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Constraining differential renormalization in abelian gauge theories

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Abstract

We present a procedure of differential renormalization at the one loop level which avoids introducing unnecessary renormalization constants and automatically preserves abelian gauge invariance. The amplitudes are expressed in terms of a basis of singular functions. The local terms appearing in the renormalization of these functions are determined by requiring consistency with the propagator equation. Previous results in abelian theories, with and without supersymmetry, are discussed in this context.

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Differential regularization and renormalization (DR) [1] was introduced as a renormalization method in coordinate space compatible with gauge and chiral symmetry. In a series of papers this method has been further developed [2, 3, 4] and successfully applied to different theories [5–13][1]. However, it might be considered unsatisfactory the fact that Ward identities among renormalized Green functions are only satisfied when the different renormalization scales are conveniently adjusted. Instead, one would like that the gauge symmetry were automatically preserved, as occurs in dimensional regularization and renormalization [16].

In this letter we present a procedure to constrain the scales in DR at one loop while preserving abelian gauge invariance. This is done in two steps. First, each diagram is written in terms of a set of independent functions (with different number of propagators and/or different tensor structure). Second, the singular functions of this set are renormalized in a way which does not depend on the diagram where they appear. The local terms are fixed by the requirement that DR be compatible with the equation defining the propagator in the space of distributions. The propagator equation also allows to treat tadpole diagrams with the usual DR rules. In this manner one obtains renormalized Green functions which depend on just one arbitrary constant (the renormalization group scale) and, as we shall see, fulfil Ward identities in abelian gauge theories. The non-abelian case will be studied elsewhere.

After describing the method, we discuss the renormalization of the one-loop vacuum polarization in massive scalar QED, which is the simplest example requiring all the ingredients of the constrained procedure. The complete one-loop renormalization of this theory will be presented in Ref. [17]. Then we review the one-loop Ward identities of massive QED [4] and massless QED in an arbitrary gauge [8], the corresponding ABJ anomaly [1, 8] and the evaluation of $(g-2)_l$ in supergravity, where supersymmetry is also preserved [12].

DR renormalizes diagrams by replacing singular expressions by derivatives of well-behaved distributions (*differential reduction*). These derivatives are understood in the sense of distribution theory, *i.e.*, they are prescribed to act formally by parts on test functions (*formal integration by parts*). In practice, to carry out this programme one has to manipulate singular expressions. This gives rise to ambiguities which are usually taken care of by keeping arbitrary renormalization scales for different diagrams (or pieces of diagrams). The scales are adjusted at the end to enforce the Ward identities (which is equivalent to the addition of finite counterterms). In this letter we show that four rules are sufficient to formally manipulate and renormalize the singular expressions,

[1] Different versions of differential renormalization can be found in [14, 15].

avoiding the introduction of unnecessary scales. The resulting renormalized amplitudes automatically satisfy the Ward identities. These rules are summarised as follows:

1. Differential reduction, where we distinguish two cases:

   (a) Functions with singular behaviour worse than x^{-4} are reduced to derivatives of 'logarithmically' singular functions without introducing extra dimensionful constants [18]. For example,

   $$\frac{1}{x^6} = \frac{1}{8}\Box\frac{1}{x^4} \quad . \tag{1}$$

   (b) Logarithmically singular functions are written as derivatives of regular functions. At one loop we have the usual DR identity [1]

   $$\frac{1}{x^4} = -\frac{1}{4}\Box\frac{\log x^2M^2}{x^2} \quad , \tag{2}$$

   which introduces a unique dimensionful constant (the renormalization group scale).

2. Formal integration by parts. In particular,

   $$[\partial F]^R = \partial F^R \ , \tag{3}$$

   where F is an arbitrary function and R stands for renormalized.

