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Fermion zero-modes of a new constrained instanton in Yang-Mills-Higgs theory

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Abstract

Self-consistent *Ansätze* are presented for the left- and right-handed isodoublet fermion zero-modes of the constrained instanton $\mathrm{I}^\star$ in the vacuum sector of euclidean $SU(2)$ Yang-Mills-Higgs theory. These left- and right-handed fermion wave functions do not coincide and, most likely, have maxima at different positions. This may be important for the fermion zero-mode contribution to the euclidean 4-point Green's function in chiral Yang-Mills-Higgs theory and the high-energy behaviour of fermion-fermion scattering processes.

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1 Introduction

Recently, a self-consistent *Ansatz* has been presented [1] for a new constrained instanton (finite action, localized, 4-dimensional euclidean solution) in the topological charge-zero sector of $SU(2)$ Yang-Mills-Higgs theory. A numerical approximation to the bosonic fields of this instanton $\mathrm{I}^\star$ has been obtained previously [2], but the complete numerical solution for the profile functions of the *Ansatz* is not available yet. In this paper, we consider the response of massless, chiral fermions to the generic bosonic fields of $\mathrm{I}^\star$, with the fermions in the fundamental (isodoublet) representation of the $SU(2)$ gauge group. The appropriate *Ansätze* for the fermion zero-modes are constructed and some partial solutions of the reduced Weyl equations are obtained analytically.

The outline of this paper is then as follows. In Section 2, the euclidean chiral $SU(2)$ Yang-Mills-Higgs theory considered is reviewed. In Section 3, the bosonic fields of the instanton $\mathrm{I}^\star$ are discussed, together with the relevant symmetries. In Section 4, these symmetries are used to construct the *Ansätze* for the left- and right-handed isodoublet fermion zero-modes. In Section 5, the two Weyl equations are evaluated, which prove the *Ansätze* to be self-consistent. Also, in Section 5, the related Dirac eigenvalue equation is established. The Dirac eigenvalues display the phenomenon of spectral flow, when evaluated over a non-contractible loop of bosonic field configurations passing through the classical vacuum and $\mathrm{I}^\star$ [3, 4]. Two of these Dirac eigenvalues are expected to cross at $\mathrm{I}^\star$ and give a pair of normalizable fermion zero-modes, one of each chirality. In Section 6, some partial solutions of the reduced Weyl equations in the $\mathrm{I}^\star$ background are established analytically. An approximate solution by courtesy of supersymmetry is also presented. The most important result, though, is that the left- and right-handed fermion zero-modes are shifted with respect to one another. This traces back to the conjectured structure of the bosonic fields of $\mathrm{I}^\star$, where two cores (with Higgs field vanishing) are kept apart in a balancing act between the repulsive Yang-Mills force and the attractive Higgs force, the core separation $d^\star$ being of order M_W^{-1} (the Yukawa range of the massive W vector bosons). Here, and in the following, natural units are used with $\hbar = c = 1$.

The main focus of the present paper is on the fermion solutions themselves, but in Section 7 we mention one possible application of the $\mathrm{I}^\star$ fermion zero-modes in the chiral $SU(2)$ Yang-Mills-Higgs theory with *two* flavors of massless left-handed fermions, namely their contribution to the euclidean 4-point Green's function relevant to the forward elastic scattering amplitude. The relative shift of the two types of fermion zero-modes gives a non-trivial phase factor in the Fourier transformed euclidean 4-point function. In Section 8, finally, we argue that this phase factor, analytically continued to Minkowski spacetime, may signal unusual behaviour at center-of-mass scattering energies of order M_W/α_w, where $\alpha_w \equiv g^2/4\pi$ stands for the fine-structure constant of the $SU(2)$ gauge fields.

2 Theory

In this section we briefly review the theory considered in this paper, mainly in order to establish notation. The theory considered can be viewed as a simplified version of the electroweak standard model, with $SU(2)$ gauge fields, a single isodoublet of complex scalars and N_f isodoublets of massless left-handed Weyl fermions. The euclidean action for this chiral $SU(2)$ Yang-Mills-Higgs theory [5, 6] is given by

$$A_{\text{YMH}} = \int_{\mathbb{R}^4} \mathrm{d}^4x \left[\frac{1}{4}\left(W^a_{\mu\nu}\right)^2 + |D_\mu\Phi|^2 + \lambda\left(|\Phi|^2 - \frac{v^2}{2}\right)^2 + \sum_{f=1}^{N_f} \Psi^\dagger_f \, i\,\bar{\sigma}^\mu D_\mu \Psi_f \right] , \tag{2.1}$$

with

$$\begin{aligned} D_\mu \,\cdot\; &\equiv \; (\,\partial_\mu + g\,W_\mu\,)\;\cdot \;, \\ W_{\mu\nu} &\equiv \; W^a_{\mu\nu}\,T^a \equiv \partial_\mu W_\nu - \partial_\nu W_\mu + g\,[\,W_\mu\,, W_\nu\,]\;, \\ W_\mu &\equiv \; W^a_\mu\;T^a \equiv W^a_\mu\,\frac{\tau^a}{2\,i}\;. \end{aligned}$$

Here, the τ^a, $a = 1, 2, 3$, denote the "weak isospin" Pauli matrices

$$\tau^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \tau^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \tau^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and the σ^μ, $\bar{\sigma}^\mu$, $\mu = 1, 2, 3, 4$, the spin matrices

$$\sigma^\mu \equiv (\,i\sigma^1, i\sigma^2, i\sigma^3, \mathbb{1}\,)\;, \quad \bar{\sigma}^\mu \equiv (-i\sigma^1, -i\sigma^2, -i\sigma^3, \mathbb{1}\,) = \sigma^{\mu\,\dagger}\;,$$

where the matrices σ^a have the same entries as the τ^a above. The chiral fermions are massless, whereas the three vector bosons W^a have mass $M_W \equiv \frac{1}{2}\,g\,v$ and the single Higgs scalar H mass $M_H \equiv \sqrt{2\,\lambda}\,v$. Except for the last two sections of this paper, we always consider a single flavor of chiral fermions ($N_f = 1$) and drop the flavor index f.

In order to obtain non-singular instanton solutions for the bosonic fields, the action A_{YMH} has to be supplemented by a constraint term A_{C} [7], which is the 4-dimensional integral of an appropriate local operator O_d of canonical mass dimension $d > 4$ (for example, $\mathrm{Tr}\,(W_{\mu\nu})^4$ with $d = 8$). See Ref. [2] for how this works in practice.

For later use, we introduce some further spinor conventions, which follow basically those of Ref. [8]. The euclidean Dirac matrices γ^μ, $\mu = 1, 2, 3, 4$, have anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\delta^{\mu\nu}$$

and can be represented by the following hermitian 4×4 matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} , \tag{2.2}$$

in terms of the 2×2 matrices σ^μ, $\bar{\sigma}^\mu$ given above. Defining the chirality operator

$$\gamma^5 \equiv \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

and the corresponding projection operators

$$P_L \equiv \frac{1}{2} \left(\mathbb{1} - \gamma^5 \right) , \quad P_R \equiv \frac{1}{2} \left(\mathbb{1} + \gamma^5 \right) ,$$

a general Dirac spinor Ψ_{D} can be written as the composite of two Weyl spinors

$$\Psi_{\mathrm{D}} = \begin{pmatrix} \Psi_{R\,\alpha} \\ \Psi_L{}^{\dot{\alpha}} \end{pmatrix} , \tag{2.3}$$

with indices $\alpha = 1,\, 2$, and $\dot{\alpha} = 1,\, 2$ (van der Waerden notation). For the spin matrices this gives the following index structure:

$$\sigma^{\mu\,\dot{\alpha}\alpha} , \quad \bar{\sigma}^\mu{}_{\alpha\dot{\alpha}} ,$$

and we refer to the Appendix A of Ref. [9] for further details.

