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A global anomaly from the Z-string

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Abstract

The response of isodoublet fermions to classical backgrounds of essentially 2-dimensional boson fields in $SU(2)$ Yang-Mills-Higgs theory is investigated. In particular, the spectral flow of Dirac eigenvalues is calculated for a non-contractible sphere of configurations passing through the vacuum and the Z-string (the embedded vortex solution). Also, a non-vanishing Berry phase is established for adiabatic transport "around" the Z-string. These results imply the existence of a new type of global (non-perturbative) gauge anomaly in $SU(2)$ Yang-Mills-Higgs quantum field theory with a single doublet of left-handed fermions. Possible extensions to other chiral gauge field theories are also discussed.

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1 Introduction

A special class of static classical field configurations in 3+1 dimensional Yang-Mills-Higgs theory consists of those configurations that are constant in a fixed spatial direction. The configuration space of these essentially 2-dimensional fields has non-trivial structure. For $SU(2)$ Yang-Mills-Higgs theory in particular, a non-contractible sphere has been constructed [1] with the classical vacuum at the bottom and the embedded Nielsen-Olesen vortex solution or Z-string [2, 3] at the top. In this paper we calculate for isodoublet fermions the spectral flow of the Dirac Hamiltonian when the bosonic background fields range over this non-contractible sphere. Also, we show that adiabatic transport of the first-quantized fermion states along a curve of these bosonic background fields close to and around the Z-string gives rise to a non-vanishing Berry phase [4], with the value π.

This curve close around the Z-string can be pulled down into the vacuum. The result is a non-contractible loop of vacuum configurations of the classical boson fields. The second-quantized fermionic vacuum then exhibits a Möbius bundle structure over this particular gauge orbit. This corresponds to a global (non-perturbative) gauge anomaly of the chiral gauge field theory, which is analogous to, but not the same as, the Witten anomaly [5]. The reasoning behind these last steps has been reviewed in [6], to which the reader is referred for further details. The main goal in this paper is to establish, at the first-quantized level, the spectral flow and the Berry phase. This is done, in detail, for chiral $SU(2)$ Yang-Mills-Higgs theory with a single doublet of left-handed fermions. The obvious question how this extends to other chiral gauge theories will be discussed briefly at the end of this article. At this point, we should mention that the results of the present paper are discussed in a $3+1$ dimensional context, but they may also be relevant to lower dimensional theories.

The outline of this paper is as follows. In Section 2, the non-contractible sphere of classical $SU(2)$ gauge and Higgs fields is presented. Actually, three related sets of configurations are constructed, with the topology of a balloon, a sphere and a disc. In Section 3, the spectral flow of Dirac eigenvalues over the "balloon" is calculated numerically. In Section 4, a Berry phase factor -1 is established for non-trivial loops on the "balloon" and on the "sphere". In Section 5, this Berry phase factor is pulled down over the "disc" to a non-contractible loop in the vacuum. The results from Sections 3 to 5 are valid at the first-quantized level. In Section 6, finally, the resulting global gauge anomaly of the chiral $SU(2)$ Yang-Mills-Higgs quantum field theory and possible extensions to other chiral gauge field theories are discussed. This last Section is reasonably self-contained. Four Appendices give some details of the calculation, alternative derivations and miscellaneous results. Natural units ($\hbar = c = 1$) are used throughout.

2 Classical background fields

As mentioned in the introduction, our main focus is on $SU(2)$ Yang-Mills-Higgs theory, which corresponds to the electroweak standard model in the limit of vanishing weak mixing angle θ_w. The usual Minkowski space-time is modified to have for the spatial coordinate x^3 a very large, but finite, range ℓ_3. The boson fields of the theory are the $SU(2)$ Gauge field $W_\mu(x)$ and the complex Higgs doublet field $\Phi(x)$. The energy density for static (time-independent) classical boson fields, in the temporal gauge $W_0 = 0$, is given by

$$e_{\mathrm{b}} = \frac{1}{4\,g^2}\,\left(W^a_{mn}\right)^2 + |D_m\Phi|^2 + \lambda\left(|\Phi|^2 - \frac{v^2}{2}\right)^2 , \tag{1}$$

with the covariant derivative and field strength

$$\begin{aligned} D_m\Phi &\equiv (\partial_m + W_m)\,\Phi \equiv \left(\partial_m + W^a_m\,\frac{\tau_a}{2\,i}\right)\Phi\,, \\ W_{mn} &\equiv \partial_m W_n - \partial_n W_m + [W_m, W_n] \equiv W^a_{mn}\,\frac{\tau_a}{2\,i}\,, \end{aligned}$$

where the indices m, n and a run over the values 1, 2, 3, and τ_a are the Pauli matrices

$$\tau_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The semiclassical masses of the three degenerate W vector bosons[1] and the single Higgs scalar boson are $M_W = \frac{1}{2}\,g\,v$ and $M_H = \sqrt{2\lambda}\,v$, respectively.

In this article we consider static field configurations that are constant in the x^3 direction and have the corresponding component of the gauge field vanishing. Effectively, one has a 2-dimensional theory with energy density (1), where the spatial indices take only the values $m = 1, 2$. These two spatial dimensions can also be described by the cylindrical coordinates ρ and φ, defined in terms of the cartesian coordinates by

$$(x^1,\, x^2,\, x^3) \equiv (\rho\cos\varphi,\, \rho\sin\varphi,\, z)\,.$$

Furthermore, we consider only field configurations with *finite* string tension (energy per unit of length in the x^3 direction). Of course, fields with finite, non-zero string tension σ have very large total energy $E_{\mathrm{b}} = \sigma\,\ell_3$. The total energy E_{b} of these essentially 2-dimensional fields can be zero only if the string tension σ vanishes exactly.

A two parameter family of such SU(2) gauge and Higgs fields has been given [1] that represents a non-contractible sphere (NCS) in configuration space. The field configurations

[1]The Z_μ field in the electroweak context for $\theta_w = 0$ is simply defined as the $a = 3$ component of the $SU(2)$ gauge fields W^a_μ.

$W \equiv \sum_{m=1,2} W_m \mathrm{d}x^m$ and Φ are defined as follows

$$\begin{aligned} W &= -f\, \mathrm{d}U\, U^{-1}\,, \\ \Phi &= \frac{v}{\sqrt{2}}\, h\, U \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{2}$$

with the $SU(2)$ valued functions

$$\begin{aligned} U(\mu,\nu,\varphi) &= \Omega\, M\,, \\ M(\mu,\nu,\varphi) &= \begin{pmatrix} \sin\mu \\ \cos\mu\, \sin\nu \\ \cos\mu\, \cos\nu\, \sin\varphi \\ \cos\mu\, \cos\nu\, \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} -i\tau_1 \\ -i\tau_2 \\ -i\tau_3 \\ \mathbb{1}_2 \end{pmatrix}, \\ \Omega(\mu,\nu) &= M(\mu,\nu,0)^{-1} = \sin\mu\; i\tau_1 + \cos\mu\, \sin\nu\; i\tau_2 + \cos\mu\, \cos\nu\; \mathbb{1}_2\,, \end{aligned} \tag{3}$$

where $\mu,\nu \in [-\pi/2, \pi/2]$ parametrize the sphere and $\varphi \in [0, 2\pi]$ is the azimuthal coordinate. The map $U(\mu,\nu,\varphi)$ is topologically non-trivial ($S_2 \times S_1 \to S_3$, with winding number $n = 1$). The field configurations (2) at the top of the sphere ($\mu = \nu = 0$) correspond to the Z-string, which is the Nielsen-Olesen vortex [2] of the $U(1)$ Abelian Higgs model embedded [3] in the $SU(2)$ Yang-Mills-Higgs theory. The profile functions $f = f(\rho)$ and $h = h(\rho)$ solve the reduced field equations (for the ansatz at $\mu = \nu = 0$) with boundary conditions

$$f(0) = h(0) = 0\ , \quad f(\infty) = h(\infty) = 1\ . \tag{4}$$

These conditions guarantee that the fields (2) are regular at the origin and pure gauge at infinity.

The W fields vanish identically at the bottom of the sphere ($|\mu| = \pi/2$ or $|\nu| = \pi/2$), but the Higgs field is still different from its vacuum value. This Higgs configuration can be connected to the vacuum V by an additional line segment on which the function $h(\rho)$ is interpolated to the constant 1. Taking the total range for μ and ν to be $[-\pi, \pi]$, we define for $[\mu\nu] \equiv \max(|\mu|, |\nu|) \geq \pi/2$

$$\begin{aligned} W &= 0\,, \\ \Phi &= \frac{v}{\sqrt{2}} \left(1 - (1-h)\, \sin[\mu\nu] \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{5}$$

The family of field configurations (2)–(5) has the topology of a balloon (Fig. 1a), and, with a slight abuse of terminology, we call this whole set of configurations the Z-NCS$'$ (the prime is to remind us of the balloon string).