3. *Delta function renormalization rule*:

   $$[F(x,x_1,...,x_n)\delta(x-y)]^R = [F(x,x_1,...,x_n)]^R\delta(x-y) \ . \tag{4}$$

4. The general validity of the *propagator equation*:

   $$F(x,x_1,...,x_n)(\Box^x - m^2)\Delta_m(x) = F(x,x_1,...,x_n)(-\delta(x)) \ , \tag{5}$$

   where $\Delta_m(x) = \frac{1}{4\pi^2}\frac{mK_1(mx)}{x}$ and K_1 is a modified Bessel function [19]. This is a valid mathematical identity between tempered distributions if F is well-behaved enough. This rule formally extends its range of applicability to an arbitrary function.

The last rule will prove essential in our procedure. For instance, the 'engineering' tensor decomposition into trace and traceless parts is not compatible with it and will be modified by the addition of a finite local term.

To evaluate the diagrams we express them in terms of a set of *basic functions* using only algebraic manipulations (including four-dimensional Dirac algebra) and the Leibnitz rule for derivatives. To one loop these functions can be classified according to their number of propagators and their derivative structure. For the examples we discuss we need functions with one, two and three propagators:

$$\begin{aligned}
\mathrm{A} &= \Delta(x)\delta(x) \ , &(6)\\
\mathrm{B}[\mathcal{O}] &= \Delta(x)\mathcal{O}^x\Delta(x) \ , &(7)\\
\mathrm{T}[\mathcal{O}] &= \Delta(x)\Delta(y)\mathcal{O}^x\Delta(x-y) \ , &(8)
\end{aligned}$$

where $\Delta(x) = \frac{1}{4\pi^2}\frac{1}{x^2}$ is the massless propagator and $\mathcal{O}$ is a differential operator. In massive theories the following basic functions are also required:

$$\begin{aligned}
\bar{\mathrm{A}} &= \bar{\Delta}(x)\delta(x) \ , &(9)\\
\bar{\mathrm{B}}[\mathcal{O}] &= \Delta(x)\mathcal{O}^x\bar{\Delta}(x) \ , &(10)\\
\bar{\mathrm{T}}[\mathcal{O}] &= \Delta(x)\Delta(y)\mathcal{O}^x\bar{\Delta}(x-y) \ , &(11)
\end{aligned}$$

where $\bar{\Delta}(x) = \frac{1}{4}\frac{1}{4\pi^2}\log x^2m^2$ appears when the massive propagator is expanded in the mass m. Such expansion allows to properly separate pieces with different degree of singularity. The same type of functions also appear in massless theories if the photon propagator is written in a general Lorentz gauge. Note that A, $\bar{\mathrm{A}}$ and B functions are singular, and $\bar{\mathrm{B}}$ and T ($\bar{\mathrm{T}}$) are singular for $n \geq 2$ ($n \geq 4$), where n is the order of the differential operator $\mathcal{O}$.

We renormalize these basic functions using systematically rules 1 to 4. For example,

$$\begin{aligned}
\mathrm{T}[\Box] &= \Delta(x)\Delta(y)\Box^x\Delta(x-y)\\
&= -[\Delta(x)]^2\delta(x-y)\\
&= -\mathrm{B}[1](x)\delta(x-y) \ ; &(12)\\
\mathrm{T}^R[\Box] &= -\mathrm{B}^R[1](x)\delta(x-y)\\
&= \frac{1}{4}\frac{1}{(4\pi^2)^2}\Box^x\frac{\log x^2M^2}{x^2}\delta(x-y) \ . &(13)
\end{aligned}$$