The Dirac eigenvalue equation

$$i\,\gamma^\mu D_\mu\, \Psi_{\mathrm{D}} = \epsilon\, \Psi_{\mathrm{D}} \tag{2.4}$$

now reduces to two coupled equations for the Weyl spinors Ψ_L and Ψ_R. For vanishing eigenvalue ϵ, these equations decouple, giving the two fundamental Weyl equations

$$i\,\bar{\sigma}^\mu D_\mu^{(L)}\, \Psi_L = 0 \,, \quad i\,\sigma^\mu D_\mu^{(R)}\, \Psi_R = 0 \,. \tag{2.5}$$

The suffixes (L) and (R) on the covariant derivatives in (2.5) are to remind us that for *massless* fermions the gauge group representations of Ψ_L and Ψ_R may differ [10], which will be important in Sect. 5.

3 Bosonic fields

The fields considered in this paper have an axial $SO(2)$ symmetry, which makes it useful to introduce cylindrical coordinates defined in terms of the cartesian coordinates by

$$(\rho\cos\varphi, \rho\sin\varphi, z, \tau) \equiv (x^1, x^2, x^3, x^4) \ ,$$

and a triad of isospin matrices

$$\mathbf{u} \equiv \cos\varphi\, T^1 + \sin\varphi\, T^2 \ , \quad \mathbf{v} \equiv -\sin\varphi\, T^1 + \cos\varphi\, T^2 \ , \quad \mathbf{w} \equiv T^3 \ ,$$

defined in terms of the isospin $\frac{1}{2}$ representation $T^a \equiv \tau^a/(2\,i)$.

The bosonic fields of the constrained instanton $\mathrm{I}^\star$ are then given by

$$\begin{aligned}
g\,W_\rho &= \frac{1}{\rho}\,\{\,C_1\ \mathbf{u} + C_2\ \mathbf{v} + C_6\ \mathbf{w}\,\}\ , \quad g\,W_\varphi = \frac{1}{\rho}\,\{\,C_3\ \mathbf{u} + C_4\ \mathbf{v} + C_5\ \mathbf{w}\,\}\ ,\\
g\,W_z &= \frac{1}{z}\,\{\,C_7\ \mathbf{u} + C_8\ \mathbf{v} + C_9\ \mathbf{w}\,\}\ , \quad g\,W_\tau = \frac{1}{\tau}\,\{\,C_{10}\,\mathbf{u} + C_{11}\,\mathbf{v} + C_{12}\,\mathbf{w}\,\}\ ,\\
\Phi &= \frac{v}{\sqrt{2}}\begin{pmatrix}(\,iH_1 + H_2\,)\,e^{-i\varphi}\\ iH_3 + H_4\end{pmatrix}\ ,
\end{aligned} \tag{3.1}$$

where the signs of the coefficient functions C_5, C_8, C_{11}, H_1 and H_2 have been changed compared to Ref. [1] and H_4 has replaced H_0. An important characteristic of $\mathrm{I}^\star$ is a certain discrete symmetry D_1 (to be discussed further in the next section), which implies for the real coefficient functions C_i and H_j the following reflection properties:

$$\begin{aligned}
C_i(\rho,z,\tau) &= +\,C_i(\rho,-z,\tau)\ , \quad i = 1,4,5,7,10\ ,\\
C_i(\rho,z,\tau) &= -\,C_i(\rho,-z,\tau)\ , \quad i = 2,3,6,8,9,11,12\ ,\\
H_j(\rho,z,\tau) &= -\,H_j(\rho,-z,\tau)\ , \quad j = 1,4\ ,\\
H_j(\rho,z,\tau) &= +\,H_j(\rho,-z,\tau)\ , \quad j = 2,3\ .
\end{aligned} \tag{3.2}$$

Further boundary conditions on the coefficient functions, in particular those at infinity, can be found in Ref. [1]. As mentioned before, we have at this moment only an approximation [2] to the behavior of the coefficient functions $C_i(\rho,z,\tau)$, $H_j(\rho,z,\tau)$ and not the full numerical solution of the reduced field equations.

This completes our brief discussion of the constrained instanton $\mathrm{I}^\star$. For later use, we also give the bosonic fields of the well-known instanton $\mathrm{I}_{\mathrm{BPSTH}}$ [11, 12], or I for short,

$$g\,W_\mu = -f(x)\ \partial_\mu U\ U^{-1}\ , \quad \Phi = \frac{v}{\sqrt{2}}\,h(x)\,U\begin{pmatrix}0\\1\end{pmatrix}\ , \quad U = \hat{x}^\mu\,\tau^\mu\ , \tag{3.3}$$

with the radial coordinate $x \equiv \sqrt{x^\mu\, x^\mu}$, the unit-vector $\hat{x}^\mu \equiv x^\mu / x$ and the isospin matrices $\tau^\mu \equiv (i\vec{\tau}, \mathbb{1})$. These fields have $SO(4)$ rotation invariance and can certainly be brought to the axial form (3.1). Moreover, the radial profile functions f and h are known analytically in the limit $\tilde{\rho}\, v \to 0$

$$f(x) = h(x)^2 = \frac{x^2}{x^2 + \tilde{\rho}^2} \ , \tag{3.4}$$

with $\tilde{\rho}$ the scale parameter of the solution as fixed by the constraint (a tilde distinguishes this scale from the axial coordinate ρ). Note that the Higgs field vanishes at the origin, independently of the gauge, and that the BPST gauge fields (3.3), (3.4) are self-dual

$${}^\star W_{\kappa\lambda} \equiv \frac{1}{2}\, \epsilon_{\kappa\lambda\mu\nu}\, W_{\mu\nu} = + W_{\kappa\lambda} \ , \tag{3.5}$$

with $\epsilon_{\kappa\lambda\mu\nu}$ the completely antisymmetric pseudotensor ($\epsilon_{1234} = 1$). For the anti-instanton $\bar{\mathrm{I}}$ the matrix U in (3.3) is replaced by $U^\dagger$ and the resulting gauge fields are anti-self-dual, with a minus sign on the right-hand side of (3.5). The (anti-)self-duality property of the contrained BPSTH (anti-)instanton solution is lost for finite values of $\tilde{\rho}$ [7].

The topological charge (Pontryagin number)

$$Q \ \equiv \int_{\mathbb{R}^4} \mathrm{d}^4 x \ q \ \equiv \ \frac{1}{32\,\pi^2} \int_{\mathbb{R}^4} \mathrm{d}^4 x \ \ {}^\star W^a_{\kappa\lambda}\, W^a_{\kappa\lambda} \tag{3.6}$$

takes the value (-1) $+1$ for the BPSTH (anti-)instanton ($\bar{\mathrm{I}}$) I, whereas the topological charge is 0 for $\mathrm{I}^\star$. In fact, the instanton $\mathrm{I}^\star$ may be viewed as an unstable di-atomic molecule built out of an approximate instanton I and anti-instanton $\bar{\mathrm{I}}$. Note, however, that this picture has not yet been established rigorously. It follows, instead, from the observation that the $z = 0$ slice of $\mathrm{I}^\star$ corresponds to the static (time-independent) sphaleron $\mathrm{S}^\star$ [13], for which a numerical solution of the field equations has indeed established the molecule-like structure (and not the ring-like structure allowed in principle; see Fig. 1 of Ref. [1] for a sketch of the two alternatives). Henceforth, we take the molecule-like structure of $\mathrm{I}^\star$ for granted, with a distance $d^\star = \mathrm{O}(M_W^{-1})$ between the two Higgs zeros on the τ-axis.