The fermion calculations of the next Section will be performed, for purely technical reasons, on the background fields (2, 3) with the matrix Ω eliminated by a global gauge

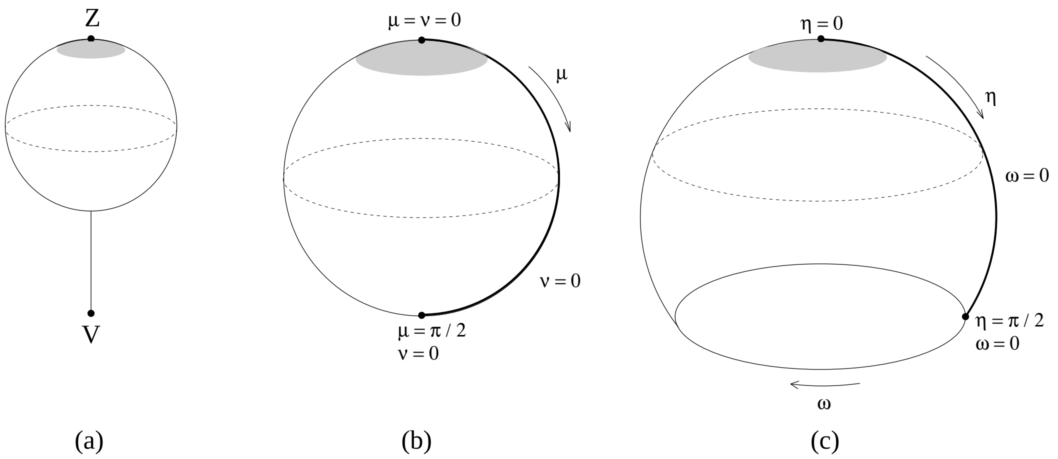

Figure 1: Sketch of (a) the Z-NCS$'$, (b) the Z-NCS and (c) the Z-NCD.

transformation. The "balloon" then has a hole, because the rim of the square $|\mu|, |\nu| \leq \pi/2$ is no longer mapped into a single configuration. Making an appropriate global gauge transformation of the fields (5), one can say that the balloon (Fig. 1a) will be temporarily replaced by a "flask".

The energy functional over the Z-NCS$'$ fields (2, 3) contains, as shown in [1], the factor $\cos^2\mu\cos^2\nu$ with a positive coefficient, so that the manifest maximum energy configuration corresponds to the Z-string at $\mu = \nu = 0$. This was the reason to introduce separately the "string" of configurations (5) that ties the "balloon" to the vacuum. A more unified set of configurations, for $\mu, \nu \in [-\pi/2, \pi/2]$, is the following:

$$\begin{aligned} W &= -f\,\mathrm{d}U\,U^{-1}\,, \\ \Phi &= \frac{v}{\sqrt{2}}\,\Omega\,\left[\,h\,M\,+\,(1-h)\,\left(\sin\mu\,(-i\tau_1)\,+\,\cos\mu\,\sin\nu\,(-i\tau_2)\right)\right]\begin{pmatrix}0\\1\end{pmatrix}, \end{aligned} \tag{6}$$

with U, M and Ω as given in (3). The family of field configurations (6) has the topology of a sphere (Fig. 1b), with the Z-string solution at $\mu = \nu = 0$ and the vacuum V at $[\mu\nu] = \pi/2$, and we call this set of configurations the Z-NCS. Clearly, the fields of the Z-NCS$'$ and the Z-NCS can be interpolated continuously. Remark that in both cases the local and global $SU(2)$ gauge freedom has been eliminated by the gauge fixing conditions

$$W_\rho(\rho,\varphi) \;=\; 0\,,$$

$$\Phi(\rho=\infty,\varphi=0) \;=\; \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} . \tag{7}$$

Finally, we introduce a set of configurations with the topology of a disc, parametrized by a radial variable $\eta \in [0,\pi/2]$ and an angular variable $\omega \in [0,2\pi]$. The Z-string sits at the middle of the disc ($\eta=0$), whereas the rim ($\eta=\pi/2$) corresponds to a non-contractible loop of vacuum configurations parametrized by ω (Fig. 1c). This set of configurations is called the Z non-contractible disc (Z-NCD), even though, properly speaking, only the vacuum loop is non-contractible. Specifically, the field configurations are the following:

$$\begin{aligned} W &= -\sin^2\eta\, \mathrm{d}U_V\, U_V^{-1} \,-\, f\cos^2\eta\, \mathrm{d}U_Z\, U_Z^{-1} \,-\, f\sin\eta\cos\eta\, \left(\mathrm{d}U_V\, U_Z^{-1} \,+\, \mathrm{d}U_Z\, U_V^{-1}\right) , \\ \Phi &= \frac{v}{\sqrt{2}}\left[\sin\eta\, U_V \,+\, h\cos\eta\, U_Z\right]\begin{pmatrix}0\\1\end{pmatrix} , \end{aligned} \tag{8}$$

with the SU(2) matrices

$$\begin{aligned} U_V(\eta,\omega,\rho,\varphi) &= \Omega_D\, M_V , \\ U_Z(\eta,\omega,\varphi) &= \Omega_D\, M_Z , \\ \Omega_D(\eta,\omega) &= \sin\eta\,(\cos\omega\, i\tau_1 - \sin\omega\, i\tau_2) + \cos\eta\, \mathbb{1}_2 , \\ M_Z(\varphi) &= \sin\varphi\,(-i\tau_3) + \cos\varphi\, \mathbb{1}_2 , \\ M_V(\omega,\rho,\varphi) &= \begin{pmatrix} \cos\omega + \sin^2\frac{\omega}{2}\,(1+\cos\bar\theta) \\ -\frac{1}{2}\sin\omega\,(1-\cos\bar\theta) \\ +\sin\frac{\omega}{2}\,\sin\bar\theta\cos\varphi \\ -\sin\frac{\omega}{2}\,\sin\bar\theta\sin\varphi \end{pmatrix} \cdot \begin{pmatrix} -i\tau_1 \\ -i\tau_2 \\ -i\tau_3 \\ \mathbb{1}_2 \end{pmatrix} , \end{aligned} \tag{9}$$

where $\varphi \in [0,2\pi]$ is the usual azimuthal angle and $\bar\theta \in [0,\pi]$ a compactified radial coordinate for which we take $\bar\theta = \pi\rho^2/(\rho^2+\pi)$. The map $U_V(\pi/2,\omega,\rho,\varphi)$ is topologically non-trivial (effectively $S_1 \times S_2 \to S_3$, with winding number $n=1$). Note that the fields (8) are manifestly regular at $\rho=0$ because of the boundary conditions (4) and the factor $\sin\bar\theta \sim \rho^2$ multiplying $\cos\varphi$ and $\sin\varphi$ in M_V.

The fields of the $\omega=0$, $\eta \in [0,\pi/2]$ ray of the Z-NCD are identical to those of the $\nu=0$, $\mu \in [0,\pi/2]$ meridian of the Z-NCS, see Fig. 1bc . Loosely speaking, the rest of the Z-NCD wraps around the Z-NCS. In fact, the configurations on the caps close to the Z-string can be interpolated continuously between the Z-NCS and the Z-NCD, if one identifies $\mu=\eta\cos\omega$ and $\nu=-\eta\sin\omega$ for $\eta \sim 0$. Note also that the circles of constant η on the Z-NCD (8) do not correspond to gauge orbits, except for the case of $\eta=\pi/2$ (this can be seen most easily for the Higgs field, with Ω_D eliminated and ω infinitesimal). For the Z-NCS (6) the gauge is fixed completely and there are no gauge orbits at all.

Having presented these three related sets of bosonic field configurations, we will first investigate the response of the fermions for the simplest case, the Z-NCS$'$.

3 Spectral flow in the Z-NCS$'$ background

Consider a single $SU(2)$ doublet of left-handed fermions $(u_L, d_L)^{\mathrm{T}}$ and two singlets of right-handed fermions (u_R and d_R) in the background of the Z-NCS$'$ boson fields. The fermion fields are combined into one doublet

$$\Psi(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix},$$

where u and d are the complete, 4-component Dirac fields. The fermions are coupled to the Higgs by a Yukawa term in the Lagrangian (strictly speaking, the Lagrangian density)

$$\mathcal{L} = -i\,\bar\Psi\,\gamma^\mu D_\mu \Psi + g_{\mathrm{Y}}\left(\bar\Psi_L\,\Phi_M\,\Psi_R + \bar\Psi_R\,\Phi_M^\dagger\,\Psi_L\right)\,, \tag{10}$$

where the covariant derivative is defined as $D_\mu \equiv \partial_\mu + W_\mu P_L$, with the projection operator $P_L \equiv \frac{1}{2}(\mathbb{1} - \gamma_5)$, and Φ_M is the Higgs field written as a matrix

$$\Phi_M(x) = \begin{pmatrix} \Phi_2^*(x) & \Phi_1(x) \\ -\Phi_1^*(x) & \Phi_2(x) \end{pmatrix}.$$

In our case, Φ_M is obtained by simply omitting the isospinor $(0,1)^{\mathrm{T}}$ in (2). The two fermions have equal mass $m = g_{\mathrm{Y}}\,v/\sqrt{2}$, due to the $SU(2)_L$ gauge and $SU(2)_R$ custodial symmetry transformations

$$\begin{aligned}
\Psi_L(x) &\to \Lambda_L(x)\,\Psi_L(x)\,, \quad \Psi_R(x) \to \Lambda_R\,\Psi_R(x)\,, \\
W_\mu(x) &\to \Lambda_L(x)\,\left(W_\mu(x) + \partial_\mu\right)\,\Lambda_L^{-1}(x)\,, \\
\Phi_M(x) &\to \Lambda_L(x)\,\Phi_M(x)\,\Lambda_R^{-1}\,,
\end{aligned} \tag{11}$$

with $\Lambda_L(x),\,\Lambda_R \in SU(2)$. As far as the fermions are concerned, the Minkowski metric $\eta^{\mu\nu}$ is taken to have signature $(+\,-\,-\,-)$ and $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}$. The Dirac matrices γ^0 and $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ are taken to be hermitian and the γ^m antihermitian.