Our main observation is that the propagator equation (rule 4) can be further used to relate the different basic functions. Thus, by requiring that their renormalization be compatible with these relations, we shall completely fix the finite local terms (or scales) which appear in the differentially renormalized functions. Let us illustrate how to do this with one example, the renormalization of the basic function $\mathrm{T}[\partial_\mu\partial_\nu]$. Using rule 4 for a massless propagator,

$$F\Box\Delta(x) = -F\delta(x) \ , \tag{14}$$

we can write

$$\begin{aligned} \mathrm{B}[\partial_\mu](x)\delta(y) &= -\partial^x_\mu\Box^y\mathrm{T}[1] + \Box^y\mathrm{T}[\partial_\mu] - 2\partial^x_\mu\partial^y_\sigma\mathrm{T}[\partial_\sigma] - \partial^x_\mu\mathrm{T}[\Box] \\ &\quad + 2\partial^y_\sigma\mathrm{T}[\partial_\mu\partial_\sigma] + \mathrm{T}[\partial_\mu\Box] \ . \end{aligned} \tag{15}$$

The last basic function on the right-hand side can be easily reduced to

$$\mathrm{T}[\partial_\mu\Box] = \frac{1}{2}(\partial^x_\mu - \partial^y_\mu)\mathrm{T}[\Box] \ , \tag{16}$$

where rules 3 and 4 have been used. Now, we decompose the basic function $\mathrm{T}[\partial_\mu\partial_\nu]$ into trace and traceless parts, adding an arbitrary (for the moment) local term to take into account the possible ambiguity introduced by this operation:

$$\mathrm{T}[\partial_\mu\partial_\nu] = \frac{1}{4}\delta_{\mu\nu}\mathrm{T}[\Box] + \mathrm{T}[\partial_\mu\partial_\nu - \frac{1}{4}\delta_{\mu\nu}\Box] + \frac{1}{64\pi^2}\, b\,\delta(x)\delta(y)\delta_{\mu\nu} \ . \tag{17}$$

The traceless part is finite because of the tensor structure and is not further renormalized. $\mathrm{T}[1]$ and $\mathrm{T}[\partial_\mu]$ are also finite. On the other hand, the left-hand side of Eq. (15) is

$$\begin{aligned} \mathrm{B}[\partial_\mu](x)\delta(y) &= \Delta(x)\partial^x_\mu\Delta(x)\delta(y) \\ &= \frac{1}{2}\partial^x_\mu\Delta(x)^2\delta(y) \\ &= \frac{1}{2}\partial^x_\mu\mathrm{B}[1](x)\delta(y) \ . \end{aligned} \tag{18}$$

So, Eq. (15) reads for renormalized functions

$$\begin{aligned} &\frac{1}{2}\partial^x_\mu\mathrm{B}^R[1](x)\delta(y) = \\ &\quad -\partial^x_\mu\Box^y\mathrm{T}[1] + \Box^y\mathrm{T}[\partial_\mu] - 2\partial^x_\mu\partial^y_\sigma\mathrm{T}[\partial_\sigma] - \frac{1}{2}\partial^x_\mu\mathrm{T}^R[\Box] + 2\partial^y_\sigma\mathrm{T}[\partial_\mu\partial_\sigma - \frac{1}{4}\delta_{\mu\sigma}\Box] \\ &\quad + \frac{1}{8}\frac{1}{4\pi^2}\, b\,\partial^y_\mu(\delta(x)\delta(y)) \ . \end{aligned} \tag{19}$$

Since both members of this equation are finite, we can integrate on x using the integration by parts prescription[2]:

$$\begin{aligned} 0 &= \int \mathrm{d}x\, (\Box^y\mathrm{T}[\partial_\mu] + 2\partial^y_\sigma\mathrm{T}[\partial_\mu\partial_\sigma - \frac{1}{4}\delta_{\mu\sigma}\Box]) + \frac{1}{8}\frac{1}{4\pi^2}\, b\,\partial^y_\mu\delta(y) \\ &= \frac{1}{16\pi^2}(\frac{1}{4} + \frac{1}{2}b)\,\partial^y_\mu\delta(y) \ . \end{aligned} \tag{20}$$

[2]Techniques for performing this sort of integrals can be found in Appendix A of Ref. [1] and Appendix C of Ref. [12].

