^2 \\
& + \left(\frac{\partial_z C_5}{\rho} - \frac{C_3 C_8}{\rho z} - \frac{C_4 C_7}{\rho z}\right)^2 + \left(\frac{\partial_\tau C_5}{\rho} - \frac{C_3 C_{11}}{\rho\tau} - \frac{C_4 C_{10}}{\rho\tau}\right)^2 \Bigg\}
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
& a_{\mathrm{HKIN}} = \\
& \frac{v^2}{2}\Bigg\{ \left(\partial_\rho H_0 - \frac{H_1 C_1}{2\rho} - \frac{H_2 C_2}{2\rho} - \frac{H_3 C_6}{2\rho}\right)^2 + \left(\partial_z H_0 - \frac{H_1 C_7}{2z} + \frac{H_2 C_8}{2z} - \frac{H_3 C_9}{2z}\right)^2 \\
& + \left(\partial_\tau H_0 - \frac{H_1 C_{10}}{2\tau} + \frac{H_2 C_{11}}{2\tau} - \frac{H_3 C_{12}}{2\tau}\right)^2 + \left(\partial_\rho H_1 + \frac{H_0 C_1}{2\rho} - \frac{H_2 C_6}{2\rho} + \frac{H_3 C_2}{2\rho}\right)^2 \\
& + \left(\partial_z H_1 + \frac{H_0 C_7}{2z} - \frac{H_2 C_9}{2z} - \frac{H_3 C_8}{2z}\right)^2 + \left(\partial_\tau H_1 + \frac{H_0 C_{10}}{2\tau} - \frac{H_2 C_{12}}{2\tau} - \frac{H_3 C_{11}}{2\tau}\right)^2 \\
& + \left(\partial_\rho H_2 + \frac{H_0 C_2}{2\rho} + \frac{H_1 C_6}{2\rho} - \frac{H_3 C_1}{2\rho}\right)^2 + \left(\partial_z H_2 - \frac{H_0 C_8}{2z} + \frac{H_1 C_9}{2z} - \frac{H_3 C_7}{2z}\right)^2 \\
& + \left(\partial_\tau H_2 - \frac{H_0 C_{11}}{2\tau} + \frac{H_1 C_{12}}{2\tau} - \frac{H_3 C_{10}}{2\tau}\right)^2 + \left(\partial_\rho H_3 + \frac{H_0 C_6}{2\rho} - \frac{H_1 C_2}{2\rho} + \frac{H_2 C_1}{2\rho}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left(\partial_z H_3 + \frac{H_0 C_9}{2z} + \frac{H_1 C_8}{2z} + \frac{H_2 C_7}{2z}\right)^2 + \left(\partial_\tau H_3 + \frac{H_0 C_{12}}{2\tau} + \frac{H_1 C_{11}}{2\tau} + \frac{H_2 C_{10}}{2\tau}\right)^2 \\
&+ \quad \left(\frac{H_0 C_4}{2\rho} + \frac{H_1}{\rho} - \frac{H_1 C_5}{2\rho} - \frac{H_3 C_3}{2\rho}\right)^2 + \left(\frac{H_0 C_3}{2\rho} - \frac{H_2}{\rho} + \frac{H_2 C_5}{2\rho} + \frac{H_3 C_4}{2\rho}\right)^2 \\
&+ \quad \left(\frac{H_0 C_5}{2\rho} + \frac{H_1 C_4}{2\rho} - \frac{H_2 C_3}{2\rho}\right)^2 + \left(\frac{H_1 C_3}{2\rho} + \frac{H_2 C_4}{2\rho} - \frac{H_3 C_5}{2\rho}\right)^2 \Bigg\} \qquad (34)
\end{aligned}
$$

$$
a_{\text{HPOT}} = \lambda \frac{v^4}{4} \left(H_0^2 + H_1^2 + H_2^2 + H_3^2 - 1\right)^2 . \qquad (35)
$$

For brevity, we have not made the functions C_i and H_j explicit by inserting (9), but if one does one readily verifies that the action density is finite everywhere and vanishes at infinity. In addition, we can fix the gauge with conditions (14).