4 Fermion Ansätze

The *Ansätze* for the fermionic fields follow directly from the continuous and discrete symmetries of $\mathrm{I}^\star$. The axial $SO(2)$ symmetry for an isodoublet of left-handed Weyl fermions is generated by the operator

$$K_3 \equiv -i\, \partial_\varphi + \frac{\sigma^3}{2} + \frac{\tau^3}{2}\, (P_L - P_R) \ , \tag{4.1}$$

with the projection operators P_L and P_R taking the values 1 and 0, respectively. Introducing the kets $|L, s_3, t_3\rangle$, with s_3 and t_3 the eigenvalues of $\sigma^3/2$ and $\tau^3/2$, respectively, the four $K_3 = 0$ eigenvectors are [2]

$$|L, 1/2, -1/2\rangle \; , \;\; |L, -1/2, 1/2\rangle \; , \;\; e^{+i\varphi}\,|L, -1/2, -1/2\rangle \; , \;\; e^{-i\varphi}\,|L, 1/2, 1/2\rangle \; . \tag{4.2}$$

Using the column vector notation

$$\begin{pmatrix} \begin{pmatrix} \times \\ \times \end{pmatrix}_{\text{isospin}} \\ \begin{pmatrix} \times \\ \times \end{pmatrix}_{\text{isospin}} \end{pmatrix}_{\text{spin}} \; ,$$

the $K_3 = 0$ eigenvectors (4.2) can be written as

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \; \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{+i\varphi} \end{pmatrix} , \; \begin{pmatrix} e^{-i\varphi} \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{4.3}$$

The general *Ansatz* for an axisymmetric left-handed Weyl isodoublet is then

$$\Psi_L = \begin{pmatrix} F_{L\uparrow}\, e^{-i\varphi} \\ G_{L\uparrow} \\ G_{L\downarrow} \\ F_{L\downarrow}\, e^{+i\varphi} \end{pmatrix} , \tag{4.4}$$

in terms of four complex functions of the axial coordinates ρ, z and τ (of course, $F_{L\uparrow}$ and $F_{L\downarrow}$ must be proportional to ρ for continuity of the fields).

We now turn to the role of the discrete symmetry of I*. As discussed in Ref. [1], this symmetry transformation D_1 consists of three parts:

1. the coordinate transformation $\rho \to \rho$, $\varphi \to \pi - \varphi$, $z \to -z$, $\tau \to \tau$;

2. the complex conjugation of the fields W, Φ, Ψ;

3. the global gauge transformation $W \to \Gamma\, W\, \Gamma^{-1}$, $\Phi \to \Gamma\, \Phi$, $\Psi \to \Gamma\, \Psi$, with parameter $\Gamma = -\mathbb{1}$ in the center of the gauge group $SU(2)$.

[2] These eigenvectors also appear in a different context, namely the static Z-string solution of the electroweak standard model. There, the four eigenvectors combine into pairs by the action of an additional reflection symmetry R_1, which is due to the 2-dimensional nature of the fields [14].

It is straightforward to check that the discrete symmetry D_1 of the scalar isodoublet

$$\Phi = \begin{pmatrix} F\, e^{-i\varphi} \\ G \end{pmatrix}$$

gives back the structure (3.1), (3.2) of the $\mathrm{I}^\star$ scalar fields. For the case of a fermion isodoublet, the *Ansatz* (4.4) is restricted as follows

$$\begin{aligned} F_{L\uparrow} &= i\,\frac{\rho\, z}{X^2}\, k_1 + \frac{\rho}{X}\, k_2 \,, \quad G_{L\uparrow} = i\, k_3 + \frac{z}{X}\, k_4 \,, \\ F_{L\downarrow} &= i\,\frac{\rho\, z}{X^2}\, l_1 \, + \frac{\rho}{X}\, l_2 \,, \quad G_{L\downarrow} = i\, l_3 \, + \frac{z}{X}\, l_4 \,, \end{aligned} \tag{4.5}$$

in terms of eight real functions k_j and l_j , $j = 1, \cdots , 4$, which are even in z

$$k_j(\rho, z, \tau) = k_j(\rho, -z, \tau) \,, \quad l_j(\rho, z, \tau) = l_j(\rho, -z, \tau) \,, \tag{4.6}$$

and the definition

$$X^2 \equiv x^2 + x_a^2 \equiv \rho^2 + z^2 + \tau^2 + x_a^2 \,,$$

for an arbitrary, but fixed, scale parameter x_a. The regular functions k_j, l_j have furthermore boundary conditions

$$\lim_{|x|\to\infty} k_j = \lim_{|x|\to\infty} l_j = 0 \tag{4.7}$$

and must give a normalizable spinor Ψ_L. This completes the *Ansatz* for the left-handed isodoublet fermion zero-mode in the $\mathrm{I}^\star$ background.

The right-handed isodoublet fermion zero-mode is constructed in the same way. The $K_3 = 0$ eigenvectors are now

$$|R, 1/2, 1/2\rangle \,, \quad |R, -1/2, -1/2\rangle \,, \quad e^{+i\varphi}\, |R, -1/2, 1/2\rangle \,, \quad e^{-i\varphi}\, |R, 1/2, -1/2\rangle \,. \tag{4.8}$$

Hence, the *Ansatz* is

$$\Psi_R = \begin{pmatrix} G_{R\uparrow} \\ F_{R\uparrow}\, e^{-i\varphi} \\ F_{R\downarrow}\, e^{+i\varphi} \\ G_{R\downarrow} \end{pmatrix} \,, \tag{4.9}$$

with the following restrictions from the discrete symmetry D_1:

$$\begin{aligned} F_{R\uparrow} &= i\,\frac{\rho\, z}{X^2}\, m_1 + \frac{\rho}{X}\, m_2 \,, \quad G_{R\uparrow} = i\, m_3 + \frac{z}{X}\, m_4 \,, \\ F_{R\downarrow} &= i\,\frac{\rho\, z}{X^2}\, n_1 \, + \frac{\rho}{X}\, n_2 \,, \quad G_{R\downarrow} = i\, n_3 \, + \frac{z}{X}\, n_4 \,, \end{aligned} \tag{4.10}$$

where the real functions m_j and n_j obey the same conditions (4.6), (4.7) as the functions k_j and l_j. Precisely which right-handed Weyl equation is solved by this *Ansatz* will be explained in the next section.

For future reference, the appropriate *Ansätze* for the fermion zero-modes of the 3-dimensional sphaleron $\mathrm{S}^\star$ [13] follow from the $\mathrm{I}^\star$ *Ansätze* by setting $z=0$ and implementing a further discrete symmetry D_2 ($z=0:\ \rho\to\rho,\ \varphi\to\varphi+\pi,\ \tau\to-\tau$).

5 Weyl and Dirac equations

Consider the left-handed Weyl equation corresponding to the action (2.1)

$$i\,\bar\sigma^\mu D_\mu\,\Psi_L = i\begin{pmatrix} D_\tau - iD_z & -e^{-i\varphi}(D_\varphi + iD_\rho) \\ e^{i\varphi}(D_\varphi - iD_\rho) & D_\tau + iD_z \end{pmatrix}\begin{pmatrix}\Psi_{L\uparrow}\\ \Psi_{L\downarrow}\end{pmatrix} = 0 \tag{5.1}$$

in the axisymmetric gauge field background (3.1). This can be evaluated readily for the fermion *Ansatz* (4.4), with the resulting four equations

$$\begin{aligned}
&\left[\left(\partial_\tau - \frac{C_9}{2\,z}\right) \mp i\left(\partial_z + \frac{C_{12}}{2\,\tau}\right)\right] F_{L{\uparrow \atop \downarrow}} - \left[\frac{1}{2\,\rho}\,(C_1 - C_4) \mp \frac{i}{2\,\rho}\,(C_2 + C_3)\right] F_{L{\downarrow \atop \uparrow}} - \\
&\left[i\left(\partial_\rho - \frac{C_5}{2\,\rho}\right) \pm \frac{C_6}{2\,\rho}\right] G_{L{\downarrow \atop \uparrow}} - \left[i\left(\frac{C_{10}}{2\,\tau} - \frac{C_8}{2\,z}\right) \pm \left(\frac{C_{11}}{2\,\tau} + \frac{C_7}{2\,z}\right)\right] G_{L{\uparrow \atop \downarrow}} = 0\,, \\
&\left[\left(\partial_\tau + \frac{C_9}{2\,z}\right) \mp i\left(\partial_z - \frac{C_{12}}{2\,\tau}\right)\right] G_{L{\uparrow \atop \downarrow}} - \left[\frac{1}{2\,\rho}\,(C_1 + C_4) \pm \frac{i}{2\,\rho}\,(C_2 - C_3)\right] G_{L{\downarrow \atop \uparrow}} - \\
&\left[i\left(\partial_\rho + \frac{1}{\rho} + \frac{C_5}{2\,\rho}\right) \mp \frac{C_6}{2\,\rho}\right] F_{L{\downarrow \atop \uparrow}} - \left[i\left(\frac{C_{10}}{2\,\tau} + \frac{C_8}{2\,z}\right) \mp \left(\frac{C_{11}}{2\,\tau} - \frac{C_7}{2\,z}\right)\right] F_{L{\uparrow \atop \downarrow}} = 0\,,
\end{aligned} \tag{5.2}$$

where for the case of the $\mathrm{I}^\star$ instanton the functions F_L and G_L have to be replaced by (4.5) and the gauge field functions C_i by the appropriate expressions in terms of the functions f_i of Ref. [1].