Then this equation fixes

$$b = -\frac{1}{2} \ . \tag{21}$$

Note that the engineering trace-traceless decomposition, commonly used in the literature of DR, is not compatible with the propagator equation in the case of logarithmic singularities (for quadratic singularities it is consistent because of rule 1a). Hence, contraction of indexes does not commute with renormalization. Generically one must simplify all the tensor and Dirac structure before identifying the basic functions to be renormalized. The addition of the local term is equivalent to using a different mass scale M' in the renormalization of the T[$\Box$] coming from the trace-traceless decomposition, and then fixing $\log \frac{M^2}{M'^2} = b = -\frac{1}{2}$. We prefer, however, to use the language of local terms to avoid confusion with the usual *ad hoc* adjustment of renormalization scales.

With the same technique one can determine the renormalization of all the basic functions. In general, besides the massless propagator equation, Eq. (14), one needs

$$F \Box \bar{\Delta}(x) = F \Delta(x) \ . \tag{22}$$

Both Eq. (14) and Eq. (22) are a consequence of the massive propagator equation, Eq. (5), and equivalent to the equation for the photon propagator in a general gauge,

$$F\,(\delta_{\mu\nu}\Box - (1-\frac{1}{a})\partial_\mu\partial_\nu)\;\Delta_{\nu\rho}(x) = -F\;\delta_{\mu\rho}\delta(x) \ , \tag{23}$$

where $\Delta_{\mu\nu}(x) = 1/16\pi^2(\delta_{\mu\nu}\Box + (a-1)\partial_\mu\partial_\nu)\log x^2\mu^2$.

The propagator equation can be further employed to 'separate' the tadpole functions into two-point functions, which can then be treated with the usual DR prescriptions:

$$\begin{aligned} \mathrm{A} &= \Delta(x)\delta(x) \\ &= -\Delta(x)\Box\Delta(x) \\ &= -\mathrm{B}[\Box] \ . \end{aligned} \tag{24}$$

In Table 1 we gather the renormalized expressions of the basic functions required in the applications below. In massive theories it is usually more convenient to work with compact expressions involving modified Bessel functions [4]. The corresponding DR identities can be obtained by expanding the propagators in the mass parameter, using Table 1 and resumming the result. In practice, one uses recurrence relations among Bessel functions (see Appendix C of Ref. [12]) and then adds the necessary local terms to agree with Table 1. Table 2 collects the massive renormalization identities used in this paper.

$$
\begin{aligned}
\mathrm{A}^R &= 0 \\
\bar{\mathrm{A}}^R &= -\Box\bar{\mathrm{B}}[1] + 2\partial_\sigma\bar{\mathrm{B}}[\partial_\sigma] + \tfrac{1}{4}\tfrac{1}{(4\pi^2)^2}\Box\tfrac{\log x^2M^2}{x^2} \\
\mathrm{B}^R[1] &= -\tfrac{1}{4}\tfrac{1}{(4\pi^2)^2}\Box\tfrac{\log x^2M^2}{x^2} \\
\mathrm{B}^R[\Box] &= 0 \\
\mathrm{B}^R[\partial_\mu\partial_\nu] &= -\tfrac{1}{12}\tfrac{1}{(4\pi^2)^2}(\partial_\mu\partial_\nu - \tfrac{1}{4}\delta_{\mu\nu}\Box)\Box\tfrac{\log x^2M^2}{x^2} \\
\bar{\mathrm{B}}^R[\Box] &= \mathrm{B}^R[1] \\
\bar{\mathrm{B}}^R[\partial_\mu\partial_\nu] &= -\tfrac{1}{16}\tfrac{1}{(4\pi^2)^2}(\Box\tfrac{\log x^2M^2}{x^2}\delta_{\mu\nu} + 2\partial_\mu\partial_\nu\tfrac{1}{x^2}) \\
\mathrm{T}^R[\Box] &= \tfrac{1}{4}\tfrac{1}{(4\pi^2)^2}\Box\tfrac{\log x^2M^2}{x^2}\delta(x-y) \\
\mathrm{T}^R[\partial_\mu\partial_\nu] &= (\tfrac{1}{16}\tfrac{1}{(4\pi^2)^2}\Box\tfrac{\log x^2M^2}{x^2}\delta(x-y) \\
&\quad - \tfrac{1}{32}\tfrac{1}{4\pi^2}\delta(x)\delta(y))\delta_{\mu\nu} + \mathrm{T}[\partial_\mu\partial_\nu - \tfrac{1}{4}\delta_{\mu\nu}\Box] \\
\bar{\mathrm{T}}^R[\Box\Box] &= \mathrm{T}^R[\Box] \\
\bar{\mathrm{T}}^R[\Box\partial_\mu\partial_\nu] &= \mathrm{T}^R[\partial_\mu\partial_\nu]
\end{aligned}
$$