The constraint operator (3, 4) gives for the ansatz fields (8) a term in the total action of the form

$$
A_{\text{C}} = \frac{\kappa}{g^2} \left(\frac{4\pi}{M_W^4} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} dz \int_0^{\infty} d\rho\, \rho\, q_{\text{P}}^2 - \frac{8\pi^2 c}{(M_{\text{W}}\varrho)^4} \right) , \qquad (36)
$$

with Pontryagin density q_{P} given by

$$
\begin{aligned}
\rho^2 z\tau\, q_{\text{P}} &= C_1 \overleftrightarrow{D}_{\tau z} C_3 + C_2 \overleftrightarrow{D}_{\tau z} C_4 + C_5 \overleftrightarrow{D}_{\tau z} C_6 + C_7 \overleftrightarrow{D}_{\rho\tau} C_3 \\
&+ C_4 \overleftrightarrow{D}_{\rho\tau} C_8 + C_5 \overleftrightarrow{D}_{\rho\tau} C_9 + C_3 \overleftrightarrow{D}_{\rho z} C_{10} + C_{11} \overleftrightarrow{D}_{\rho z} C_4 + C_{12} \overleftrightarrow{D}_{\rho z} C_5 \\
&+ \rho\, \overrightarrow{\partial}_\rho \left[(C_5 - 1)(C_7 C_{11} - C_8 C_{10}) + C_9 (C_3 C_{11} + C_4 C_{10}) - C_{12} (C_3 C_8 + C_4 C_7) \right] \\
&+ z\, \overrightarrow{\partial}_z \left[(1 - C_5)(C_2 C_{10} + C_1 C_{11}) - C_6 (C_3 C_{11} + C_4 C_{10}) + C_{12} (C_1 C_4 - C_2 C_3) \right] \\
&+ \tau\, \overrightarrow{\partial}_\tau \left[(C_5 - 1)(C_1 C_8 + C_2 C_7) + C_6 (C_3 C_8 + C_4 C_7) - C_9 (C_1 C_4 - C_2 C_3) \right] ,
\end{aligned}
\qquad (37)
$$

in terms of

$$
\overleftrightarrow{D}_{\tau z} \equiv \overleftarrow{\partial}_\tau\, \tau z\, \overrightarrow{\partial}_z - \overleftarrow{\partial}_z\, z\tau\, \overrightarrow{\partial}_\tau
$$

and similarly for $\overleftrightarrow{D}_{\rho\tau}$ and $\overleftrightarrow{D}_{\rho z}$, where the partial derivatives $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ operate to the left and to the right, respectively. Again, we have to insert (9) and implement the gauge fixing conditions (14).

It is possible to get more compact expressions for the Pontryagin density and action-densities, see Appendix C. With (31, 36) one then obtains the final expression for the total action (1–6) evaluated with the ansatz fields. Furthermore, the Pontryaginindex (7) can be shown to vanish for the fields of the ansatz. This can be established most easily by use of the Chern-Simons current

$$j_\mu \equiv -\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\left[W_\nu\left(\partial_\alpha W_\beta + \frac{2}{3}\,W_\alpha W_\beta\right)\right], \tag{38}$$

whose divergence gives the Pontryagin density $\partial_\mu j_\mu = q_{\mathrm{P}}$, so that

$$Q_{\mathrm{P}} = \frac{1}{8\pi^2}\oint_{S^3_\infty} d\Sigma_\mu\, j_\mu\ .$$

For the ansatz fields at infinity (8, 9, 13) the Chern-Simons current vanishes identically and

$$Q_{\mathrm{P}}(W_{\mathrm{I}^\star}) = 0\ . \tag{39}$$

This result for the topological charge Q_{P} also follows from the simple observation that the solution I* lies on a *continuous* path of configurations (NCL) connected to the vacuum configuration $W_{\mathrm{vac}} = 0$, which is topologically trivial $Q_{\mathrm{P}}(0) = 0$.

6 Solution

The ansatz (8–10) is self-consistent, which means that the field equations (from $\delta A/\delta W = \delta A/\delta\Phi = 0$) reduce to 16 equations, which are precisely equal to those obtained from variations δC_i and δH_j of the ansatz action (31, 36). This agrees with the so-called principle of symmetric criticality [7], which states that in the quest of stationary points it suffices, under certain conditions, to consider variations that respect the symmetries of the ansatz (rotation and reflection symmetries in our case).