The field equation derived from this Lagrangian is

$$i\,\partial_t \Psi = H\,\Psi, \tag{12}$$

with the hermitian Dirac Hamiltonian

$$\begin{aligned}
H \;=\; & -i\,\gamma^0\left(\gamma^\rho\,\partial_\rho + \rho^{-1}\,\gamma^\varphi\,\partial_\varphi + (\gamma^\rho\,W_\rho + \gamma^\varphi\,W_\varphi)P_L + \gamma^3\,\partial_z\right) \\
& + g_{\mathrm{Y}}\,\gamma^0\left(\Phi_M P_R + \Phi_M^\dagger P_L\right),
\end{aligned} \tag{13}$$

for gauge fields W_0 and W_3 vanishing and gamma matrices in cylindrical coordinates

$$\gamma^\rho \equiv \gamma^1\cos\varphi + \gamma^2\sin\varphi\,, \quad \gamma^\varphi \equiv -\gamma^1\sin\varphi + \gamma^2\cos\varphi\,.$$

The Dirac Hamiltonian (13) is real in the sense that[2]

$$H^{*} = Q\,H\,Q^{-1}\,, \tag{14}$$

with, for example, in the Majorana representation (all γ^{μ} imaginary)

$$Q = \gamma^{0}\,\gamma_5\,\tau_2\,. \tag{15}$$

Correspondingly, a vector Ψ is called Q-real if $\Psi^{*} = Q\,\Psi$. It is not difficult to prove that non-degenerate eigenvectors of a Q-real hermitian operator (for example, H) can be chosen Q-real, with an unique phase factor up to a sign. This fact will be important for the Berry phase factor later on.

For static background fields, the stationary solutions of (12),

$$\Psi(\mathbf{x},t) = \Psi(\mathbf{x})\,\exp(-i\,E\,t)\,, \tag{16}$$

are given in terms of the solutions to the eigenvalue equation

$$H\,\Psi(\mathbf{x}) = E\,\Psi(\mathbf{x})\,. \tag{17}$$

Remark that the Dirac Hamiltonian (13) depends on the parameters μ and ν through the classical background fields W and Φ of the Z-NCS$'$. These background fields can be simplified significantly if the matrix Ω in the definition (2, 3) is eliminated by a global gauge transformation, which does not affect the eigenvalues of H. As suggested in the previous Section, the resulting background may be called the Z-flask.

Eigenstates of the transformed Hamiltonian are obtained by using an ansatz for the fermion fields that takes advantage of the symmetries of the background fields. The resulting fermion fields are z-independent and possess a continuous rotation symmetry generated by

$$K_3 \equiv -i\,\partial_{\varphi} + \frac{1}{2}\,\Sigma_3 + \frac{1}{2}\,\tau_3\,(P_L - P_R)\,, \tag{18}$$

together with a certain discrete symmetry. Here, and in the following, the Pauli matrices τ_a act on isospinors, whereas the $\Sigma_a \equiv \frac{i}{2}\,\epsilon_{abc}\,\gamma_b\,\gamma_c$ act on Dirac spinors. The discrete symmetry operator R_1 consists, for $\nu = 0$, of a rotation over π around the x^1-axis and a matching isospin transformation with $i\tau_1$. The fermion ansatz and the resulting differential equations are given in the Appendix A.

Setting $\nu = 0$ first, the eigenfunctions and eigenvalues of H are obtained numerically for the case of equal masses $m = M_H = M_W$. The phenomenon of spectral flow is observed,

[2] Quantum mechanically, a linear operator O has a complex conjugate operator $O^{\star}$ defined by $O^{\star}\psi^{\star} = (O\psi)^{\star}$, for an arbitrary wave function ψ. This operator $O^{\star}$ is in general different from the adjoint $O^{\dagger}$, defined by $(\psi_2, O^{\dagger}\psi_1) = (O\psi_2, \psi_1)$. An hermitian operator has $O^{\dagger} = O$.

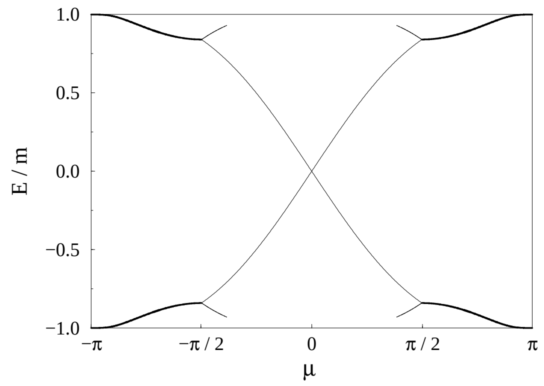

Figure 2: Energy eigenvalues for $\nu = 0$.

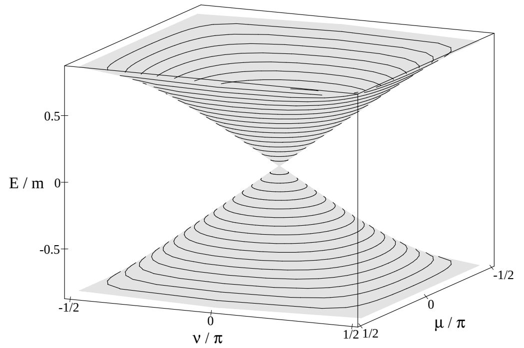

Figure 3: Energy eigenvalues on the Z-NCS$'$.

with a single pair of eigenvalues crossing through zero[3] at $\mu = 0$, which corresponds to the Z-string (Fig. 2). The relevant eigenstates have K_3 eigenvalue 0 and R_1 eigenvalues ± 1. By considering the continuity of the wave functions, it is shown in Appendix A that the levels really cross, instead of being tangent to one other. The eigenvalues for $|\mu| \geq \pi/2$ are doubly degenerate (R_1 eigenvalues ± 1).

The energy eigenvalues for $\nu \neq 0$ are obtained by taking advantage of the symmetry of the Z-NCS$'$ (or the Z-flask, for that matter). In fact, the Dirac Hamiltonians (13) on the special contour $\cos\mu \cos\nu = c$, with $c \in [0, 1]$ a constant, are related by a unitary transformation, so that the eigenvalues are the same. This unitary transformation operator N will be given in the next Section. Plotted against μ and ν, the eigenvalues display cone-like spectral flow (Fig. 3) and are everywhere non-degenerate, except for $\mu = \nu = 0$ and the rim of the μ, ν square shown. By continuity, the Z-NCS and the Z-NCD have essentially the same cone-like spectral flow.

4 Berry phase on the Z-NCS$'$ and Z-NCS

We now ask what happens to a real eigenvector of the Dirac Hamiltonian H under adiabatic transport along a loop encircling the Z-string. The transported state will differ from the original state by a phase factor, which consists of the usual dynamical phase and, potentially, the so-called Berry phase. Let the loop be parametrized by $\alpha \in [0, 2\pi]$ and choose differentiably for each α a normalized eigenstate[4] $|\Theta(\alpha)\rangle$ of $H(\alpha)$ with non-degenerate eigenvalue $E(\alpha)$. The implicit condition $|\Theta(0)\rangle = |\Theta(2\pi)\rangle$ may require com-

[3] These fermion zero modes were already discovered in [7, 8].

[4] For clarity, bra-ket notation is used in this Section.

plex phases in the definition of the eigenstates $|\Theta(\alpha)\rangle$, see below. If $|\Psi\rangle_{\text{initial}} = |\Theta(0)\rangle$ is transported along this loop adiabatically (setting $\alpha = 2\pi t/T$ and taking the total time T large), the result is

$$|\Psi\rangle_{\text{final}} = e^{-i\int_0^T \mathrm{d}t\, E(t)}\; e^{i\gamma}\, |\Psi\rangle_{\text{initial}}\,, \tag{19}$$

with the Berry phase [4]

$$\gamma = i\int_0^{2\pi} \mathrm{d}\alpha\, \langle\Theta(\alpha)|\, \frac{\partial}{\partial\alpha}\, |\Theta(\alpha)\rangle\,. \tag{20}$$

The Berry phase can be calculated directly for the special loops on the Z-NCS$'$ or Z-NCS given by $\cos\mu\cos\nu = c$, with $0 < c < 1$. In this case one has for the Dirac Hamiltonian

$$H(\mu_0,\alpha) \;=\; N(\alpha)\, H(\mu_0,0)\, N(\alpha)^{-1}\,, \tag{21}$$

with

$$N(\alpha) = \exp\left(i\,\frac{\alpha}{2}\,\tau_3\right)\,, \tag{22}$$

where $\alpha \equiv 2\pi t/T \in [0,2\pi]$ parametrizes the curve $\cos\mu\cos\nu = c$ and $H(\mu_0,0)$ is the Dirac Hamiltonian (13) corresponding to the point ($\mu = \mu_0 \equiv \arccos c$, $\nu = 0$) on the curve. As mentioned before, the eigenvalues of H are constant along these curves (Fig. 3). The general solution of the Schrödinger-like equation (12, 21) is

$$|\Psi(\mu_0,t)\rangle = e^{-i\,\tilde{E}(\mu_0)\,t}\, N(2\pi t/T)\, |\Psi(\mu_0,0)\rangle\,, \tag{23}$$

with

$$\left[H(\mu_0,0) + (\pi/T)\,\tau_3\right]\, |\Psi(\mu_0,0)\rangle = \tilde{E}(\mu_0)\, |\Psi(\mu_0,0)\rangle\,. \tag{24}$$

The normalized eigenvectors (24) approach in the adiabatic limit ($T \to \infty$) those of the initial Hamiltonian

$$H(\mu_0,0)\, |\Psi(\mu_0,0)\rangle = E(\mu_0)\, |\Psi(\mu_0,0)\rangle\,, \tag{25}$$

so that

$$\tilde{E}(\mu_0) = E(\mu_0) + (\pi/T)\, \langle\Psi(\mu_0,0)|\, \tau_3\, |\Psi(\mu_0,0)\rangle\,. \tag{26}$$

The τ_3 expectation value in (26) vanishes identically, because the state $|\Psi(\mu_0,0)\rangle$ considered has R_1 eigenvalues ±1 and the anticommutator $\{R_1,\tau_3\}$ vanishes, see Appendix A for further details. The solution (23) for $t = T$, with $N(2\pi) = -N(0) = -\mathbb{1}_2$, gives then the Berry phase

$$\gamma = \pi \tag{27}$$

for adiabatic transport along the curve $\cos\mu\cos\nu = c$.