Table 1: Renormalized expressions of basic functions.

$$
\begin{aligned}
\mathrm{A}_m^R &= \tfrac{1}{(4\pi^2)^2}\pi^2 m^2 (1-\log\tfrac{\bar{M}^2}{m^2})\delta(x) \\
\mathrm{B}_m^R[1] &= \tfrac{1}{(4\pi^2)^2}\{\tfrac{1}{2}(\Box-4m^2)\tfrac{mK_0(mx)K_1(mx)}{x} + \pi^2\log\tfrac{\bar{M}^2}{m^2}\delta(x)\} \\
\mathrm{B}_m^R[\Box] &= \tfrac{1}{(4\pi^2)^2}m^2\{\tfrac{1}{2}(\Box-4m^2)\tfrac{mK_0(mx)K_1(mx)}{x} \\
&\quad + \pi^2(2\log\tfrac{\bar{M}^2}{m^2}-1)\delta(x)\} \\
\mathrm{B}_m^R[\partial_\mu\partial_\nu] &= \tfrac{1}{(4\pi^2)^2}\{\tfrac{1}{6}\partial_\mu\partial_\nu[(\Box-4m^2)(\tfrac{mK_0(mx)K_1(mx)}{x} + \tfrac{1}{4}m^2(K_0^2(mx)-K_1^2(mx))) \\
&\quad + \pi^2(2\log\tfrac{\bar{M}^2}{m^2}-1)\delta(x)] \\
&\quad - \tfrac{1}{24}\delta_{\mu\nu}[(\Box-4m^2)(\Box-4m^2)\tfrac{mK_0(mx)K_1(mx)}{x} \\
&\quad + 2\pi^2\log\tfrac{\bar{M}^2}{m^2}\Box\delta(x) - 4\pi^2 m^2(1+3\log\tfrac{\bar{M}^2}{m^2})\delta(x)]\}
\end{aligned}
$$

Table 2: Renormalized expressions of massive basic functions, where $\mathrm{A}_m = \Delta_m(x)\delta(x)$ and $\mathrm{B}_m[\mathcal{O}] = \Delta_m(x)\mathcal{O}^x\Delta_m(x)$.

At this point any one-loop diagram can be renormalized: one just has to use the renormalized basic functions listed in the Tables.

As a simple example which contains all the ingredients of the constrained procedure let us consider in detail the vacuum polarization in massive scalar QED. The contributing diagrams are depicted in Fig. 1. Using the Feynman rules given in Ref. [12] one gets

$$
\begin{aligned}
\Pi_{\mu\nu}^{(1)}(x) &= -e^2\Delta_m(x)\overleftrightarrow{\partial}_\mu\overleftrightarrow{\partial}_\nu\,\Delta_m(x)\ , &(25)\\
\Pi_{\mu\nu}^{(2)}(x) &= -2e^2\delta_{\mu\nu}\Delta_m(x)\delta(x)\ , &(26)
\end{aligned}
$$

which expressed in terms of basic functions read

$$
\begin{aligned}
\Pi_{\mu\nu}^{(1)}(x) &= -e^2\{4\mathrm{B}_m[\partial_\mu\partial_\nu] - \partial_\mu\partial_\nu B_m[1]\}\ , &(27)\\
\Pi_{\mu\nu}^{(2)}(x) &= -2e^2\delta_{\mu\nu}\mathrm{A}_m\ . &(28)
\end{aligned}
$$