An analytic solution of the resulting non-linear partial differential equations for the functions $f_i(\rho, z, \tau)$ and $h_j(\rho, z, \tau)$ seems to be impossible and even an accurate numerical evaluation is a major enterprise, which we have to postpone for the moment. Still, it is possible to argue that there does exist a non-trivial solution, i. e. a solution different from the vacuum ($W = 0$, $|\Phi|^2 = \frac{1}{2}v^2$) in whatever complicated gauge. The crucial observation is that the boundary conditions and symmetries of the ansatz imply the existence of at least one point where the total Higgs field vanishes $\Phi = 0$, which is a gauge invariant statement.

Consider then the $H_3 = 0$ surface Σ as it comes in from infinity (far out Σ is, according to (13), given by the equation $z^2 + \tau^2 = \rho^2$). The simplest possibility is that of a single, connected surface as shown in Fig. 1a. The ansatz (9) gives, in that case, at least one point on the τ-axis with *all* coefficient functions H_j vanishing, i. e. $\Phi = 0$ there. Alternatively, the $H_3 = 0$ surface could "pinch off", as shown in Fig. 1b, in which case

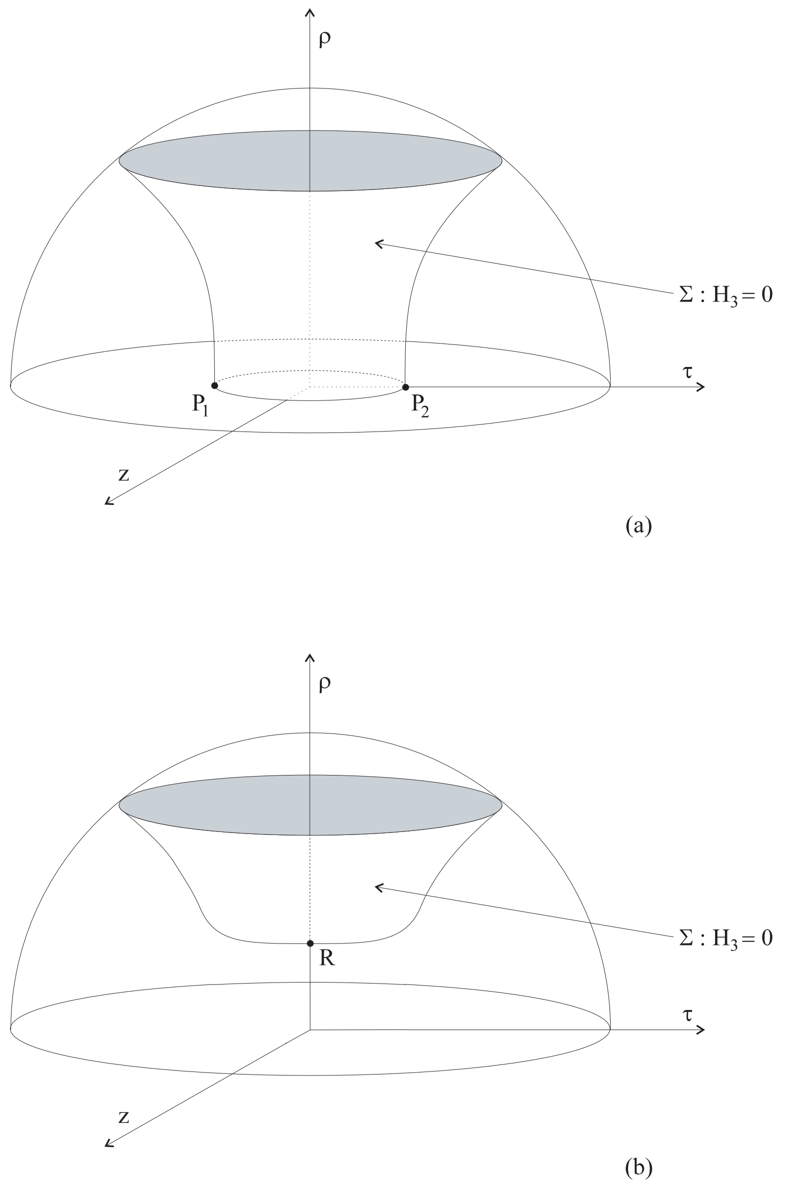

Fig. 1. Sketch of the $H_3 = 0$ surface Σ as it propagates in from the sphere at infinity. *A priori* there are two possibilities : (a) a single connected surface embedded in $\mathbb{R}^4$, with Higgs field vanishing at, generically, two points P_1 and P_2 ; (b) a "disconnected" surface in $\mathbb{R}^4$, with Higgs field vanishing at R, which corresponds to a ring of zeros in $\mathbb{R}^4$.

