KA-TP-2-'97
May '97

On Sums of Powers of Zeros of Polynomials *

Wolfdieter L a n g **

Institut für Theoretische Physik
Universität Karlsruhe
Kaiserstrasse 12, D-76128 Karlsruhe
Germany

ABSTRACT

Due to Girard's (sometimes called Waring's) formula the sum of the $r-$th power of the zeros of every one variable polynomial of degree N, $P_N(x)$, can be given explicitly in terms of the coefficients of the monic $\tilde{P}_N(x)$ polynomial. This formula is closely related to a known $N-1$ variable generalisation of Chebyshev's polynomials of the first kind, $T_r^{(N-1)}$. The generating function of these power sums (or moments) is known to involve the logarithmic derivative of the considered polynomial. This entails a simple formula for the Stieltjes transform of the distribution of zeros. Perron-Stieltjes inversion can be used to find this distribution, *e.g.* for $N \to \infty$.

Classical orthogonal polynomials are taken as examples. The results for ordinary Chebyshev $T_N(x)$ and $U_N(x)$ polynomials are presented in detail. This will correct a statement about power sums of zeros of Chebyshev's $T-$polynomials found in the literature. For the various cases (Jacobi, Laguerre, Hermite) these moment generating functions provide solutions to certain Riccati equations.

AMS MSC numbers: 30C15, 33C50, 34L30
PACS numbers: 02.50.Sk, 02.10.Nj 02.30.Gp

* The complete ps-file is also available via www under
http://www-itp.physik.uni-karlsruhe.de/~wl/preprints.html

** E-mail address: wolfdieter.lang@physik.uni-karlsruhe.de

1. Introduction and Summary

In many applications in theoretical physics one is led to eigenvalue problems. In the finite dimensional case their solution is equivalent to finding the zeros of the characteristic polynomial, $P_N(x)$. The infinite case can also be tackled, as soon as the asymptotics for large degree number N is under control. In almost all interesting cases, however, the zeros are not known. In such circumstances it is desirable to learn as much as possible about the density of zeros (eigenvalues), $\rho(N,x)$. This can be achieved if the moments of the density (considered as probability distribution function) are known. In particular, the asymptotic distribution function $lim_{N\to\infty}\ \rho(N,x)$ can be obtained if a formula for its moments is found which can be analytically continued in the degree number N. What is, in fact, required is the ordinary (as opposed to exponential) generating function of the moments, $G(N,z)$. This is because $\frac{1}{z}\ G(N,\frac{1}{z}) \ = \ N\ \chi(N,z)$, with the *Stieltjes* transform $\chi(N,z)$ of the distribution. $N\ \chi(N,z)$ is given by the logarithmic derivative of the polynomial $P_N(z)$. The distribution of zeros can be recovered *via Perron-Stieltjes* inversion from $\chi(N,z)$, which can also be given as a continued fraction approximation. In order to obtain the asymptotic density in this way one needs therefore $\chi(N,z)$ as *function* of N.

In the mathematical physics literature the moments of the distribution of zeros of polynomials (also called sum rules or *Newton* sums) have received a great deal of attention, especially after the derivation in ref. [1] of so-called recursion relations for these moments from certain types of differential equations which were assumed to hold for the characteristic polynomials. In this direction we mention ref. [2] where the class of treated differential equations was enlarged, and where further references can be found.

In this paper we shall first state, in *section 2*, an explicit formula for the sum of the $r-$th power of zeros valid for every polynomial $P_N(x)$ of degree N. This formula is nothing but *Girard*'s (sometimes called *Waring*'s) formula [3], which has been used in refs. [4,5] in order to define a $N-1$ variable generalisation of *Chebyshev*'s polynomials of the first kind, $T_r^{(N-1)}(x_1,...,x_{N-1})$. The instance $N=2$ corresponds to the ordinary *Chebyshev* polynomials. Only a slight improvement of this formula over the one found in refs.[4,5] will be given here: it will be stated as an $(N-1)-$fold nested sum with all summations made explicit. For polynomials of definite parity this formula simplifies further. The example of orthogonal polynomials will be useful. In particular the moments for the ordinary *Chebyshev* polynomials, $T_N(x)$ and $U_N(x)$, where the zeros *are* known, will be reconsidered. It turns out that the statement about the moments of the zeros of $T_N(x)$ which is found in ref.[1] is false. The sum rules found in ref. [1] are only the asymptotic ones ($N\to\infty$), contrary to the claim. It *is* true that the asymptotic values of the moments coincide for all Jacobi type polynomials. The reason for the mistake in ref.[1] is that in the Jacobi case (whence in the $T_N(x)$ case) the "recursion" formulae cannot be used to determine all moments. These formulae are incomplete due to possibly vanishing factors which multiply the moment with highest index.