Substituting the renormalized basic functions of Table 2, we obtain for each diagram

$$
\begin{aligned}
\Pi_{\mu\nu}^{(1)\,R}(x) &= -\frac{e^2}{(4\pi^2)^2}\{(\partial_\mu\partial_\nu-\delta_{\mu\nu}\Box)[\frac{1}{6}(\Box-4m^2)(\frac{mK_0(mx)K_1(mx)}{x} \\
&\quad + m^2(K_0^2(mx)-K_1^2(mx))) + \frac{1}{3}\pi^2(\log\frac{\bar{M}^2}{m^2}-2)\delta(x)]
\end{aligned}
$$

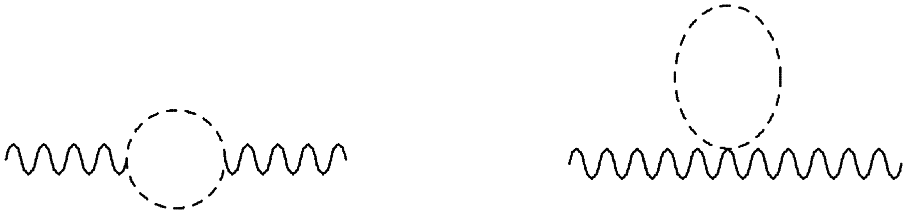

Diagram 1 Diagram 2

Figure 1: One-loop diagrams contributing to the vacuum polarization of scalar QED.

$$
\begin{aligned}
& + \delta_{\mu\nu}[2\pi^2 m^2(\log\frac{\bar{M}^2}{m^2} - 1)\delta(x)]\} \ , && (29)\\
\Pi^{(2)\,R}_{\mu\nu}(x) \quad &= \quad \frac{e^2}{(4\pi^2)^2}2\pi^2 m^2(\log\frac{\bar{M}^2}{m^2} - 1)\delta_{\mu\nu}\delta(x) \ . && (30)
\end{aligned}
$$

The longitudinal terms cancel in the sum, yielding a transverse result:

$$
\begin{aligned}
\Pi^R_{\mu\nu}(x) \quad &= \quad -\frac{e^2}{(4\pi^2)^2}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)[\frac{1}{6}(\Box - 4m^2)(\frac{mK_0(mx)K_1(mx)}{x} \\
&\quad + m^2(K_0^2(mx) - K_1^2(mx))) + \frac{1}{3}\pi^2(\log\frac{\bar{M}^2}{m^2} - 2)\delta(x)] \ . \qquad (31)
\end{aligned}
$$

In the same way, our method recovers previous DR results in abelian gauge theories, without the need to impose Ward identities *a posteriori*. Let us consider first the vacuum polarization in massive QED [4]. In terms of basic functions it reads

$$
\Pi_{\mu\nu}(x) = 4e^2\{(m^2\delta_{\mu\nu} + \frac{1}{2}\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\,\mathrm{B}_m[1] + 2\mathrm{B}_m[\partial_\mu\partial_\nu] - \delta_{\mu\nu}\mathrm{B}_m[\Box]\} \qquad (32)
$$

and the renormalized expression is

$$
\begin{aligned}
\Pi^R_{\mu\nu}(x) \quad &= \quad -\frac{4e^2}{(4\pi^2)^2}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)[\frac{1}{6}(\Box - 4m^2)(\frac{mK_0(mx)K_1(mx)}{x} \\
&\quad - \frac{1}{2}m^2(K_0^2(mx) - K_1^2(mx))) + \frac{1}{3}\pi^2(\log\frac{\bar{M}^2}{m^2} + 1)\delta(x)] \ , \qquad (33)
\end{aligned}
$$

which is again transverse.