the ansatz (9, 27) gives a ring of zeros for the Higgs field. Whichever possibility is realized is for the field equations to decide. The same two basic possibilities occur for the static sphaleron $\mathrm{S}^\star$, where the numerical solution of the field equations appears to indicate the first alternative [4]. Hence, one expects also $\mathrm{I}^\star$ to resemble a di-atomic molecule, albeit on a scale of $M_{\mathrm{W}}^{-1} \sim 10^{-8}$Å.

7 Discussion

In this paper we have presented a selfconsistent ansatz for a new constrained instanton $\mathrm{I}^\star$ in $SU(2)$ Yang-Mills-Higgs theory. Further work on the numerical solution of the reduced field equations remains to be done, but we can already make some general remarks, see also [1, 4].

The constrained instanton $\mathrm{I}^\star$ may be thought of as locating the top of the action barrier for the global $SU(2)$ anomaly [8]. The electroweak standard model has, of course, no global $SU(2)$ anomaly, the total number of left-handed fermion doublets being even. Still, the barrier structure in configuration space and the corresponding constrained instanton solution remain.

A more direct application of $\mathrm{I}^\star$ to the electroweak interactions may be the asymptotics of Feynman perturbation theory. From the outset it is important to realize [9] that the electroweak standard model has only been established in (low order) perturbation theory. High-order contributions $c_n\, g^{2n}$ to an arbitrary physical observable may be estimated [5] by saddle-point approximation of the euclidean path integral. It appears that $\mathrm{I}^\star$ provides the relevant saddle-point. In particular, the $\mathrm{I}^\star$ negative mode (which projects onto the non-contractible loop) plays a important role. Assuming there to be a single negative mode and assuming the integral over the collective coordinate ϱ to be dominated by the value $\varrho = 0$, the expected asymptotic behavior is

$$c_n\, g^{2n} \sim \frac{n!}{(A^\star_{\mathrm{YMH}}(0))^n} \sim \frac{n!}{(16\,\pi^2)^n}\, g^{2n} \;, \tag{40}$$

where for the instanton action $A^\star_{\mathrm{YMH}}(\varrho) = 16\pi^2/g^2 + \mathrm{O}(\varrho^2 v^2)$ is used [1]. If this is indeed the asymptotic behaviour of standard electroweak perturbation theory, then the series is not even Borel summable. Physics considerations (such as causality and unitarity) should tell us how to make sense of perturbation theory or, more generally, how to define electroweak field theory non-perturbatively.

Appendix A

Here, we review the non-contractible loop (NCL) given in [1] and put it in the appropriate form for the construction of the ansatz of Section 3.

The basic structure of this loop of 4-dimensional configurations is to first create and separate (to a distance d) an BPST-like instanton–anti-instanton pair, then make a complete relative isospin rotation, and finally collapse and annihilate the pair. For the present purpose we only need the isospin rotation part of the NCL with loop-parameter $\omega \in [-\pi/2, +\pi/2]$, whereas the whole NCL has ω running from $-3\pi/2$ to $+3\pi/2$.