In *section 3* the moment generating functions $G(N,z)$ will be considered. From the generating function of the generalised *Chebyshev* polynomials one finds a simple formula for the moment generating function for given N in terms of the logarithmic derivative of the polynomial $P_N(x)$ under consideration. For later applications (in *section 4*) it is important to find $G(N,z)$ as function of N in order to be able to determine, for instance, the limit $N \to \infty$. This will be done for the classical orthogonal polynomials as examples. In conjunction with *Case*'s equations [1] it will be seen that for these examples the $G(N,z)$ satisfy certain *Riccati* eqs. for each N. We comment on the $1/N$ expansion of these eqs. and show that the correct results for $G(N,z)$ are in the *Chebyshev* case not represented by their $1/N$ expansion. This explains why *Case*'s conclusion for *Chebyshev*'s T polynomials is wrong.

Finally, in *Section 4* it will be explained why the moment generating function is instrumental for the determination of the distribution of zeros of polynomials. It turns out that in the finite N case the *Stieltjes* transform of the purely discontinuous distribution of zeros is obtained directly from $G(N,1/z)/z$. The *Perron-Stieltjes* inversion formula then allows one to recover the distribution of zeros from the moment generating function. This procedure is also applicable in the $N \to \infty$ limit provided $G(N,z)$ is known as function of N and the limit exists. Then the asymptotic distribution of zeros can be found *via* the inversion formula. As examples we treat classical orthogonal polynomials with appropriately scaled variables.

2. Girard-Waring formula for the moments

Girard's formula from 1629, which is often called *Waring*'s formula (from 1782) (see ref.[3] for the dates and a reference) expresses the sum of the $r-$th power of N variables, $p_r(N)$, as polynomial in the elementary symmetric functions (polynomials) $\{\sigma_k\}_1^N$ of these variables. This formula becomes an identity if the definitions $\sigma_k := \sum_{1\le i_1<\ldots<i_k\le N} x_{i_1}\cdots x_{i_k}$ are inserted. Together with *Vieta*'s theorem on the roots it can be employed to give an explicit expression for the $r-$th power sum of the zeros of every polynomial in one variable of degree N, $P_N(x)$, in terms of the coefficients $\{\sigma_k\}_1^N$ of the corresponding monic polynomial *

$$\tilde{P}_N(x) \;=\; \sum_{k=0}^{N} (-1)^k\, \sigma_k\, x^{N-k} \quad , \quad \sigma_0 \;=\; 1 \quad . \tag{2.1}$$

Usually (see e.g.[3],p.5) *Girard*'s formula is given in a semi-explicit form, still needing certain partitions to be performed. It has been simplified in ref.[4], pp 109,110. We have simplified it a little further and the explicit formula, involving $N-1$ iterated sums, written for the (normalised) power sums of zeros $\{x_k^{(N)}\}_1^N$ of every polynomial (2.1) is:

Theorem 1: Girard's formula for the moments

For $r \in \mathbf{N}$, $N = 2, 3, ...$

$$m_r(N) \;:=\; \frac{1}{N}\sum_{k=1}^{N} (x_k^{(N)})^r \;=:\; \frac{1}{N}\, p_r(N) \;=$$

* The $N-$dependence of the coefficients, $\sigma_k = \sigma_k(N)$ will in general be suppressed.

$$= \frac{1}{N}\, \sigma_1^r \,\Big(\prod_{k=1}^{N-1} (\sum_{i_k=0}^{\lfloor b_k \rfloor} \frac{(-1)^{(N-k)i_k}}{(i_k)!})\Big) \;< N, r, \{i_j\}_1^{N-1} > \;\prod_{p=1}^{N-1} (\frac{\sigma_{p+1}}{\sigma_1^{p+1}})^{i_{N-p}} \quad ,$$

$$=: \; t_r^{(N)}(\sigma_1, ..., \sigma_N) \quad , \tag{2.2}$$

where the k-product is a shorthand notation for $N-1$ ordered sums, starting on the left-hand side (*l.h.s.*) with the i_1 sum. The $k-$th sum runs up to the greatest integer (floor function $\lfloor \cdot \rfloor$) of $b_k \equiv b_k(N, r, \{i_j\}_1^{k-1})$ which is given by $b_1 := r/N$ and

$$b_k \;:=\; \frac{r \;-\; \sum_{j=1}^{k-1} \,(N+1-j)\; i_j}{N+1-k} \quad , \quad k = 2, ..., N-1 \; . \tag{2.3}$$