The result (5.2) illustrates the axial symmetry of the fermionic and bosonic fields, with the phases $\exp(\pm\, i\,\varphi)$ factoring out. Also, the z-reflection symmetry (3.2), (4.6) of $\mathrm{I}^\star$ is respected by (5.2), with the complex structure

$$\mathrm{Even} + i\,\mathrm{Odd} = 0 \tag{5.3}$$

for the first two equations and with the role of Even and Odd switched for the last two. All in all, the *Ansatz* (4.4), (4.5) has reduced the left-handed Weyl equation (5.1) in the

I* background to eight real partial differential equations (5.2) for the eight real functions $k_j(\rho,z,\tau)$ and $l_j(\rho,z,\tau)$. In short, the *Ansatz* is self-consistent.

The analysis of the right-handed Weyl equation (2.5) proceeds in the same manner. The only subtlety is to realize [10] that if Ψ_L has a gauge transformation employing the hermitian representation $\hat{T}^a$, then the corresponding $\Psi_R \equiv i\sigma^2\,\Psi_L^\star$ uses the *conjugate* representation $-\hat{T}^{a\mathrm{T}} = -\hat{T}^{a\,\star}$. (For $SU(2)$ isodoublets, these two representations are equivalent by unitary transformation with $i\tau^2$, which will be used later on.) As we have seen above, the theory (2.1) has left-handed fermions satisfying the Weyl equation (5.1) in an axisymmetric gauge field background, explicitely

$$i\,\bar\sigma^\mu D_\mu\,\Psi_L = 0\;, \quad D_\mu \equiv \partial_\mu - i\,g\,\hat{T}^a\,W^a_\mu\;, \tag{5.4}$$

with $\hat{T}^a \equiv \tau^a/2$. We are therefore interested in solving the independent equation

$$i\,\sigma^\mu \bar{D}_\mu\,\Psi_R = 0\;, \quad \bar{D}_\mu \equiv \partial_\mu + i\,g\,\hat{T}^{a\,\star}\,W^a_\mu\;, \tag{5.5}$$

with the same $\hat{T}^a \equiv \tau^a/2$ and gauge field background W^a_μ. This is accomplished by the fermion *Ansatz* (4.9), with the resulting four equations

$$\begin{aligned}
&\left[\left(\partial_\tau + \frac{C_9}{2\,z}\right) \pm i\left(\partial_z - \frac{C_{12}}{2\,\tau}\right)\right] F_{R}{}^{\uparrow}_{\downarrow} \;-\left[\frac{1}{2\,\rho}\,(C_1 - C_4) \mp \frac{i}{2\,\rho}\,(C_2 + C_3)\right] F_{R}{}^{\downarrow}_{\uparrow} \;+ \\
&\left[i\left(\partial_\rho - \frac{C_5}{2\,\rho}\right) \pm \frac{C_6}{2\,\rho}\right] G_{R}{}^{\downarrow}_{\uparrow} \;+\left[i\left(\frac{C_{10}}{2\,\tau} + \frac{C_8}{2\,z}\right) \pm \left(\frac{C_{11}}{2\,\tau} - \frac{C_7}{2\,z}\right)\right] G_{R}{}^{\uparrow}_{\downarrow} \;= 0\;, \\[2ex]
&\left[\left(\partial_\tau - \frac{C_9}{2\,z}\right) \pm i\left(\partial_z + \frac{C_{12}}{2\,\tau}\right)\right] G_{R}{}^{\uparrow}_{\downarrow} \;-\left[\frac{1}{2\,\rho}\,(C_1 + C_4) \pm \frac{i}{2\,\rho}\,(C_2 - C_3)\right] G_{R}{}^{\downarrow}_{\uparrow} \;+ \\
&\left[i\left(\partial_\rho + \frac{1}{\rho} + \frac{C_5}{2\,\rho}\right) \mp \frac{C_6}{2\,\rho}\right] F_{R}{}^{\downarrow}_{\uparrow} \;+\left[i\left(\frac{C_{10}}{2\,\tau} - \frac{C_8}{2\,z}\right) \mp \left(\frac{C_{11}}{2\,\tau} + \frac{C_7}{2\,z}\right)\right] F_{R}{}^{\uparrow}_{\downarrow} \;= 0\;.
\end{aligned} \tag{5.6}$$

Observe that for general coefficient functions C_i the two sets of equations (5.2) and (5.6) are not equivalent (consider, for example, the combinations of $(C_1 \pm C_4)/2\rho$ and $(C_{10}/2\tau \pm C_8/2z)$ appearing together).

Now turn to the Dirac equation (2.4) in the gauge field background (3.1), using the chiral representation (2.2) of the Dirac matrices. This eigenvalue problem

$$\begin{pmatrix} 0 & i\bar\sigma\cdot D \\ i\sigma\cdot D & 0 \end{pmatrix}\begin{pmatrix} \tilde\Psi_R \\ \tilde\Psi_L \end{pmatrix} = \epsilon\begin{pmatrix} \tilde\Psi_R \\ \tilde\Psi_L \end{pmatrix} \tag{5.7}$$

is solved by taking

$$\begin{pmatrix} \tilde\Psi_R \\ \tilde\Psi_L \end{pmatrix} = \begin{pmatrix} \tau^2\,\Psi_R \\ \Psi_L \end{pmatrix}\;, \tag{5.8}$$

with the wave functions on the right-hand side given by the *Ansätze* (4.4), (4.9). The simple relation

$$\tau^2\,\tau^a\,\tau^2 = -\,\tau^{a\,\star}$$

turns $D_\mu \tilde{\Psi}_R$ into $\tau^2 \bar{D}_\mu \Psi_R$, so that we can use our previous results for the right-handed Weyl equation (5.5). The eigenvalue equation (5.7) gives then eight coupled partial differential equations with the following structure:

$$\begin{aligned}
(\partial_\tau \mp C_9/2z)\, F_{\substack{L\\R}\uparrow}\,\cdots &= \mp\,\epsilon\, F_{\substack{R\\L}\uparrow}\;, & (\partial_\tau \pm C_9/2z)\, G_{\substack{L\\R}\uparrow}\,\cdots &= \pm\,\epsilon\, G_{\substack{R\\L}\uparrow}\;,\\
(\partial_\tau \mp C_9/2z)\, F_{\substack{L\\R}\downarrow}\,\cdots &= \pm\,\epsilon\, F_{\substack{R\\L}\downarrow}\;, & (\partial_\tau \pm C_9/2z)\, G_{\substack{L\\R}\downarrow}\,\cdots &= \mp\,\epsilon\, G_{\substack{R\\L}\downarrow}\;,
\end{aligned} \tag{5.9}$$

where the expressions on the left-hand sides are given by the corresponding Weyl equation results (5.2), (5.6).

As mentioned in the Introduction, if the coefficient functions C_i are taken [1, 2] to belong to a non-contractible loop (NCL) of bosonic field configurations (loop parameter $\omega \in [-3\,\pi/2,\, 3\,\pi/2]$) passing through the vacuum ($\omega = \pm\,3\,\pi/2$) and I* ($\omega = 0$), then the eigenvalues $\epsilon(\omega)$ of (5.9) exhibit spectral flow with an odd number of pairs of opposite levels crossing [3, 4]. This has been shown to follow from a 5-dimensional mod 2 Atiyah-Singer index theorem [3]. The energy levels cross through $\epsilon = 0$ in pairs, because each non-zero eigenvalue ϵ has a matching eigenvalue $-\epsilon$, as follows from the anticommutation of $i\gamma^\mu D_\mu$ and γ^5. As long as each pair of fermion zero-modes contains both chiralities, this agrees with the standard Atiyah-Singer index theorem (see [8] and references therein)

$$n_- - n_+ = Q\;, \tag{5.10}$$

where $n_\pm$ denotes the number of normalizable zero-modes of the isospin $\frac{1}{2}$ Dirac operator with chirality ± 1, since the configurations of the NCL have topological charge $Q = 0$. Most likely, there is a single pair of levels crossing through $\epsilon = 0$ precisely at $\omega = 0$ corresponding to the I* fields, which are distinguished on the NCL by their discrete symmetry D$_1$. If so, this implies that the reduced Weyl equations (5.2), (5.6) in the I* background each have a non-trivial solution.