With the same notation as in Section 3 and defining the radial coordinate $r^2 \equiv \rho^2 + z^2$, the configurations of the NCL are for $\omega \in [-\pi/2, +\pi/2]$

$$\begin{aligned} W_\mu &= -f\, \partial_\mu U\ U^{-1} \\ \Phi &= \frac{v}{\sqrt{2}}\, h\, U \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{A.1}$$

with the $SU(2)$ matrix*

$$U = i\sigma_3\, e^{(\omega+\pi/2)i\sigma_3}\, (\hat{x}_- \cdot \sigma)\, e^{-(\omega+\pi/2)i\sigma_3}\, (\hat{x}_+ \cdot \sigma)^\dagger \tag{A.2}$$

and definitions

$$\begin{aligned} \hat{x}^\mu_\pm &\equiv x^\mu_\pm / |x_\pm| \\ x^\mu_\pm &\equiv \left(x^1, x^2, x^3, x^4 \pm d/2\right) \\ \sigma^\mu &\equiv \left(i\sigma^1, i\sigma^2, i\sigma^3, \mathbb{1}_2\right). \end{aligned}$$

The axial functions $f = f(r,\tau)$ and $h = h(r,\tau)$ have boundary conditions**

$$\begin{gathered} f(0, \pm d/2) = h(0, \pm d/2) = 0 \\ \lim_{|x|\to\infty} f = \lim_{|x|\to\infty} h = 1\,. \end{gathered} \tag{A.3}$$

This completes our review of the crucial part of the NCL, which is based on the topologically non-trivial loop of mappings $U(\omega)$ of the sphere at infinity S^3_∞ into the group manifold $SU(2) = S^3$.

The NCL configurations (A.1) are axially symmetric and in the notation of (8) the corresponding coefficient functions C_i and H_j are given by

$$\begin{aligned} C_1 = \frac{\rho f}{x_+^2 x_-^2} \Bigg\{ & \left((2\tau - d)\left(\tau^2 - \frac{d^2}{4} + z^2\right) - 2z^2 d\right) - 2zx_+^2 \sin 2\omega \\ & + (2\tau - d)\, x_+^2 \cos 2\omega - 2z\rho^2 \sin 4\omega + \rho^2 (2\tau + d) \cos 4\omega \Bigg\} \end{aligned}$$

*the functions $g_\pm$ of [1] are set to unity and a global gauge transformation (28) with $\Gamma = i\sigma_3$ has been performed.

**the additional conditions $f(r,-\tau) = f(r,\tau)$ and $h(r,-\tau) = h(r,\tau)$ of [1] are, strictly speaking, not essential.

$$C_2 = -\frac{\rho f}{x_+^2 x_-^2}\left\{2z\left(\tau d + \tau^2 - \frac{3}{4}\,d^2 + z^2\right) + (2\tau - d)\,x_+^2 \sin 2\omega \right.$$
$$\left. + 2zx_+^2 \cos 2\omega + \rho^2\,(d + 2\tau)\sin 4\omega + 2z\rho^2 \cos 4\omega \right\}$$

$$C_3 = \frac{\rho f}{x_+^2 x_-^2}\left\{2z\left(\tau d + \tau^2 - \frac{3}{4}\,d^2 + z^2\right) - (2\tau - d)\left(2\rho^2 - x_+^2\right)\sin 2\omega \right.$$
$$\left. - 2z\left(2\rho^2 - x_+^2\right)\cos 2\omega - \rho^2\,(2\tau + d)\sin 4\omega - 2z\rho^2 \cos 4\omega \right\}$$

$$C_4 = \frac{\rho f}{x_+^2 x_-^2}\left\{\left((2\tau - d)\left(\tau^2 - \frac{d^2}{4} + z^2\right) - 2z^2 d\right) + 2z\left(2\rho^2 - x_+^2\right)\sin 2\omega \right.$$
$$\left. - (2\tau - d)\left(2\rho^2 - x_+^2\right)\cos 2\omega + 2z\rho^2 \sin 4\omega - \rho^2\,(2\tau + d)\cos 4\omega \right\}$$

$$C_5 = \frac{4\rho^2 f}{x_+^2 x_-^2}\left\{\left(\tau^2 + z^2 + \frac{d^2}{4}\right) + dz \sin 2\omega + \left(\tau^2 + z^2 - \frac{d^2}{4}\right)\cos 2\omega \right\}$$

$$C_6 = \frac{4\rho^2 f}{x_+^2 x_-^2}\left\{\left(\frac{d^2}{4} - z^2 - \tau^2\right)\sin 2\omega + dz \cos 2\omega \right\}$$