This non-vanishing Berry phase (27) is essentially due to the fact that the transformation matrix $N(\alpha)$ of the Hamiltonian (21) runs from one element in the center of the group

$SU(2)$ to the other. Of course, this is only possible for fermions in $SU(2)$ representations of half-integer isospin ($I = 1/2$ for the doublet here). Note, finally, that the Berry formula (20), with the differentiable (complex) choice of eigenstates

$$|\Theta(\alpha)\rangle = e^{-i\,\alpha/2}\, N(\alpha)\, |\Psi(\mu_0, 0)\rangle\,, \tag{28}$$

gives the same result $\gamma = \pi$. The advantage of the general solution (23) is that it also applies to the case of degenerate eigenvalues $E(\alpha)$.

The result (27) for the Berry phase over these special loops on the non-contractible sphere can be extended to arbitrary loops by the following argument. As mentioned in Section 3, the Dirac Hamiltonian obeys the reality condition (14). Moreover, the eigenvalues under consideration are non-degenerate, see Fig. 3. Under these conditions, the Berry phase factor $e^{i\gamma}$ is necessarily ± 1, as follows from the remarks below (15). Since these allowed values are isolated points in $\mathbb{R}$, the Berry phase factor (which is a continuous functional of the loop) cannot change if the loop is continuously deformed. Thus, every loop on the Z-NCS (or Z-NCS$'$) that winds around the Z-string exactly once, and does not touch the degeneracy points ($\mu = \nu = 0$ or $[\mu\nu] = \pi/2$), leads to the same Berry phase factor -1. More generally, the Berry phase $\gamma_{\rm C}$ for any closed curve C on the Z-NCS with the degeneracy points omitted, which corresponds to a cylinder topologically, is given by

$$e^{\,i\gamma_{\rm C}} = (-1)^{n_{\rm C}}\,, \tag{29}$$

with $n_{\rm C}$ the winding number of the curve C on the cylinder.

The true origin of the Berry phase factor -1 is the fermion degeneracy in the Z-string background. This is clarified in Appendix B by a different derivation of the Berry phase, in terms of a solid angle in parameter space.

5 Berry phase on the Z-NCD

In the previous Section we have established a non-vanishing Berry phase $\gamma = \pi$ for loops on the Z-NCS$'$ or Z-NCS (Fig. 1ab) that circumnavigate the Z-string once. For a loop close to the degeneracy point (i. e. the Z-string) we have calculated, in Appendix B, the Berry phase more generally, in terms of an abstract solid angle in parameter space. The Berry phase factor $e^{i\gamma} = -1$ for this small loop on the Z-NCS carries over to a small loop on the Z-NCD (Fig. 1c). The reason is that, on the one hand, the background fields can be mapped into each other continuously, and, on the other hand, the Dirac Hamiltonian is essentially real, so that the Berry phase factor $e^{i\gamma}$ stays discrete (± 1). See also the discussion in the paragraph leading up to (29). The implicit assumption here is that there are no further degeneracies, which is certainly the case for loops close enough to the Z-string.

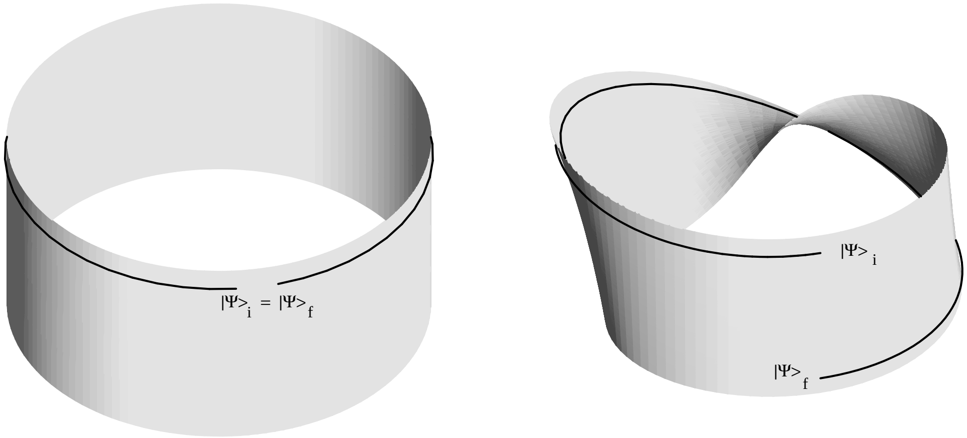

Figure 4: Cylinder and Möbius bundle over different gauge orbits, with the line representing a normalized real eigenstate of the Dirac Hamiltonian.

There is thus a Berry phase factor $e^{i\gamma} = -1$ for a small loop around the top of the Z-NCD. This small loop can be pulled towards the vacuum continuously ($\eta \to \pi/2$), with the discrete Berry phase factor remaining at the value -1. In this way we end up with a non-contractible loop of vacuum bosonic field configurations (8, 9), for $\eta = \pi/2$ and $\omega \in [0, 2\pi]$, which gives a non-trivial Berry phase factor -1. The real sub-bundle over this particular gauge orbit is *twisted*, so that instead of a trivial cylinder bundle there is, in fact, a Möbius bundle (Fig. 4). This non-trivial Berry phase factor of first-quantized fermion states over the vacuum gauge orbit is a necessary, but not sufficient, condition for the global anomaly of the purely left-handed gauge field theory to be discussed in the next Section. Note that for the numerical calculations of Section 3 the right-handed fermions were introduced in order to have a mass gap, but for the Berry phase calculation in this and the previous Section they can be left out from the start.[5] For completeness, we discuss in Appendix C the Berry phase on the non-contractible sphere with winding number $n = 2$, which leads to a gauge orbit with trivial bundle.

At this point, we like to mention two issues that deserve further study. The first issue is wether or not there is an index theorem responsible for the observed spectral flow of Dirac eigenvalues over the non-contractible sphere (Fig. 3). For the Witten anomaly [5] to be discussed in the following Section there is the so-called mod 2 Atiyah-Singer index theorem and in our case the standard Atiyah-Singer index theorem [9, 10] may be expected to play a role. In fact, the matrix M as given in (3) has the same structure as the corresponding matrix of the BPST instanton [11]. The second issue is the removal of the range limitation on the x^3 coordinate ($\ell_3 \to \infty$). A related point is the choice of boundary conditions on the fields at $x_3 = \pm\, \ell_3/2$ (implicitely we have taken free boundary conditions). All the results presented in this paper are formally independent of ℓ_3. In

[5] With massless fermions it may be necessary to have also a finite range for the x^1 and x^2 coordinates.

particular, the reality condition (14) on the Dirac Hamiltonian holds generally. Still, the existence and the nature of the ℓ_3 limit need to be established rigorously. Pending this proof, we keep the range ℓ_3 large, but finite.

6 Global gauge anomaly

One speaks of a gauge anomaly if there occurs, upon quantization of a chiral gauge field theory, an insurmountable obstacle to maintaining gauge invariance. Concretely, the quantization in the hamiltonian formulation may proceed in two steps. First, the fermions are quantized ("filling the Dirac sea") in the background of the classical boson fields. Second, the boson fields themselves are quantized. What may then happen is the following (quoting liberally from [6]): Quantizing the fermionic matter in the presence of background gauge fields, the resulting family of quantum theories in general realizes its classical gauge symmetry via a perfectly good ray representation. As far as the fermions are concerned there is nothing wrong with gauge symmetry. If, however, the phases in the ray representation are topologically unremovable, then they prevent us from implementing the symmetry in the fully quantized theory, and in particular from imposing the constraint of gauge invariance (Gauss' law) on the physical quantum states. We will now show that this is precisely what follows from the results of the previous Sections.