6 Analytic results

Before we look for explicit solutions of the reduced Weyl equations (5.2), (5.6) in the $SO(2)$ symmetric background of the instanton I*, we must verify that the known results are reproduced in the $SO(4)$ symmetric background of the BPST instanton I. For the self-dual gauge fields (3.3), (3.4) the single left-handed isodoublet fermion zero-mode [12] is indeed recovered

$$F_{L\uparrow}{}^{(\mathrm{I})} = F_{L\downarrow}{}^{(\mathrm{I})} = 0\;, \quad G_{L\uparrow}{}^{(\mathrm{I})} = -\,G_{L\downarrow}{}^{(\mathrm{I})} \propto (x^2+\tilde{\rho}^{\,2})^{-3/2}\;, \tag{6.1}$$

whereas there is no normalizable solution for the right-handed Weyl equation. This agrees with the index theorem (5.10) for $Q=1$ and also with the stronger statement $n_- = Q$, $n_+ = 0$ valid for self-dual gauge fields [8]. Similar results are obtained for the BPST anti-instanton $\bar{\mathrm{I}}$, with the single right-handed isodoublet fermion zero-mode

$$F_{R\uparrow}{}^{(\bar{\mathrm{I}})} = F_{R\downarrow}{}^{(\bar{\mathrm{I}})} = 0\;, \quad G_{R\uparrow}{}^{(\bar{\mathrm{I}})} = +\,G_{R\downarrow}{}^{(\bar{\mathrm{I}})} \propto (x^2+\tilde{\rho}^{\,2})^{-3/2}\;, \tag{6.2}$$

but without normalizable solution for the corresponding left-handed Weyl equation. We now turn to the case of the (non-self-dual) constrained instanton $\mathrm{I}^\star$ and, pending the complete numerical solution, study three aspects of the problem analytically.

6.1 Asymptotics

In this first subsection we investigate the asymptotic behaviour of the left-handed fermionic fields in the $\mathrm{I}^\star$ background. Setting in the *Ansatz* (4.4)

$$F_{L\uparrow} = F_{L\downarrow} = 0\;, \quad G_{L\uparrow} = -\,G_{L\downarrow}\;, \tag{6.3}$$

and using the appropriate boundary conditions on the gauge field functions C_i [1], the Weyl equations (5.2) give asymptotically three partial differential equations (PDEs)

$$\left(\partial_\rho + \frac{4\,\rho}{x^2}\right) G_{L\uparrow} = \left(\partial_z + \frac{4\,z}{x^2}\right) G_{L\uparrow} = \left(\partial_\tau + \frac{4\,\tau}{x^2}\right) G_{L\uparrow} = 0\;, \tag{6.4}$$

together with a fourth equation vanishing trivially at infinity

$$\frac{C_6}{2\,\rho} - \frac{C_7}{2\,z} - \frac{C_{11}}{2\,\tau} = 0\;. \tag{6.5}$$

Taking in the *Ansatz* (4.5) for $G_{L\uparrow}$ the functions

$$k_3 = k_3(x^2)\;, \quad k_4 = 0\;, \tag{6.6}$$

the three PDEs (6.4) are reduced to a single ordinary differential equation (ODE)

$$\left(\partial_{x^2} + \frac{2}{x^2}\right) k_3(x^2) = 0\;. \tag{6.7}$$

This ODE gives the asymptotic solution

$$\Psi_L^{\dot{\alpha}n\ (\mathrm{I}^\star)} \propto \epsilon^{\dot{\alpha}n}\, x^{-4}\ , \tag{6.8}$$

where the spin and isospin indices have been made explicit. The solution (6.8) should be a reasonable approximation for $|x| >> \max(M_W^{-1}, d^\star/2)$, since the theory considered has massive vector fields ($M_W = \frac{1}{2}\, g\, v$), which rapidly approach pure gauge configurations.

The index theorem (5.10), for $Q = 0$, implies the existence of a corresponding right-handed fermion zero-mode. The asymptotic right-handed fields do not have an as simple form as the left-handed ones and we will not give the explicit solution here. The somewhat surprising difference between the left- and right-handed fermion zero-modes can be traced back to the structure of the boundary conditions of the gauge fields at infinity, as determined by the $SU(2)$ matrix $U^\infty = (\hat{x}\cdot\tau)\ i\tau^3\ (\hat{x}\cdot\tau)^\dagger$ with the group factors in this particular order [1]. But the position of the angle-dependent group factors in U^∞ can be interchanged by a gauge transformation. The solution Ψ_R has therefore the same asymptotic behaviour $\mathrm{O}(x^{-4})$ as the Ψ_L solution above. These fermion zero-modes approach zero faster than the free fermion propagator $S_\mathrm{F} = \mathrm{O}(x^{-3})$, which implies that the effective I* vertex has derivative couplings (momentum factors), in contrast to the case of 't Hooft's effective I vertex [12, 15].

6.2 Symmetry axis

In this second subsection we look for solutions of the Weyl equations on the symmetry axis of I*, which in our coordinate system coincides with the τ-axis ($\rho = z = 0$). The general behaviour of the gauge field functions C_i on the τ-axis is known [1], for example

$$C_1 = C_4 = 4\frac{\rho}{X} f_1(0,0,\tau)\ , \quad C_9 = 4\frac{z}{X} f_9(0,0,\tau)\ , \tag{6.9}$$

with $X^2 = \tau^2 + x_a^2$ and x_a an arbitrary fixed scale parameter. The Weyl equations (5.2) for the left-handed fermions are then reduced to

$$\begin{aligned} &\left(\partial_\tau - \frac{C_9}{2\,z} \mp i\,\partial_z\right) F_{L\updownarrow} - i\,\partial_\rho\, G_{L\downarrow\uparrow}\ \Big|_{\rho=z=0} = 0\ , \\ &\left(\partial_\tau + \frac{C_9}{2\,z} \pm i\,\partial_z\right) G_{L\downarrow\uparrow} - \frac{C_1}{\rho}\, G_{L\updownarrow} - i\left(\partial_\rho + \frac{1}{\rho}\right) F_{L\updownarrow}\ \Big|_{\rho=z=0} = 0\ . \end{aligned} \tag{6.10}$$

Setting in the *Ansatz* (4.5)

$$k_2 = l_2 = k_4 = l_4 = 0\ , \quad k_3 = -\,l_3\ \Big|_{\rho=z=0}\ , \tag{6.11}$$

with $k_1(0,0,\tau)$ and $l_1(0,0,\tau)$ arbitrary but finite, there remains a single ODE

$$\partial_\tau\, k_3(0,0,\tau) + K_+(\tau)\, k_3(0,0,\tau) = 0\ , \tag{6.12}$$

with

$$K_\pm(\tau) \equiv \frac{2}{X}\, [\, 2\, f_1(0,0,\tau) \pm f_9(0,0,\tau)\,]\ . \tag{6.13}$$

The asymptotic behaviour of the gauge field functions on the τ-axis is [1] $f_1 \to \tau/X$ and $f_9 \to 0$, so that (6.12) gives asymptotically

$$k_3(0,0,\tau) \propto \tau^{-4}\ , \tag{6.14}$$

which agrees with (6.8). The general solution of (6.12) is

$$k_3(0,0,\tau) = k_3(0,0,0)\ \exp\left[\, - \int_0^\tau \mathrm{d}\tau'\, K_+(\tau')\,\right]\ , \tag{6.15}$$

in terms of the functions f_1 and f_9 on the τ-axis.