$$C_7 = -2\frac{\rho z f}{x_+^2 x_-^2}\left\{z\,(2\tau - d) - \left(\tau d + z^2 + \rho^2 - \tau^2 + \frac{3}{4}\,d^2\right)\sin 2\omega \right.$$
$$\left. + z\,(d + 2\tau)\cos 2\omega - \rho^2 \sin 4\omega \right\}$$

$$C_8 = 2\frac{\rho z f}{x_+^2 x_-^2}\left\{\left(\tau^2 - z^2 - \tau d + \frac{d^2}{4}\right) - \left(\tau d + z^2 + \rho^2 - \tau^2 + \frac{3}{4}\,d^2\right)\cos 2\omega \right.$$
$$\left. - z\,(2\tau + d)\sin 2\omega - \rho^2 \cos 4\omega \right\}$$

$$C_9 = 2\frac{z f}{x_+^2 x_-^2}\left\{\left(2\rho^2\tau + \tau^2 d - z^2 d - \frac{d^3}{4}\right) + 2z\rho^2 \sin 2\omega + \rho^2\,(2\tau - d)\cos 2\omega \right\}$$

$$C_{10} = 2\frac{\rho\tau f}{x_+^2 x_-^2}\left\{\left(z^2 - \tau^2 + \tau d - \frac{d^2}{4}\right) - \left(\rho^2 + \tau d - z^2 + \frac{d^2}{4} + \tau^2\right)\cos 2\omega \right.$$
$$\left. + z\,(2\tau - d)\sin 2\omega - \rho^2 \cos 4\omega \right\}$$

$$C_{11} = -2\frac{\rho\tau f}{x_+^2 x_-^2}\left\{z\,(2\tau - d) + \left(\rho^2 + \tau d - z^2 + \frac{d^2}{4} + \tau^2\right)\sin 2\omega \right.$$
$$\left. + z\,(2\tau - d)\cos 2\omega + \rho^2 \sin 4\omega \right\}$$

$$
\begin{aligned}
C_{12} &= -4\frac{f\tau}{x_+^2 x_-^2}\left\{z\left(\tau d+\rho^2\right)+\rho^2\left(\frac{d}{2}-\tau\right)\sin 2\omega+\rho^2 z\cos 2\omega\right\} \\
H_0 &= \frac{h}{x_+x_-}\left\{-zd+\rho^2\sin 2\omega\right\} \\
H_1 &= -\frac{h}{x_+x_-}\rho\left\{z+\left(\tau+\frac{d}{2}\right)\sin 2\omega+z\cos 2\omega\right\} \\
H_2 &= -\frac{h}{x_+x_-}\rho\left\{\left(\tau-\frac{d}{2}\right)-z\sin 2\omega+\left(\tau+\frac{d}{2}\right)\cos 2\omega\right\} \\
H_3 &= \frac{h}{x_+x_-}\left\{\left(\frac{d^2}{4}-z^2-\tau^2\right)+\rho^2\cos 2\omega\right\}.
\end{aligned}
\tag{A.4}
$$

Only for $\omega = 0$ (and $\omega = \pm\pi/2$) do we have the discrete symmetries described in Section 4.2. In H_0, for example, there is a term proportional to $\rho^2 \sin 2\omega$ that breaks the discrete symmetry (22), whereas the other term proportional to zd respects it.

The ansatz of Section 3 then is a generalization of the configuration (A.4) at the top ($\omega = 0$) of this particular NCL, keeping the axial and discrete symmetries present and taking over the boundary conditions at infinity. In fact, the ansatz (8, 9) has *a priori* 16 independent functions $f_i(\rho, z, \tau)$ and $h_j(\rho, z, \tau)$, whereas the maximal configuration of the NCL (A.1) has only two, namely $f(r, \tau)$ and $h(r, \tau)$. This illustrates the degree of generalization required to obtain an exact solution of the field equations.

Appendix B

Here, we list the transformation properties of the coefficient functions C_i, H_j of the ansatz (8) under the residual gauge symmetries (28, 29).