The central object for the anomaly discussion here is the gauge-covariant left-handed Dirac operator

$$\not{D}_L \equiv \gamma^\mu \left(\partial_\mu + W_\mu\right) P_L \,, \tag{30}$$

with W_μ antihermitian and in the Lie algebra of the gauge group.[6] Consider then $SU(2)$ Yang-Mills-Higgs theory with a single doublet of left-handed fermions (and no right-handed fermions) on a flat $3+1$ dimensional space-time manifold with a large, but finite, range ℓ_3 for the the spatial x^3 coordinate. In the previous Section we discovered that for a particular loop of gauge transformations of the classical bosonic vacuum fields the first-quantized fermion states acquired a topological phase factor, namely the Berry phase factor $e^{i\gamma} = -1$. More importantly, there is a *single* pair of fermionic levels crossing at $E = 0$ in the Z-string background (see Fig. 2 and Appendix A). For the second-quantized vacuum state this results in a Möbius bundle structure over the gauge orbit, which interferes with the definition of physical states, see Appendix D. Therefore, $SU(2)$ Yang-Mills-Higgs quantum field theory with a single doublet of massless left-handed fermions (Weyl spinors) is anomalous. Without exact gauge invariance, the theory is most likely inconsistent.

[6] The anomaly in the euclidean path integral formulation comes from the ambiguous determinant (product of eigenvalues) of $i\not{D}_L$. The operator $i\not{D}_L$ has, strictly speaking, no eigenvectors, as it maps left-handed fields to right-handed fields and *vice versa*.

Concretely, our loop of gauge transformations, given by (8, 9) for $\eta = \pi/2$ and loop-parameter $\omega \in [0, 2\pi]$, is independent of the spatial x^3 coordinate and the topologically non-trivial map is

$$S_1 \times S_2 \sim S_3 \longrightarrow SU(2) \sim S_3\,, \tag{31}$$

with winding number $n = 1$.[7] This may be compared to the Witten global $SU(2)$ anomaly in the hamiltonian formulation [5], where the loop of gauge transformations depends on all three spatial coordinates and the non-trivial map is

$$S_1 \times S_3 \sim S_4 \longrightarrow SU(2) \sim S_3\,, \tag{32}$$

which is the suspension of the Hopf map $S_3 \to S_2$. Both maps (for the same gauge group $G = SU(2)$) can indeed be non-contractible, since the respective homotopy groups are non-trivial $\pi_3(SU(2)) = \mathbb{Z}$ and $\pi_4(SU(2)) = \mathbb{Z}_2$, where $\mathbb{Z}$ denotes the group of integers and $\mathbb{Z}_2$ the integers modulo 2. Moreover, both non-contractible loops pick up a Berry phase factor -1, because they encircle a degeneracy point in configuration space corresponding to the Z-string [3] and the S* sphaleron [12], respectively. Thus, chiral SU(2) Yang-Mills-Higgs quantum field theory with an odd number of massless left-handed fermion doublets is ruled out on both counts.

Up till now we have considered chiral $SU(2)$ Yang-Mills-Higgs theory, but our results may also apply to theories without Higgs. It is clear that for massless fermions the Berry phase depends on the gauge fields only. As long as there remains an encircled degeneracy point (most likely, guaranteed by an index theorem), the bundle over the loop of vacuum gauge fields stays twisted, even in the absence of the Higgs fields. Hence, pure $SU(2)$ gauge field theory with an odd number of massless left-handed fermion doublets is also expected to suffer from the new global anomaly (in addition to the Witten anomaly, of course).

We can make two further generalizations. First, consider for the background fields of the non-contractible sphere (6) massless left-handed fermions in $SU(2)$ representations larger than the doublet representation (isospin $I = 1/2$). A straightforward, but tedious, calculation gives for a single irreducible representation of isospin I and dimension $n = 2\,I + 1$ the following number of pairs of fermionic levels crossing at $E = 0$:

$$N_{\text{pair}} = \begin{cases} n^2/\,4 & \text{if} \;\; n = 2\,l \\ (n^2 - 1)\,/\,4 & \text{if} \;\; n = 2\,l + 1\,, \end{cases} \tag{33}$$

with l a positive integer. There is then an odd number of pairs crossing for isospin values

$$I = \frac{4\,k + 1}{2}\,, \quad k = 0, 1, 2, \ldots\;. \tag{34}$$

[7]For the case of winding number $n = 2$ there are two pairs of fermionic levels crossing at $E = 0$ (see Appendix C) and the second-quantized vacuum bundle is trivial. In this Section only winding number $n = 1$ is considered (later on n will denote the dimension of the $SU(2)$ representation of the fermions).

Therefore, $SU(2)$ Yang-Mills(-Higgs) quantum field theory with massless left-handed fermions in a single irreducible representation of isospin (34) has a twisted vacuum bundle and the theory suffers from the new global anomaly.[8] Second, consider gauge groups G other than $SU(2)$ and massless left-handed fermions in corresponding representations. For *any* compact connected simple Lie group G the third homotopy group is again non-trivial

$$\pi_3(G) = \mathbb{Z}\,. \tag{35}$$

Furthermore, it is known that any continuous mapping of S_3 into G can be continuously deformed into a mapping into an $SU(2)$ subgroup [9]. Hence, all constructions in $SU(2)$ Yang-Mills theory carry over to theories with gauge group $G \supset SU(2)$ and the same twist factor -1 appears for each embedded fermion doublet (or other anomalous isospinor). If the gauge group G acts on left-handed fermion fields only, a further condition for the new global anomaly is then that the *total* number $N_{I=2k+1/2}$ of irreducible representations (34) of the $SU(2)$ subgroup considered be odd

$$(-1)^{N_{I=2k+1/2}} = -1\,. \tag{36}$$

Chiral gauge field theories fulfilling the conditions (35, 36) suffer thus from the same global anomaly (twisted bundle) due to the Z-string-like fermion degeneracy as the doublet $SU(2)$ Yang-Mills-Higgs theory.

Remarkably, the same isospinors (34) are singled out by the Witten anomaly. In fact, for $SU(2)$ generators T_a with algebra $[\,T_a\,,\,T_b\,] = \epsilon_{abc}\,T_c$ the Witten anomaly [5] requires $N_0 \equiv -2\,\mathrm{Tr}\,(T_3)^2 = n\,(n^2-1)/6$ to be an odd integer, corresponding to representations of dimension $n = 4\,k+2$ and isospin $I = (4\,k+1)/2$ for $k = 0, 1, 2, \ldots$.[9] However, the new global anomaly does rule out theories allowed in principle by Witten's global anomaly, because $\pi_4(G)$ is non-trivial only in certain cases, whereas (35) holds generally. A simple example is $SU(3)$ Yang-Mills(-Higgs) theory with an odd number of massless left-handed fermion triplets. This theory has no genuine Witten anomaly (the fourth homotopy group of $SU(3)$ is trivial), but does satisfy the conditions (35, 36) for the new global anomaly. Of course, chiral $SU(3)$ gauge theory with *any* number of left-handed fermion triplets also has the *local* (perturbative) Bardeen anomaly [14]. This brings us to the following question: which chiral Yang-Mills theories are safe from perturbative Bardeen anomalies (either intrinsically or by cancellations between the different representations), but do suffer from the new global anomaly? The group-theoretic results of [15] appear to rule

[8] It is important to verify this result by other methods, for example euclidean path integrals (paying attention to the relevant fermion zero modes).

[9] N_0 turns out to be equal to the number of fermion zero modes [10] in the 4-dimensional background of the BPST instanton [11]. Physically, this connection is made plausible by Goldstone's derivation [13] of the Witten $SU(2)$ anomaly. Note that $N_0(n)$ is in general different from $N_{\mathrm{pair}}(n)$ as given by (33), even though both find the same isospin values (34) through $(-1)^{N_0} = (-1)^{N_{\mathrm{pair}}} = -1$.

out all compact simple gauge groups G with $\pi_4(G) = 0$. This leaves essentially the $Sp(N)$ groups ($Sp(1) = SU(2)$), which are free from perturbative anomalies, but can have the new global anomaly (and the Witten anomaly) provided condition (36) holds.

To summarize, the main idea of this paper is to consider in 3+1 dimensional chiral Yang-Mills(-Higgs) quantum field theory in the hamiltonian formulation a non-contractible loop of gauge transformations with reduced dependence on the spatial coordinates. A non-trivial Berry phase factor -1 can only occur if the loop encircles a configuration with degenerate fermions (in this paper, the embedded Z-string). Depending on the fermion representations present, this may then result in a twisted bundle over the gauge orbit, which signals the presence of a new kind of global gauge anomaly in the theory.

One of us (FRK) acknowledges stimulating discussions with the participants of the 9$^{\text{th}}$ Workshop on Physics Beyond the Standard Model, Bad Honnef, March 3–6, 1997.

A Fermion ansatz and numerical solution

In this Appendix we give the details of the ansatz for isodoublet fermions in the Z-NCS′ background and present the numerical solution. Also, we show that the energy levels really cross at the top of the non-contractible sphere. The relevance of this will be explained at the end.