In supersymmetric QED the vacuum polarization is the sum of the spinor QED diagram and twice (two complex scalars for each Dirac spinor) the scalar QED diagrams. In terms of basic functions this gives directly a transverse result depending on one basic function only:

$$
\Pi_{\mu\nu}(x) = -2e^2(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)\mathrm{B}_m[1] \ . \qquad (34)
$$

In this case one could use an engineering trace-traceless decomposition in renormalizing each diagram. The gauge-non-invariant terms vanish in the total sum due to supersymmetry cancellations. The complete result would thus be the same as the one obtained from the renormalized functions of Table 2:

$$
\begin{aligned}
\Pi^R_{\mu\nu}(x) &= -\frac{e^2}{(4\pi^2)^2}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)[(\Box - 4m^2)\frac{mK_0(mx)K_1(mx)}{x} \\
&\quad + 2\pi^2 \log\frac{\bar{M}^2}{m^2}\delta(x)] \ . \qquad (35)
\end{aligned}
$$

Next we consider the QED vertex Ward identity between the electron self-energy and the electron-electron-photon vertex in an arbitrary Lorentz gauge, which was studied in Ref. [8]. In this case and the next one the masses play no relevant role, as far as renormalization is concerned, so we consider massless electrons for simplicity. In order to respect the Ward identity for both the a-dependent and a-independent pieces (where a is the gauge parameter), the authors of Ref. [8] had to impose two relations among different scales:

$$
\log\frac{M_\Sigma}{M_V} = \frac{1}{4} \ , \qquad (36)
$$

$$
\lambda \equiv \log\frac{M_\Sigma}{M'_\Sigma} = 3 \ , \qquad (37)
$$

where M_V and M_Σ and M'_Σ appear in the vertex and in the two pieces of the electron self-energy, respectively. With the constrained method we find

$$
\begin{aligned}
\Sigma^R(x) &= e^2\{\frac{1}{4}\frac{1}{(4\pi^2)^2}\ \partial\!\!\!/\Box\frac{\log x^2M^2}{x^2} \\
&\quad + (a-1)[\frac{1}{4}\frac{1}{(4\pi^2)^2}\ \partial\!\!\!/\Box\frac{\log x^2M^2}{x^2} + \frac{1}{16\pi^2}\ \partial\!\!\!/\delta(x)]\} \qquad (38)
\end{aligned}
$$

for the electron self-energy, and

$$
\begin{aligned}
V^R_\mu(x,y) &= ie^3\{-2\gamma_b\gamma_\mu\gamma_a(\partial^x_a\partial^y_b\bar{\mathrm{T}}[\Box] + \partial^x_a\bar{\mathrm{T}}[\partial_b\Box] - \partial^y_b\bar{\mathrm{T}}[\partial_a\Box]) + 4\gamma_a\mathrm{T}[\partial_a\partial_\mu - \frac{1}{4}\delta_{a\mu}\Box] \\
&\quad - \frac{1}{4}\frac{1}{(4\pi^2)^2}\gamma_\mu\Box\frac{\log x^2M^2}{x^2}\delta(x-y) - \frac{1}{8}\frac{1}{4\pi^2}\gamma_\mu\delta(x)\delta(y) \\
&\quad + (a-1)[\gamma_\rho\gamma_a\gamma_\mu\gamma_b\gamma_\sigma(\partial^x_a\partial^y_b\bar{\mathrm{T}}[\partial_\rho\partial_\sigma] + \partial^x_a\bar{\mathrm{T}}[\partial_b\partial_\rho\partial_\sigma] - \partial^y_b\bar{\mathrm{T}}[\partial_a\partial_\rho\partial_\sigma]) \\
&\quad - \frac{1}{4}\frac{1}{(4\pi^2)^2}\gamma_\mu\Box\frac{\log x^2M^2}{x^2}\delta(x-y)]\} \qquad (39)
\end{aligned}
$$

for the vertex. These renormalized amplitudes automatically satisfy the Ward identity

$$
(\partial^x_\mu + \partial^y_\mu)V^R_\mu(x,y) = ie\Sigma^R(x-y)(\delta(x)-\delta(y)) \ , \qquad (40)
$$

as can be seen by integrating on y.