The transformations of the gauge field coefficients functions C_i ($i = 1, \ldots, 12$) are the following:

$$
\begin{aligned}
\begin{pmatrix} C_1 \\ C_2 \\ C_6 \end{pmatrix} &\rightarrow \begin{pmatrix} T_{\rho 1} \\ T_{\rho 2} \\ T_{\rho 3} \end{pmatrix} + R \begin{pmatrix} C_1 \\ C_2 \\ C_6 \end{pmatrix} \\
\begin{pmatrix} C_3 \\ C_4 \\ 1 - C_5 \end{pmatrix} &\rightarrow \qquad\qquad R \begin{pmatrix} C_3 \\ C_4 \\ 1 - C_5 \end{pmatrix} \\
\begin{pmatrix} C_7 \\ -C_8 \\ C_9 \end{pmatrix} &\rightarrow \begin{pmatrix} T_{z 1} \\ T_{z 2} \\ T_{z 3} \end{pmatrix} + R \begin{pmatrix} C_7 \\ -C_8 \\ C_9 \end{pmatrix} \\
\begin{pmatrix} C_{10} \\ -C_{11} \\ C_{12} \end{pmatrix} &\rightarrow \begin{pmatrix} T_{\tau 1} \\ T_{\tau 2} \\ T_{\tau 3} \end{pmatrix} + R \begin{pmatrix} C_{10} \\ -C_{11} \\ C_{12} \end{pmatrix},
\end{aligned}
\tag{B.1}
$$

with the $SO(3)$ matrix

$$
R \equiv \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix},
$$

defined in terms of

$$
\begin{aligned}
R_{ca} &\equiv \frac{1}{\Omega^2} \Big[\omega_c \omega_a \left(1 - \cos\Omega\right) - \epsilon_{cab} \omega_b \Omega \sin\Omega + \delta_{ca} \Omega^2 \cos\Omega \Big] \\
T_{\bar{\alpha} c} &\equiv \frac{1}{\Omega^2} \left(\bar{\alpha}\, \partial_{\bar{\alpha}}\, \omega_a\right) \left[\omega_c \omega_a \left(\frac{\sin\Omega}{\Omega} - 1 \right) + \epsilon_{cab} \omega_b \left(1 - \cos\Omega\right) - \delta_{ca} \Omega \sin\Omega \right] \\
\Omega^2 &\equiv \omega_1^2 + \omega_2^2 + \omega_3^2 \,,
\end{aligned}
$$

where $\bar{\alpha}$ stands for the variables ρ, z, τ, and the indices a, b, c run over the values 1, 2, 3 (as always, there is the summation convention of repeated indices).

The transformations of the Higgs field coefficient functions H_j $(j = 0, \ldots, 3)$ are the following :

$$
\begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix} \rightarrow R_{\mathrm{H}} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix},
\tag{B.2}
$$

with the $SO(4)$ matrix

$$
R_{\mathrm{H}} = \cos\left(\frac{\Omega}{2}\right) \mathbb{1}_4 + \sin\left(\frac{\Omega}{2}\right) \frac{\vec{\omega} \cdot \vec{S}}{\Omega},
$$

defined in terms of the 4-dimensional identity matrix $\mathbb{1}_4$ and

$$S_1 \equiv \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \qquad S_2 \equiv \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \qquad S_3 \equiv \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.$$

Appendix C

Here, we give the ansatz actiondensities a_{WKIN}, a_{HKIN}, a_{HPOT} and Pontryagin density q_{P} in a compact, transparent notation.

First, define the 3-dimensional coordinates

$$(y_1, y_2, y_3) \equiv (\rho, z, \tau)\,. \tag{C.1}$$

Second, introduce the following "isovectors" given in terms of the coefficient functions C_i of the ansatz (8) :

$$\vec{k}_1 \equiv \begin{pmatrix} C_1 \\ C_2 \\ C_6 \end{pmatrix} \qquad \vec{k}_2 \equiv \begin{pmatrix} C_7 \\ -C_8 \\ C_9 \end{pmatrix} \qquad \vec{k}_3 \equiv \begin{pmatrix} C_{10} \\ -C_{11} \\ C_{12} \end{pmatrix} \qquad \vec{k}_4 \equiv \begin{pmatrix} C_3 \\ C_4 \\ 1-C_5 \end{pmatrix}, \tag{C.2}$$

which are motivated by the transformation properties (B.1) found in Appendix B. Third, define the "field strengths" and "covariant derivatives" as