The classical background fields considered here are given by the $\nu = 0$ slice of Z-NCS′ (2, 3), with the matrix Ω eliminated by a global gauge transformation. Furthermore, we take in this Appendix the 2-dimensional version of the Dirac Hamiltonian (13), that is without the term $-i\,\gamma^0\,\gamma^3\,\partial_z$. The resulting Dirac Hamiltonian commutes with

$$K_3 \equiv L_3 + \frac{1}{2}\,\Sigma_3 + \frac{1}{2}\,\tau_3\,(P_L - P_R) \tag{A.1}$$

and

$$R_1 \equiv \exp\left[-i\pi (L_1 + \frac{1}{2}\,\Sigma_1 + \frac{1}{2}\,\tau_1)\right], \tag{A.2}$$

where L_k and Σ_k are the standard orbital and spin angular momentum operators $L_k \equiv -i\,\epsilon_{klm}\,x_l\,\partial_m$ and $\Sigma_k \equiv \frac{i}{2}\,\epsilon_{klm}\,\gamma_l\,\gamma_m$. The action of R_1 on the 2-dimensional coordinates is simply to change the azimuthal angle φ into $-\varphi$. Since K_3 and R_1 do not commute, it is in general not possible to find common eigenfunctions. However, common eigenfunctions do exist in the subspace of vanishing K_3 eigenvalue, since $\{K_3, R_1\} = 0$. These eigenfunctions are

$$\begin{aligned}\Psi_1(\rho,\varphi) \;\; = \;\; & i\,G_L\,(|L\uparrow d\rangle - |L\downarrow u\rangle) + F_L\,(e^{i\varphi}|L\downarrow d\rangle - e^{-i\varphi}|L\uparrow u\rangle) + \\ & G_R\,(|R\downarrow d\rangle + |R\uparrow u\rangle) + i\,F_R\,(e^{-i\varphi}|R\uparrow d\rangle + e^{i\varphi}|R\downarrow u\rangle)\,,\end{aligned}$$

$$
\begin{aligned}
\Psi_2(\rho,\varphi) &= i\,G_L\,(|L\uparrow d\rangle + |L\downarrow u\rangle) - F_L\,(e^{i\varphi}|L\downarrow d\rangle + e^{-i\varphi}|L\uparrow u\rangle) + \\
&\quad G_R\,(|R\downarrow d\rangle - |R\uparrow u\rangle) - i\,F_R\,(e^{-i\varphi}|R\uparrow d\rangle - e^{i\varphi}|R\downarrow u\rangle)\,, \qquad \text{(A.3)}
\end{aligned}
$$

where G_L, G_R, F_L and F_R are real functions of ρ (with suitable boundary conditions) and the kets stand for constant normalized eigenvectors of γ_5, Σ_3 and τ_3, in an obvious notation. The eigenfunctions have the following eigenvalues:

$$
\begin{aligned}
K_3\,\Psi_1 &= 0\,, \qquad & R_1\,\Psi_1 &= +\Psi_1\,, \\
K_3\,\Psi_2 &= 0\,, \qquad & R_1\,\Psi_2 &= -\Psi_2\,.
\end{aligned} \qquad \text{(A.4)}
$$

If Ψ_1 is inserted into the eigenvalue equation (17) of the Dirac Hamiltonian, the following equations are obtained:

$$
\begin{aligned}
\partial_\rho G_L &= -\frac{f}{\rho}\cos^2\mu\,G_L + \frac{f}{2\rho}\sin 2\mu\,F_L + m\,h\cos\mu\,G_R + m\,h\sin\mu\,F_R + E\,F_L\,, \\
\partial_\rho F_L &= -\frac{1}{\rho}\,F_L + \frac{f}{\rho}\cos^2\mu\,F_L + \frac{f}{2\rho}\sin 2\mu\,G_L + m\,h\cos\mu F_R - m\,h\sin\mu\,G_R - E\,G_L\,, \\
\partial_\rho F_R &= -\frac{1}{\rho}\,F_R + m\,h\cos\mu\,F_L + m\,h\sin\mu\,G_L + E\,G_R\,, \\
\partial_\rho G_R &= -m\,h\sin\mu\,F_L + m\,h\cos\mu\,G_L - E\,F_R\,, \qquad \text{(A.5)}
\end{aligned}
$$

with $f(\rho)$ and $h(\rho)$ the profile functions [2, 16] of the Z-string gauge and Higgs fields, respectively, and $m = g_Y\,v/\sqrt{2}$ the fermion mass in the Higgs vacuum. For $\mu = 0$ one recovers the Z-string fermion zero mode [7, 8], in terms of two functions G_L and G_R ($F_L = F_R = 0$).

It is not necessary to investigate the second ansatz Ψ_2 seperately. All solutions with R_1 eigenvalue -1 can be obtained from those with R_1 eigenvalue $+1$ by applying the operator $-\Sigma_3\gamma_5$, since $-\Sigma_3\gamma_5$ anticommutes with R_1 and H. Furthermore, it it sufficient to consider the case $\mu \geq 0$. If μ is replaced by $-\mu$ in (A.5), the change in the equations can be compensated by changing simultaneously the eigenvalue E to $-E$ and the functions F_L, F_R to $-F_L$, $-F_R$. This shows that simple sign changes can turn a solution for μ, with energy E, into a solution for $-\mu$, with energy $-E$.

For the numerical calculations the quantities ρ, m, E, F_i and G_i are rescaled so that they become dimensionless

$$
\tilde\rho \equiv \rho\,M_W\,, \quad \tilde m \equiv m\,M_W^{-1}\,, \quad \tilde E \equiv E\,M_W^{-1}\,, \quad \tilde F_i \equiv F_i\,M_W^{-3/2}\,, \quad \tilde G_i \equiv G_i\,M_W^{-3/2}\,. \qquad \text{(A.6)}
$$

The tildes are dropped in the following. The numerical results for the functions G_L, G_R, F_L and F_R are shown in Figs. A.1 and A.2, for the case of equal masses $m = M_H = 1$, and the corresponding energy eigenvalues in Fig. 2 of the main text. The functions are

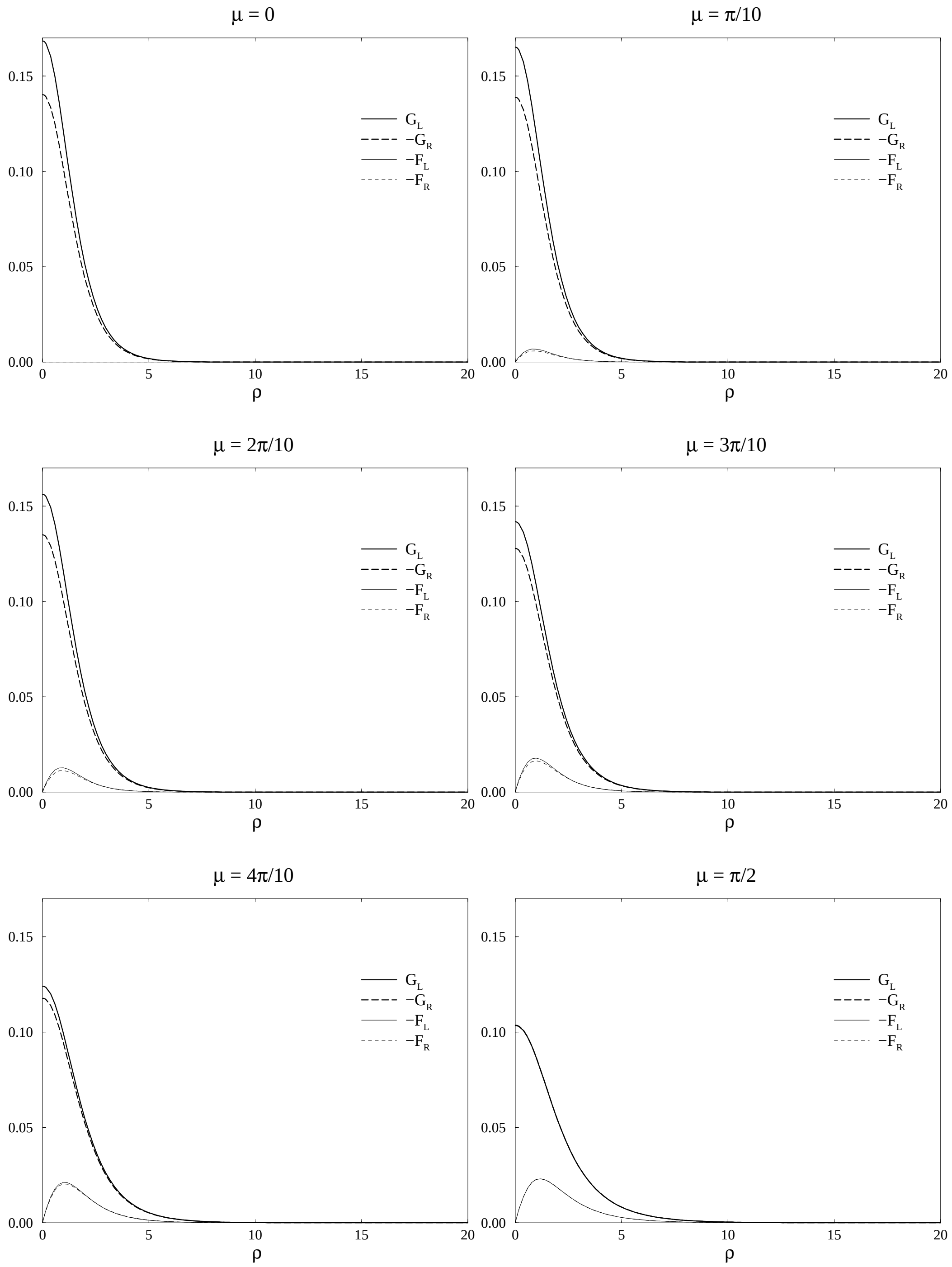

Figure A.1: Fermion profile functions for $0 \le \mu \le \pi/2$ and $\nu = 0$.