The chiral triangle anomaly in QED was discussed in Refs. [1, 8]. The renormalized triangle diagram depended on the relation between two renormalization scales. By adequately choosing these scales one could respect either vector current or axial current conservation, but not both. Imposing conservation of the vector current, the correct value of the axial anomaly resulted. In contrast, in the constrained DR method everything is determined and a non-ambiguous result is obtained:

$$\begin{aligned} \mathrm{T}^R_{\mu\nu\lambda}(x,y) &= ie^3\{-2\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\lambda\gamma_\nu\gamma_a\gamma_b\gamma_c)\partial^x_c\partial^y_a\mathrm{T}[\partial_b] + 16(\epsilon_{\lambda\mu ab}\partial^x_a\partial^y_\nu - \epsilon_{\lambda\nu ab}\partial^x_\mu\partial^y_a)\mathrm{T}[\partial_b] \\ &\quad + 16\epsilon_{\lambda b\mu a}\partial^x_a\mathrm{T}[\partial_\nu\partial_b - \frac{1}{4}\delta_{\nu b}\Box] - 16\epsilon_{\lambda\nu ba}\partial^y_a\mathrm{T}[\partial_\mu\partial_b - \frac{1}{4}\delta_{\mu b}\Box] \\ &\quad + \frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda a}(\partial^x_a - \partial^y_a)(\delta(x)\delta(y))\}\ , \end{aligned} \tag{41}$$

where the index λ corresponds to the axial vertex. Then (see appendix B of Ref. [1])

$$\partial^x_\mu T^R_{\mu\nu\lambda}(x,y) = 0\ , \tag{42}$$

$$\partial^y_\nu T^R_{\mu\nu\lambda}(x,y) = 0\ , \tag{43}$$

$$-(\partial^x_\lambda + \partial^y_\lambda)T^R_{\mu\nu\lambda}(x,y) = \frac{ie^3}{2\pi^2}\epsilon_{\mu\nu\lambda\rho}\partial^x_\lambda\partial^y_\rho(\delta(x)\delta(y))\ , \tag{44}$$

so the vector Ward identities are directly preserved while the axial one is broken, giving the known result for the anomaly.

Finally, let us comment briefly on the calculation of the $(g-2)_l$ in unbroken supergravity performed in Ref. [12]. There, the symmetry to be preserved was supersymmetry, which implies a vanishing anomalous magnetic moment [20]. This result was obtained thanks to the use of the propagator equation to explicitly relate diagrams with different topology. Then, only one type of singular basic function, $\mathrm{T}[\Box]$, appeared. Although engineering trace-traceless decompositions were performed at intermediate steps, this did not affect the total sum because the extra local terms cancel, as occurs in the vacuum polarization in supersymmetric QED discussed above. Using the renormalized basic functions in Tables 1 and 2, $(g-2)_l$ also vanishes, although the contribution of each diagram is different, as is the total graviton contribution.

Summarizing, we have proposed a procedure of differential renormalization to one loop which only introduces a single renormalization scale. We have verified that the renormalized amplitudes so obtained automatically satisfy the Ward identities of abelian gauge symmetry in known examples, that the chiral anomaly is correctly treated, and that supersymmetry is preserved in a relatively complex calculation. In practice, one just needs to use the renormalized functions of Tables 1 and 2.

In principle, the method could be generalized to higher loops. However this is not straightforward. New more complicated functions emerge, which could be tackled with the systematic method of Ref. [3]. Still, one should properly constrain the local terms using rules 1 to 4 or some consistent extension of them.

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