$$\vec{K}_{\alpha\beta} \equiv \frac{\partial}{\partial y_\alpha}\left(\frac{\vec{k}_\beta}{y_\beta}\right) - \frac{\partial}{\partial y_\beta}\left(\frac{\vec{k}_\alpha}{y_\alpha}\right) + \frac{\vec{k}_\alpha}{y_\alpha} \times \frac{\vec{k}_\beta}{y_\beta}\,, \tag{C.3}$$

$$D_\alpha \vec{k}_4 \equiv \frac{\partial}{\partial y_\alpha}\vec{k}_4 + \frac{\vec{k}_\alpha}{y_\alpha} \times \vec{k}_4\,, \tag{C.4}$$

where the indices α, β take the values 1, 2, 3. With these definitions the Yang-Mills actiondensity (33) and Pontryagin density (37) become simply

$$a_{\mathrm{WKIN}} = \frac{1}{2g^2}\left\{ \frac{1}{2}\sum_{\alpha,\beta=1}^{3} \left|\vec{K}_{\alpha\beta}\right|^2 + \sum_{\alpha=1}^{3}\left|\frac{1}{\rho}\,D_\alpha\,\vec{k}_4\right|^2 \right\} \tag{C.5}$$

$$q_{\mathrm{P}} = \frac{1}{2\rho}\sum_{\alpha,\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\,\frac{\partial}{\partial y_\gamma}\left(\vec{k}_4 \cdot \vec{K}_{\alpha\beta}\right). \tag{C.6}$$

For the Higgs actiondensity it turns out to be useful to introduce "4-vectors", together with an implicit euclidean metric of positive signature. First, define the following 4-vector in terms of the coefficient functions H_j of the ansatz (8) :

$$H^\mu \equiv \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix}. \tag{C.7}$$

Second, introduce the "covariant derivatives"

$$\mathcal{D}_\alpha \equiv \frac{\partial}{\partial y_\alpha}\,\mathbb{1}_4 + \frac{\vec{k}_\alpha}{y_\alpha}\cdot\frac{\vec{S}}{2}\;, \tag{C.8}$$

with $\alpha = 1,\, 2,\, 3$, and the 4×4 matrices

$$S_1 \equiv \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \qquad S_2 \equiv \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \qquad S_3 \equiv \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \tag{C.9}$$

which appeared already in the transformations (B.2) of Appendix B. Third, define the matrix

$$K_4 \equiv \vec{k}_4\cdot\vec{S} + T_3\,, \tag{C.10}$$

where

$$T_3 \equiv \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \tag{C.11}$$

commutes with all S_a. With these definitions the Higgs actiondensities (34) and (35) become

$$a_{\text{HKIN}} = \frac{v^2}{2}\left\{ \sum_{\alpha=1}^{3} (\,\mathcal{D}_\alpha^{\mu\nu}\, H^\nu\,)^2 + \left(\frac{1}{2\rho}\,K_4^{\mu\nu} H^\nu\right)^2 \right\} \tag{C.12}$$

$$a_{\text{HPOT}} = \lambda\,\frac{v^4}{4}\,(\,H^\mu H^\mu - 1\,)^2\;, \tag{C.13}$$

where the last term in a_{HKIN} mixes the two types of "scalars" ($\vec{k}_4$ and H^μ) of the effective $SO(3)$ Yang-Mills-Higgs theory found.

References

[1] F. Klinkhamer, Nucl. Phys. **B407** (1993) 88

[2] A. Belavin, A. Polyakov, A. Schwartz and Yu. Tyupkin, Phys. Lett. **59B** (1975) 85

[3] M. Atiyah, V. Drinfeld, N. Hitchin and Yu. Manin, Phys. Lett. **65A** (1978) 185

[4] F. Klinkhamer, Nucl. Phys. **B410** (1993) 343

[5] L. Lipatov, Sov. Phys. JETP **45** (1977) 216

[6] I. Affleck, Nucl. Phys. **B191** (1981) 429

[7] R. Palais, Comm. Math. Phys. **69** (1979) 19

[8] E. Witten, Phys. Lett. **117 B** (1982) 324

[9] M. Veltman, *Diagrammatica – The Path to Feynman Rules*, Cambridge UP (1994)