Unfortunatly, the exact functions f_1 and f_9 are not known. We can use, instead, the approximate functions from the non-contractible loop construction as summarized in the Appendix A of Ref. [1]. This approximation gives on the τ-axis

$$f_1(0,0,\tau) = f(0,\tau)\,\frac{X}{2}\,\frac{2\,\tau}{(\tau^2 - d^2/4)}\ ,\quad f_9(0,0,\tau) = f(0,\tau)\,\frac{X}{2}\,\frac{d}{(\tau^2 - d^2/4)}\ , \tag{6.16}$$

in terms of a single function $f = f(r,\tau)$, with $r^2 \equiv \rho^2 + z^2$, which has the following symmetry property and boundary conditions

$$f(r,\tau) = f(r,-\tau)\ ,\quad f(0,\pm d/2) = 0\ ,\quad \lim_{|x|\to\infty} f(r,\tau) = 1\ . \tag{6.17}$$

A useful trial function for f comes from the product of BPST functions (3.4)

$$f(r,\tau) = \frac{x_+^2}{x_+^2 + \tilde\rho^{\,2}}\ \frac{x_-^2}{x_-^2 + \tilde\rho^{\,2}}\ , \tag{6.18}$$

with $x_\pm^2 \equiv r^2 + (\tau \pm d/2)^2$. The result for $K_+(\tau)$ as defined in (6.13),

$$K_+(\tau) = f(0,\tau)\,\frac{(4\,\tau + d)}{(\tau^2 - d^2/4)}\ , \tag{6.19}$$

has no definite symmetry properties under $\tau \to -\tau$, nor has the solution $k_3(\tau)$ of (6.12). In fact, the solution (6.15) in this approximation can be seen to have a major peak at $\tau = +\,d/2$ and a minor one at $\tau = -\,d/2$. The relative height of this "satellite" peak

$$\frac{k_3(0,0,-d/2)}{k_3(0,0,+d/2)} = \exp\left[\int_{-d/2}^{+d/2} \mathrm{d}\tau'\, K_+(\tau')\right] , \tag{6.20}$$

evaluated with (6.18), (6.19), vanishes for $\tilde\rho \to 0$ and approaches unity for $\tilde\rho \to \infty$.

The same analysis for the right-handed Weyl equation (5.6), setting in the *Ansatz* (4.10)

$$m_2 = n_2 = m_4 = n_4 = 0\;, \quad m_3 = -\,n_3\;\Big|_{\rho=z=0}\;, \tag{6.21}$$

leads to the ODE

$$\partial_\tau\, m_3(0,0,\tau) + K_-(\tau)\, m_3(0,0,\tau) = 0\;, \tag{6.22}$$

with $K_-(\tau)$ as defined by (6.13). The asymptotic right-handed solution $m_3(0,0,\tau)$ goes as τ^{-4}, just as for the left-handed solution (6.14). But the major peak of $m_3(0,0,\tau)$, for the approximate functions (6.16), now can be seen to occur at $\tau = -\,d/2$ and the minor one at $\tau = +\,d/2$, which is the mirror image of the left-handed fermionic fields.

The discrete symmetry D_2 of $\mathrm{I}^\star$ also requires [1] the exact function $f_1(0,0,\tau)$ to be antisymmetric in τ. Assuming furthermore the exact function $f_9(0,0,\tau)$ to be strictly symmetric in τ, the equations (6.12) and (6.22) imply the following relation between the left- and right-handed fermion zero-modes:

$$|\Psi_L(0,0,\tau)| = |\Psi_R(0,0,-\tau)|\;, \tag{6.23}$$

with the normalization at the origin taken equal. Most likely, this is realized with Ψ_L concentrated around $\tau = +\,d^\star/2$ and Ψ_R around $\tau = -\,d^\star/2$, at least for small enough values of the scale parameter $\tilde\rho$. Anyway, for non-vanishing $f_9(0,0,\tau)$ the left- and right-handed fermion zero-modes on the τ-axis are certainly different, since they solve, under the same boundary conditions, different equations, namely (6.12) and (6.22). This difference between left- and right-handed fermion zero-modes of $\mathrm{I}^\star$ may have important physics implications, as will be discussed in Sects. 7 and 8.

6.3 Approximate solution

In this third and last subsection on analytic results, we obtain an approximation to the left-handed fermion zero-mode of $\mathrm{I}^\star$ valid in the limit $\tilde\rho\, v \to 0$ and $\lambda \sim g^2 \to 0$. The idea is to enlarge the classical theory (2.1) temporarily by introducing Wino fields $\Lambda^{\dot\alpha a}$ with appropriate supersymmetric interactions.[3] The extended theory [9] involves the vector superfield $\hat{V}$ (component fields in the Wess-Zumino gauge: W^a_μ, $\Lambda^{\dot{\alpha}a}$, D^a) and the scalar superfield $\hat{\Phi}$ (component fields: $\Psi_L^{\dot{\alpha}n}$, Φ^n, F^n), where the auxiliary fields D^a and F^n can be eliminated on-shell.

The purely bosonic instanton I* remains a solution of the extended theory in the limiting case mentioned at the beginning of this subsection. Furthermore, a global supersymmetry transformation (parameter ξ_α) does not change these bosonic fields, but does generate fermionic fields Ψ_L and Λ^a, which automatically solve the field equations, in particular [9]

$$i\,\bar{\sigma}^\mu D_\mu\,\Psi_L = \sqrt{2}\,g\,\Lambda^{a\,\dagger}\,T^a\,\Phi\;. \tag{6.24}$$

The right-hand side of (6.24) vanishes by duality [8, 16] close enough to the $\tau = +\,d^\star/2$ core (positive topological charge density q), whereas it is only suppressed by the Higgs zero close enough to the $\tau = -\,d^\star/2$ core (negative q). From the explicit supersymmetry transformation [9] we obtain, therefore, an approximate solution of the left-handed Weyl equation (2.5) in the I* background

$$\Psi^{\;\dot{\alpha}n}_{L\,\mathrm{approx}} = A_L\; i\,\sigma^{\mu\,\dot{\alpha}1}\,\xi_1\;(D_\mu\,\Phi)^n\;\;, \tag{6.25}$$

with $A_L = A_L(\rho, z, \tau)$ an additional amplitude factor which can become small if necessary and with ξ_1 one component of a constant spinor ξ_α which provides the proper normalization and mass dimension of Ψ_L.

It is straightforward to verify that (6.25) respects the continuous and discrete symmetries of the *Ansatz* (4.4), (4.5), provided $A_L(\rho, z, \tau)$ is even in z. This approximation thus reduces the eight unknown *Ansatz* functions k_j and l_j to one, A_L. Concretely, the functions k_j, l_j are replaced by A_L and specific combinations of the known bosonic coefficient functions C_i, H_j and their derivatives.

The approximate solution for the right-handed fermion zero-mode has essentially the same structure as (6.25), now with the operator $\bar{\sigma}\cdot D$ and amplitude function A_R. One expects these approximations, with suitable amplitude functions A_L and A_R, to be quite reasonable as long as $\tilde{\rho} << d^\star/2$, so that the cores are well separated. More importantly, the appearance of distinct composite fields $\sigma\cdot D\,\Phi$ and $\bar{\sigma}\cdot D\,\Phi$ explains, in a way, the intrinsic difference between the left- and right-handed fermion zero-modes found in the previous subsection.

This completes the discussion of the I* fermion zero-modes. We now turn to one possible application.

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[3] See Ref. [16] for a related discussion of the fermion zero-modes due to the BPSTH instanton in genuinely supersymmetric gauge theories. Note that also I* may be relevant to dynamical supersymmetry breaking.