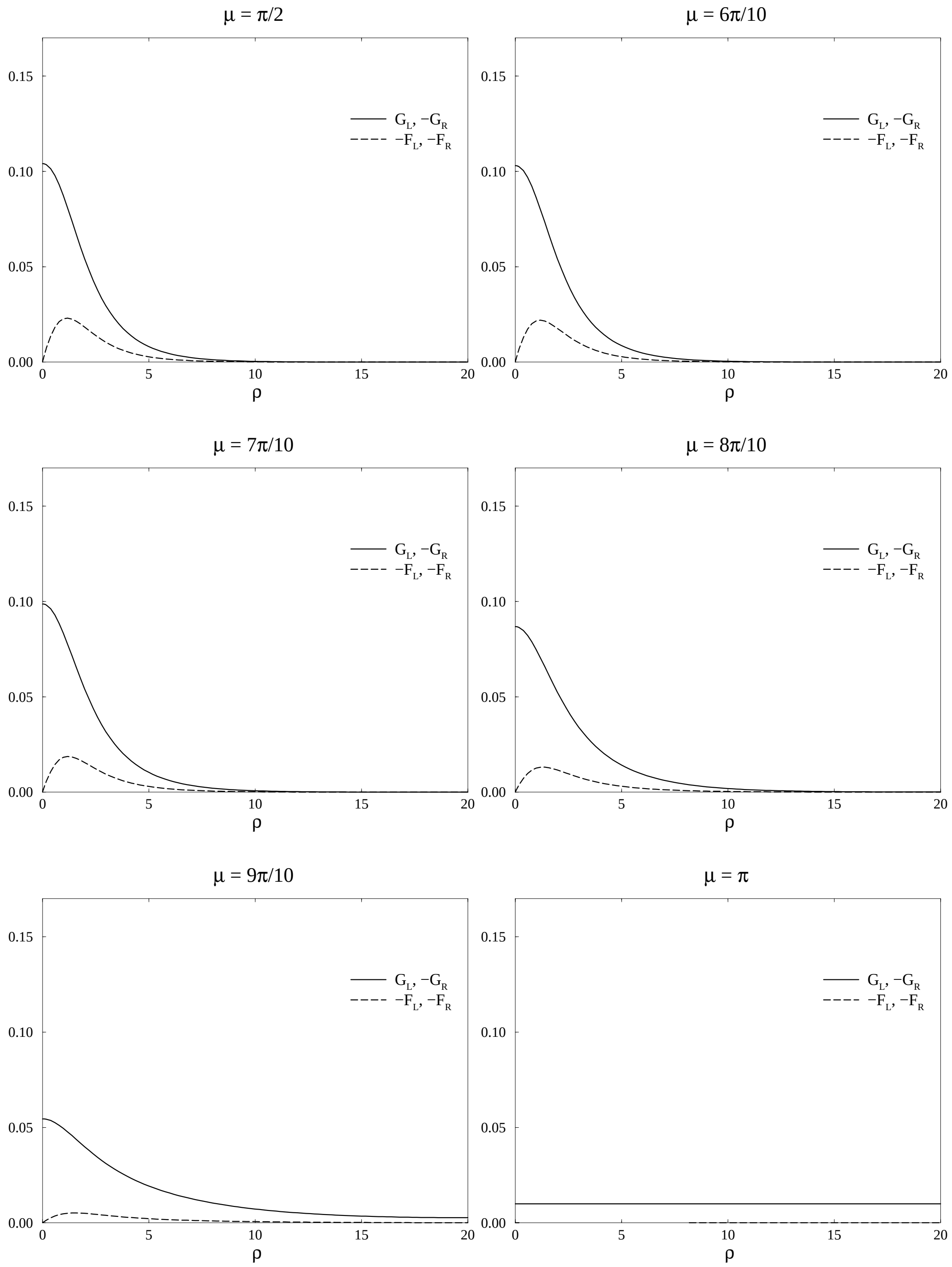

Figure A.2: Fermion profile functions for $\pi/2 \leq \mu \leq \pi$ and $\nu = 0$.

calculated for values of ρ between $\rho_{\min} = 10^{-3}$ and $\rho_{\max} = 20$ and are normalized as follows

$$||\Psi||^2 \equiv \int_0^{2\pi} \mathrm{d}\varphi \int_{\rho_{\min}}^{\rho_{\max}} \mathrm{d}\rho\, \rho\; \Psi^\dagger(\rho,\varphi)\Psi(\rho,\varphi) = 1\,. \tag{A.7}$$

This is then the reason that in Fig. A.2 the functions G_L and G_R for $\mu = \pi$ take on constant, but nonzero, values. If the interval of normalization were $[0,\infty)$, they would vanish identically.

In order to decide if the energy levels cross at $\mu = 0$ or merely touch, we consider the continuity of the wavefunction. In fact, the behaviour of $E(\mu)$ is determined completely by demanding that the wavefunction varies continuously with μ. To see this, write the ansatz (A.3) in a more general form

$$\begin{aligned}\Psi(\rho,\varphi) \;=\;\; & i\,G_{L1}\,|L\uparrow d\rangle + F_{L1}\,e^{i\varphi}|L\downarrow d\rangle + i\,G_{L2}\,|L\downarrow u\rangle + F_{L2}\,e^{-i\varphi}|L\uparrow u\rangle + \\ & G_{R1}\,|R\downarrow d\rangle + i\,F_{R1}\,e^{-i\varphi}|R\uparrow d\rangle + G_{R2}\,|R\uparrow u\rangle + i\,F_{R2}\,e^{i\varphi}|R\downarrow u\rangle\,. \end{aligned} \tag{A.8}$$

Solutions of the eigenvalue equation $H\,\Psi = E\,\Psi$ can be expressed in terms of the solutions G_L, G_R, F_L and F_R of (A.5) for $\mu \geq 0$ and $E \geq 0$ as follows

	G_{L1}	G_{L2}	G_{R1}	G_{R2}	F_{L1}	F_{L2}	F_{R1}	F_{R2}
$+\mu, +E$	$+G_L$	$-G_L$	$+G_R$	$+G_R$	$+F_L$	$-F_L$	$+F_R$	$+F_R$
$+\mu, -E$	$+G_L$	$+G_L$	$+G_R$	$-G_R$	$-F_L$	$-F_L$	$-F_R$	$+F_R$
$-\mu, +E$	$+G_L$	$+G_L$	$+G_R$	$-G_R$	$+F_L$	$+F_L$	$+F_R$	$-F_R$
$-\mu, -E$	$+G_L$	$-G_L$	$+G_R$	$+G_R$	$-F_L$	$+F_L$	$-F_R$	$-F_R$

. (A.9)

We now look at the over-all signs of these eight functions. The signs of the four basic functions G_L, G_R, F_L and F_R can be read from Fig. (A.1). The resulting signs of G_{L1}, G_{L2}, G_{R1}, G_{R2}, F_{L1}, F_{L2}, F_{R1} and F_{R2} are given – in this order – by the following table:

$E > 0$	$(+,+,-,+,-,-,-,+)$		$(+,-,-,-,-,+,-,-)$
$E = 0$		$(+,-,-,-,0,0,0,0)$ $(+,+,-,+,0,0,0,0)$	
$E < 0$	$(+,-,-,-,+,-,+,+)$		$(+,+,-,+,+,+,+,-)$
	$\mu < 0$	$\mu = 0$	$\mu > 0$

If the wavefunction is traced from negative to positive values of μ, say, the relative signs of the eight functions cannot change abruptly. From the sign table above, it then follows that the energy levels really cross. For time-dependent background fields ($\mu = \mu(t)$, $\nu = 0$) this results in pair creation. The role of particle production in anomalies has been emphasized in [6] and, evidently, pair production also plays a role in the global gauge anomaly of Section 6 of this paper.

B Berry phase close to the Z-string

In this Appendix we calculate the Berry phase for loops on the Z-NCS that are very close to the Z-string, using the original method of [4].

In that paper [4], Hamiltonians are considered which depend on three external parameters X_a. Discussed are loops in parameter space that are near the point ($X_a = 0$) at which the Hamiltonian has a degenerate eigenvalue. If two states are involved in this degeneracy, then the Berry phase can be calculated within the subspace spanned by these two states (projection operator Π). Setting the degenerate eigenvalue to zero, the Hamiltonian in the restricted space has the following form:

$$\Pi\, H\, \Pi = \frac{1}{2} \begin{pmatrix} X_3 & X_1 - i\,X_2 \\ X_1 + i\,X_2 & -X_3 \end{pmatrix} . \tag{B.1}$$

It is shown in [4] that the Berry phase corresponding to a simple loop $X_a(\alpha)$, $\alpha \in [0, 2\pi]$, is given by

$$\gamma = \pm \frac{1}{2}\, \Omega\,, \tag{B.2}$$

where Ω is the solid angle under which the loop is seen from the point of degeneracy ($X_a = 0$) and the sign depends on the state considered. If the loop lies in a plane through the origin, then there are two possibilities: either the origin is encircled once by the loop and $\gamma = \pm\pi$ or the origin lies outside the loop and $\gamma = 0$.