7 Euclidean Green's function

In this section and the next, the theory (2.1) is specialized to $N_f = 2$ flavors of massless left-handed fermions. At any rate, the $N_f = 1$ theory suffers from a global (non-perturbative) gauge anomaly and is, most likely, inconsistent [3]. Consider, then, the following euclidean 4-point Green's function evaluated by functional integration:

$$\begin{aligned} G_{\mathrm{E}}(x_1,x_2,x_3,x_4) \;\; &\equiv \;\; Z^{-1} \int \Big[DW\, D\Phi\, D\Psi_1\, D\Psi_1^{\dagger}\, D\Psi_2\, D\Psi_2^{\dagger} \Big] \\ &\quad \times \exp\big(- A_{\mathrm{YMH}}\,[\,W,\, \Phi,\, \Psi_f\,]\,\big)\, \Psi_1(x_1)\, \Psi_2(x_2)\, \Psi_1^{\dagger}(x_3)\, \Psi_2^{\dagger}(x_4) \;\; , \end{aligned} \tag{7.1}$$

with the constant normalization factor Z, the functional measure $[DW \,\cdots]$ including the necessary gauge-fixing factors and the euclidean action functional A_{YMH} as given by (2.1). Spin and isospin indices in (7.1) have been suppressed. (Note that our choice of Green's function is by no means exclusive; an alternative to (7.1) would be the 4-point function in terms of gauge invariant composite fields $\Phi^{\dagger}\Psi_f$ and $\Psi_f^{\dagger}\Phi$.)

The only hope of doing the functional integral analytically, at least in weak coupling, appears to be the saddle-point approximation. But in addition to the vacuum stationary point ($W = \Psi_f = \Psi_f^{\dagger} = 0$, $\Phi =$ constant), which gives rise to the standard Feynman perturbation theory [17, 18], there is now another stationary point to consider, namely the constrained instanton $\mathrm{I}^{\star}$. The leading non-perturbative contribution to (7.1) follows from inserting the classical $\mathrm{I}^{\star}$ fields and the corresponding fermion zero-modes. Of course, there is still the integration over the scale parameter $\tilde{\rho}$ of the solution and the (bosonic) collective coordinates, viz. the over-all position of the instanton $\mathrm{I}^{\star}$, the orientation of the $\mathrm{I}^{\star}$ axis (in Sect. 3 taken to be along the τ-axis) and the combined global gauge and custodial [19] $SU(2)$ transformations of the $\mathrm{I}^{\star}$ fields. The result is

$$\begin{aligned} G_{\mathrm{E}}^{(\mathrm{I}^{\star})}(x_1,x_2,x_3,x_4) \;\; &\propto \;\; \int_0^{\infty} \mathrm{d}\tilde{\rho} \int_{\mathbb{R}^4} \mathrm{d}^4 x_{\,\mathrm{I}^{\star}} \int_{S^3} \mathrm{d}^3\Omega_{\mathrm{I}^{\star}} \int_{SU(2)} \mathrm{d}^3 U_{\mathrm{I}^{\star}} \;\; n(\tilde{\rho}) \\ &\quad \times \exp\Big(- A_{\mathrm{YMH}}^{(\mathrm{I}^{\star})}(\tilde{\rho}) \Big) \;\; \Psi_1^{(\mathrm{I}^{\star})}(x_1 - x_{\,\mathrm{I}^{\star}})\, \Psi_2^{(\mathrm{I}^{\star})}(x_2 - x_{\,\mathrm{I}^{\star}}) \\ &\quad \times \Psi_1^{\dagger\,(\mathrm{I}^{\star})}(x_3 - x_{\,\mathrm{I}^{\star}})\, \Psi_2^{\dagger\,(\mathrm{I}^{\star})}(x_4 - x_{\,\mathrm{I}^{\star}}) \;\; , \end{aligned} \tag{7.2}$$

with the factor n containing the Jacobian of the scale parameter $\tilde{\rho}$ [7, 16]. The Ψ_f in the integrand here correspond to the fermion zero-modes (4.4), (4.5), with the dependence on $\tilde{\rho}$ suppressed, and the $\Psi_f^{\dagger}$ follow from (4.9), (4.10) by evaluating $\Psi_{Rf}^{\mathrm{T}}\, i\sigma^2$, as discussed in Sect. 5. In principle, it is now straightforward to take the Fourier transform of this 4-point function.

At this point it may be of interest to compare our result with earlier calculations, which employ an *approximate* stationary point consisting of a BPSTH instanton and anti-instanton pair $\mathrm{I}\,\bar{\mathrm{I}}$; see [20] and references therein. The result corresponding to (7.2) is given by Eq. (2.22) of Ref. [20]

$$\begin{aligned} G_{\mathrm{E}}^{(\mathrm{I}\bar{\mathrm{I}})}(x_1,x_2,x_3,x_4) \;\propto\; & \int \mathrm{d}\tilde{\rho}_{\mathrm{I}}\, \mathrm{d}\tilde{\rho}_{\bar{\mathrm{I}}}\, \mathrm{d}^4x_{\mathrm{I}}\, \mathrm{d}^4x_{\bar{\mathrm{I}}}\, \mathrm{d}^3U_{\mathrm{I}}\, \mathrm{d}^3U_{\bar{\mathrm{I}}}\;\; n_{\mathrm{I}}(\tilde{\rho}_{\mathrm{I}})\, n_{\bar{\mathrm{I}}}(\tilde{\rho}_{\bar{\mathrm{I}}}) \\ & \times \exp\Big(- A_{\mathrm{YMH}}^{(\mathrm{I}\bar{\mathrm{I}})}\Big(\, \tilde{\rho}_{\mathrm{I}}\,,\, \tilde{\rho}_{\bar{\mathrm{I}}}\,,\, x_{\mathrm{I}} - x_{\bar{\mathrm{I}}}\,,\, U_{\mathrm{I}}U_{\bar{\mathrm{I}}}^{-1}\Big)\Big) \\ & \times \Psi_1^{(\mathrm{I})}(x_1 - x_{\mathrm{I}})\, \Psi_2^{(\mathrm{I})}(x_2 - x_{\mathrm{I}})\, \Psi_1^{\dagger\,(\bar{\mathrm{I}})}(x_3 - x_{\bar{\mathrm{I}}}) \\ & \times \Psi_2^{\dagger\,(\bar{\mathrm{I}})}(x_4 - x_{\bar{\mathrm{I}}}) \;\;, \end{aligned} \tag{7.3}$$

with U_{I} a global isospin rotation (and custodial [19] symmetry transformation) of the I fields as given in (3.3), (3.4), where x^μ is replaced by $(x^\mu - x^\mu_{\mathrm{I}})$ and $\tilde{\rho}$ by $\tilde{\rho}_{\mathrm{I}}$, and similarly for $\bar{\mathrm{I}}$. Two remarks are in order. First, the fermion wave functions in (7.3) are taken to be the standard functions (6.1) for the BPSTH instanton, up to a global isospin rotation, and similarly (6.2) for the anti-instanton. This is certainly a reasonable approximation close to the cores, but asymptotically (far away from both instanton and anti-instanton) we expect the correct fermion wave functions to go as $\mathrm{O}(x^{-4})$, just as found in Sect 6.1. Second, the unrestricted integration over the instanton positions x_{I} and $x_{\bar{\mathrm{I}}}$ in (7.3) is problematic, since an overlapping $\mathrm{I}\,\bar{\mathrm{I}}$ pair in the attractive channel loses the fermion zero-modes altogether [4]. The solution would be to fix the relative isospin rotation to the repulsive channel (for example $U_{\mathrm{I}}\, U_{\bar{\mathrm{I}}}^{-1} = i\tau^3$ for $x^\mu_{\mathrm{I}} - x^\mu_{\bar{\mathrm{I}}} = d\,\delta^{\mu 4}$), for which case there are, most likely, still fermion zero-modes, as follows from the discussion on spectral flow at the end of Sect. 5. This amounts to integrating over a particular "mountain ridge", which has its minimum ("mountain pass") for a configuration in the neighbourhood of $\mathrm{I}^\star$ (see, for example, Fig. 1a of Ref. [2]). With our expression (7.2) we seem to have found (by solving the classical field equations) the *lowest possible* "mountain pass", namely the exact stationary point $\mathrm{I}^\star$.