These considerations apply to our case, with the Z-NCS fields as the external parameters. In the background of the Z-string ($\mu = 0$, $\nu = 0$), we have indeed two degenerate fermion zero modes $|\Psi^{\pm}(0,0)\rangle$, called Ψ_1 and Ψ_2 in Appendix A. We now parametrize the Dirac Hamiltonians (13) by $\mu_0 \equiv \arccos c$ and $\alpha \equiv 2\pi t/T \in [0, 2\pi]$, for the contour $\cos\mu \cos\nu = c$ on the Z-NCS. See also (21) in the main text. Since we only need curves that are close to $\mu_0 = 0$, we can expand $H(\mu_0, \alpha)$ to first order in μ_0. A straightforward calculation gives then

$$\langle \Psi^{\pm} |\, H\, | \Psi^{\pm} \rangle = \mu_0\, E^{(1)} \begin{pmatrix} \cos\alpha & i\sin\alpha \\ -i\sin\alpha & -\cos\alpha \end{pmatrix} \; + \; \mathrm{O}(\mu_0^2)\,, \tag{B.3}$$

with $E^{(1)}$ a non-vanishing constant, which is expressed in terms of the bosonic and fermionic profile functions of the Z-string by

$$E^{(1)} = 4\,\pi\, M_W \int_0^\infty \mathrm{d}\rho\, \rho \left(\frac{f}{\rho}\, G_L^2 - 2\, m\, G_L\, G_R \right) , \tag{B.4}$$

where all quantities inside the integral are dimensionless as in (A.6, A.7). The same result holds for the Z-NCS′, with the factor m in the second term of the integrand of $E^{(1)}$ replaced by $m\,h$. Comparing (B.3) with the general form (B.1), we have

$$X_1(\alpha) = 0\,, \quad X_2(\alpha) = -2\,\mu_0 E^{(1)} \sin\alpha\,, \quad X_3(\alpha) = 2\,\mu_0 E^{(1)} \cos\alpha\,. \tag{B.5}$$

This curve $X_a(\alpha)$ lies in a plane through the origin and for μ_0 fixed encircles it once as α runs from 0 to 2π, so that the Berry phase factor is $e^{i\gamma} = -1$. More general closed curves C near the Z-string give the result (29) in the main text.

C Berry phase on the Z^2-NCS′

In this Appendix we calculate for isodoublet fermions the spectral flow and the Berry phase over a non-contractible sphere with winding number $n = 2$.

The Z-NCS′ (2, 3) can be generalized by replacing φ with $n\,\varphi$, so that the matrix function U has winding number $n \in \mathbf{Z}$. The resulting set of bosonic field configurations is called the Z^n-NCS′. The axial symmetry of the fermion fields is then generated by

$$K_3^{(n)} \equiv -i\,\partial_\varphi + \frac{1}{2}\,\Sigma_3 + \frac{n}{2}\,\tau_3\,(P_L - P_R)\,. \tag{C.1}$$

As an example, we consider the case $n = 2$. The Dirac eigenvalues for the $\nu = 0$ slice of the Z^2-NCS′ are now doubly degenerate, corresponding to the $K_3^{(2)}$-eigenvalues $\pm\frac{1}{2}$, and have no bifurcation at $\mu = \pm\,\pi/2$, see Fig. C.1. At $\mu = 0$ there is a four-fold degeneracy, of course. It can be shown that the levels there really cross, by the same type of argument as used in Appendix A.

As regards the Berry phase γ, there is an important difference compared to the case of winding number $n = 1$. The configuration space of the Dirac field has to be restricted to an eigenspace of $K_3^{(2)}$ with non-degenerate energies. However, the relevant $K_3^{(2)}$ eigenstates are *not* real and therefore the Berry phase factor $e^{i\gamma}$ is *not* restricted to the values ± 1. Indeed, a numerical calculation based on (23) shows that $e^{i\gamma}$ depends on the loop of Hamiltonians chosen and is in general complex. What is more, the Berry phase depends on whether the matrix Ω in (2, 3) is included or not. This is shown in Fig. C.2, where for both cases the Berry phase of the $K_3^{(2)} = +\frac{1}{2}$ eigenstate is plotted against μ_0, which corresponds to the point where the $\cos\mu\cos\nu = c$ contour intersects the positive μ-axis

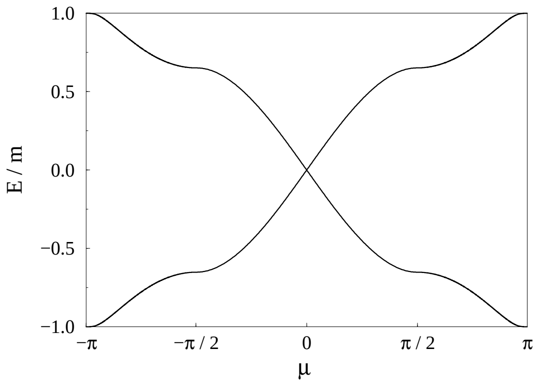

Figure C.1: Energies for $n = 2$, $\nu = 0$.

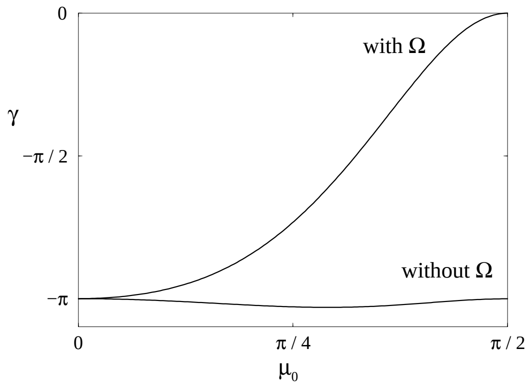

Figure C.2: Berry phase for $n = 2$.

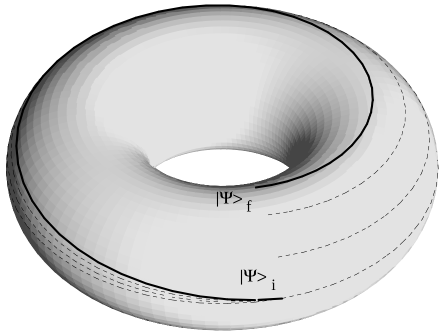

Figure C.3: Torus bundle for a slice of the Z^2-NCS$'$, with different Berry phases.

($\mu_0 = \arccos c$). Interpolation between these two alternatives (with or without Ω) is also possible, so that *a priori* any intermediate value of the Berry phase γ can be obtained. For the Z^2-NCS$'$ with μ_0 fixed and $\alpha \in [0, 2\pi]$ there is, in fact, a complex torus bundle (Fig. C.3). Still, the true Z^2-NCS$'$ does involve the matrix Ω and the Berry phase actually disappears as the loop is pulled towards the vacuum ($\mu_0 \to \pi/2$ in Fig. C.2). Hence, we expect that the vacuum loop of the Z^2-NCD, given by (8, 9) with $\eta = \pi/2$ and φ replaced by $2\,\varphi$, has a trivial Berry phase factor $+1$, corresponding to a cylinder bundle over the $n = 2$ gauge orbit (Fig. 4).

D Gauss' law and the global anomaly

In this Appendix we give a brief discussion of the physical state condition (Gauss' law) and the problem that arises due to the global gauge anomaly.

Consider the hamiltonian quantization, in the temporal gauge, of a $3+1$ dimensional

Yang-Mills theory with chiral fermions (the Higgs field is not important for the present discussion). In the Schrödinger representation the states Ψ are functionals of the gauge fields $W_m(\mathbf{x})$ and infinite component column vectors in the fermionic Hilbert space. Physical states Ψ_{phys} obey the condition [13]

$$ {}^{\mathrm{F}}\mathcal{G}_U \, \Psi_{\text{phys}} \left[W_m^U \right] = \Psi_{\text{phys}} \left[W_m \right] , \tag{D.1} $$

with $W_m^U \equiv U\,(W_m + \partial_m)\,U^{-1}$ the gauge transform of the field W_m and ${}^{\mathrm{F}}\mathcal{G}_U$ the corresponding unitary fermionic transformation. The condition (D.1) incorporates the physical requirement of gauge invariance and its infinitesimal version, for $U(\mathbf{x}) = \mathbb{1} + \theta(\mathbf{x})$ with $\theta(\mathbf{x})$ in the Lie algebra of the gauge group G, gives the so-called non-Abelian Gauss' law. The gauge transformation function $U(\mathbf{x}) \in G$ is considered to be topologically trivial, so that there are no additional phase factors in (D.1). As emphasized in [6], this physical state condition (D.1) respects the real structure of the second-quantized Hilbert bundle.

The global gauge anomaly shows up in the following way. The starting point is the existence of a loop of 3-dimensional gauge transformations $U(\omega) \in G$, with $\omega \in [0, 2\pi]$ and $U(0) = U(2\pi) = \mathbb{1}$, that gives a Berry phase factor -1 for one particular state $\bar{\Psi}_{\text{phys}}$ (in this paper, the vacuum state). This non-trivial Berry phase factor may come from having an *odd* number of pairs of fermionic levels crossing at $E = 0$. The result

$$ {}^{\mathrm{F}}\mathcal{G}_{U(2\pi)} \, \bar{\Psi}_{\text{phys}} \left[W_m^{U(2\pi)} \right] = - \; {}^{\mathrm{F}}\mathcal{G}_{U(0)} \, \bar{\Psi}_{\text{phys}} \left[W_m^{U(0)} \right] = - \, \bar{\Psi}_{\text{phys}} \left[W_m \right] \tag{D.2} $$

is, however, incompatible with (D.1). In other words, Gauss' law cannot be implemented continuously over the *whole* of the space of Yang-Mills connections and the theory is said to have a global anomaly. See [6, 13] for further details and discussion. Here, we only remark that there may be different types of gauge transformation loops $U(\omega)$ which produce a factor -1 in (D.2), see in particular (31) and (32) in the main text.

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