Returning to the Fourier transform of the 4-point function (7.2), we specialize to forward momenta $p_1 = p_3$ and $p_2 = p_4$. Also, the corresponding isospin indices are matched ($n_1 = n_3$ and $n_2 = n_4$) and averaged over. Ultimately, this Green's function will be relevant to the forward elastic scattering amplitude, which in turn controls the total cross-section for this process. The crucial observation now is that the expected relative shift of the Ψ_f and $\Psi_f^\dagger$ zero-modes (see Sect. 6.2) gives rise to a non-trivial momentum phase factor in the integrand. Assuming the average Ψ_f and $\Psi_f^\dagger$ fermion wave functions to be displaced by a distance $D^\star$ along the symmetry axis (the τ-axis in our original coordinate system), the combined exponential factor in the integrand is

$$\exp\left[\, i\, E\, D^\star\, \cos\theta - A^\star_{\mathrm{YMH}}\,\right] \;, \tag{7.4}$$

with $A^\star_{\rm YMH} = A^\star_{\rm YMH}(\tilde\rho)$ the bosonic action of the constrained instanton $\mathrm{I}^\star$, E the norm of the total momentum vector $p_1^\mu + p_2^\mu$, and θ the angle between this vector and the $\mathrm{I}^\star$-axis. Performing the integration over the $\mathrm{I}^\star$ orientation gives the approximate result

$$\tilde G_{\rm E}^{(\mathrm{I}^\star)}(p_1,p_2,p_1,p_2) \;\propto\; \int_{\tilde\rho_{\rm c}}^\infty {\rm d}\tilde\rho \; N \; \exp\left(-A^\star_{\rm YMH}\right) \, \frac{J_1(E\,D^\star)}{E\,D^\star} \; [\,1+{\rm O}(\tilde\rho\,M_W)\,] \;, \qquad (7.5)$$

with E the euclidean invariant $|p_1+p_2|$, J_1 the Bessel function of order 1, and $N = N(\tilde\rho,|p_1|,|p_2|)$ a factor containing both the Jacobian of the scale parameter $\tilde\rho$ and the remaining fermion wave functions. The precise form of the integrand in (7.5) depends on the details of the fermion zero-mode wave functions in (7.2) and we can only give its generic structure for small enough values of the scale parameter $\tilde\rho$ (as indicated symbolically by the square bracket term in the integrand). Generally speaking, the convergence of the integral (7.5) at the upper end is assured by the Higgs contribution ${\rm O}(\tilde\rho^2\,v^2)$ to the action, whereas the integral at the lower end has a temporary cut-off $\tilde\rho_{\rm c}$ (quantum corrections are expected [12] to bring in further factors of $\tilde\rho$, thereby assuring the convergence).

The effective fermion shift parameter $D^\star = D^\star(\tilde\rho,|p_1|,|p_2|,E)$ is implicitely defined by (7.2) and (7.5). The value of $D^\star$ is expected to be close to that of the distance $d^\star = d^\star(\tilde\rho)$ between the Higgs zeros of $\mathrm{I}^\star$, at least for small enough values of the scale parameter $\tilde\rho$ and the momenta p_1, p_2. (For finite values of $\tilde\rho$, the fermion zero-mode wave functions develop a satellite peak (6.20), which tends to reduce the value of $D^\star$.) As a first approximation, we consider N and $D^\star$ in (7.5) to depend only on the scale parameter $\tilde\rho$ and replace $D^\star$ by $d^\star(\tilde\rho)$. This approximation will be used in the next section.

8 Threshold energy

In this last section we discuss possible implications of the non-perturbative euclidean 4-point Green's function found above. The scale integration in (7.5), with $D^\star$ replaced by $d^\star(\tilde\rho)$, can be performed either numerically or by saddle-point approximation [4]. Making, afterwards, the analytic continuation from euclidean to minkowskian momenta ($E \to i\sqrt{s}$), results for $\sqrt{s}\,d^\star >> 1$ in a leading exponential factor

$$\exp\left[\,\sqrt{s}\,d^\star(\hat\rho) - A^\star_{\rm YMH}(\hat\rho)\,\right] \;, \qquad (8.1)$$

with $\hat\rho$ the dominant value of $\tilde\rho$ in the integration (7.5), for which we take $\hat\rho\,v \sim 1$ (or at least $\hat\rho\,M_W << 1$). The exponential suppression in (8.1) disappears for large enough center-of-mass scattering energy

$$(\sqrt{s})_{\rm threshold} \sim \frac{A^\star_{\rm YMH}(\hat\rho)}{d^\star(\hat\rho)} = \left(\frac{A^\star_{\rm YMH}(\hat\rho)}{16\,\pi^2/g^2}\right)\,\left(\frac{2\,M_W^{-1}}{d^\star(\hat\rho)}\right)\;\tilde E_{{\rm S}^\star}\;, \qquad (8.2)$$

with the definition

$$\tilde{E}_{\mathrm{S}^\star} \equiv 2\,\pi\,\frac{M_W}{\alpha_w}\,,$$

which is close to the actual energy of the sphaleron $\mathrm{S}^\star$ [13]. The energy $\tilde{E}_{\mathrm{S}^\star}$ is approximately 20 TeV in the electroweak standard model.[4]

The main uncertainty in (8.2) comes from the value of the core distance $d^\star(\hat{\rho})$, but $2\,M_W^{-1}$ seems to be a reasonable first estimate. Recall, however, that we have made a drastic simplification by setting the effective fermion shift parameter $D^\star$ equal to the distance $d^\star$ between the zeros of the Higgs field. The ultimate high-energy behaviour remains unclear, as long as the exact fermion zero-mode wave functions and the resulting shift parameter $D^\star$ have not been determined. Still, the estimate (8.2) may be expected to give the energy scale for non-perturbative contributions to the Green's functions.

The particular non-perturbative contribution (7.5) to the forward Green's function appears to be relevant to both fermion number $(B+L)$ conserving and non-conserving [15] processes. Indeed, there exists no suitable fermion number projection operator which can be used to "split" the $\mathrm{I}^\star$ contribution to the forward Green's function (except, possibly, in the low-energy limit; cf. Ref. [20]), because of the simple fact that in a theory with a fermion number anomaly the fermion number is in general not well defined. Given this new, unexpected contribution to the Green's function, it is conceivable that chiral Yang-Mills-Higgs theory above the "non-perturbative threshold" (8.2) is radically different from the standard picture based on low-order Feynman perturbation theory.

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[4] $\mathrm{I}^\star$-like configurations may also play a role, though perhaps a less dramatic one, in non-chiral (vector-like) Yang-Mills theories such as QCD.

References

[1] F. Klinkhamer and J. Weller, Nucl. Phys. B 481 (1996) 403.

[2] F. Klinkhamer, Nucl. Phys. B 407 (1993) 88.

[3] E. Witten, Phys. Lett. B 117 (1982) 324.

[4] F. Klinkhamer, Nucl. Phys. B 376 (1992) 255.

[5] C. Yang and R. Mills, Phys. Rev. 96 (1954) 191.

[6] P. Higgs, Phys. Rev. 145 (1966) 1156.

[7] I. Affleck, Nucl. Phys. B 191 (1981) 429.

[8] R. Jackiw and C. Rebbi, Phys. Rev. D 16 (1977) 1052.

[9] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, 1983), Chapter 7.

[10] M. Peskin and D. Schroeder, *Introduction to Quantum Field Theory* (Addison-Wesley, Reading, 1995), Chapter 19.

[11] A. Belavin, A. Polyakov, A. Schwartz and Yu. Tyupkin, Phys. Lett. B 59 (1975) 85.

[12] G. 't Hooft, Phys. Rev. D 14 (1976) 3432; (E) D 18 (1978) 2199.

[13] F. Klinkhamer, Nucl. Phys. B 410 (1993) 343.

[14] F. Klinkhamer and C. Rupp, Nucl. Phys. B 495 (1997) 172.

[15] G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8.

[16] I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. B 241 (1984) 493.

[17] R. Feynman, Phys. Rev. 76 (1949) 749, 769; 80 (1950) 440.

[18] G. 't Hooft, Nucl. Phys. B 33 (1971) 173; B 35 (1971) 167.

[19] M. Shifman and A. Vainshtein, Nucl. Phys. B 362 (1991) 21.

[20] V. Khoze and A. Ringwald, Nucl. Phys. B 355 (1991) 351.