A combinatorial factor is defined as

$$< N, r, \{i_j\}_1^{N-1} > := \frac{r\; (r-1-\sum_{j=1}^{N-1}\; (N-j) i_j)!}{(r - \sum_{j=1}^{N-1}\; (N+1-j)\; i_j)!} \quad . \tag{2.4}$$

For $r = 0$ one has $m_0(N) \equiv 1$. $m_r(1) \;=\; \sigma_1^r$. The polynomial $t_r^{(N)}(\sigma_1, ..., \sigma_N)$ has been introduced in order to make contact with the $N-1$ variable generalisation of Chebyshev's polynomial of the first kind, $T_r^{(N-1)}(\sigma_1, ..., \sigma_{N-1})$, which is, following refs. [4,5], obtained by *

$$T_r^{(N-1)}(\sigma_1, ..., \sigma_{N-1}) \;:=\; t_r^{(N)}(\sigma_1, ..., \sigma_{N-1}, \sigma_N = 1) \quad , \tag{2.5}$$

Therefore, by appropriate scaling one has

$$T_r^{(N-1)}(s_1, ..., s_{N-1}) \;=\; t_r^{(N)}(\sigma_1, ..., \sigma_N)/\sigma_N^{\,r/N} \quad , \tag{2.6a}$$

with

$$s_k := \sigma_k/\sigma_N^{\,k/N} \quad , \quad k = 1, ..., N-1 \quad . \tag{2.6b}$$

The moments (2.2) can therefore be expressed in terms of $\sigma_N^{r/N}\; T_r^{(N-1)}(s_1, ..., s_{N-1})$ with the definition (2.6b). For $N = 1 : m_r(1) \;=\; \sigma_1^r$, and we put $m_r(0) \equiv 1$. Thus $t_r^{(1)}(\sigma_1) \;=\; \sigma_1^r$, $T_r^{(0)} \equiv 1$, and the $N = 2$ instance of (2.2) is

$$t_r^{(2)}(\sigma_1, \sigma_2) \;=\; \frac{1}{2}\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^k \;\frac{r}{r-k}\; \binom{r-1}{k} \;\sigma_2^k\; \sigma_1^{r-2k} \quad , \quad T_r^{(1)}(\sigma_1) \;=\; T_r(\sigma_1/2) \quad , \tag{2.7}$$

with the explicit sum representation of Chebyshev's $T_r(x)$ polynomial. For later reference we quote the coefficients of the monic $\tilde{T}_N(x)$ polynomial according to (2.1)

$$\sigma_{2k+1}^{(\tilde{T})} \equiv 0 \quad , \quad \sigma_{2k}^{(\tilde{T})} \;=\; (-1)^k \;\frac{N}{N-k}\; \binom{N-k}{k}\; 2^{-2k} \quad . \tag{2.8}$$

---

* $t_r^{(N)}(\sigma_1, ..., \sigma_N)$ occurs in the literature also as generalised $V_r^{(N)}-$(or second type) *Lucas* polynomial (see ref.[6]). Note that if the $\{\sigma_i\}_1^N$ are expressed in terms of N variables $\{x_1, ..., x_N\}$ then the $\{\sigma_i\}_1^{N-1}$ of $T_r^{(N-1)}$ are obtained from σ_i, $i = 1, ..., N$, after the replacement $x_N = (x_1 \cdots x_{N-1})^{-1}$ due to $\sigma_N = 1$.

For $N=3$ one has

$$t_r^{(3)}(\sigma_1,\sigma_2,\sigma_3) \;=\; \frac{1}{3}\;\sigma_1^r \sum_{i=0}^{\lfloor \frac{r}{3}\rfloor} \frac{1}{i!} \sum_{j=0}^{\lfloor \frac{r-3i}{2}\rfloor} \frac{(-1)^j}{j!}\; \frac{r\;(r-1-2i-j)!}{(r-3i-2j)!}\; (\frac{\sigma_2}{\sigma_1^2})^j\; (\frac{\sigma_3}{\sigma_1^3})^i , \quad (2.9a)$$

$$T_r^{(2)}(\sigma_1,\sigma_2) \;=\; \frac{1}{3} \sum_{i=0}^{\lfloor \frac{r}{3}\rfloor} \frac{1}{i!} \sum_{j=0}^{\lfloor \frac{r-3i}{2}\rfloor} \frac{(-1)^j}{j!}\; \frac{r\;(r-1-2i-j)!}{(r-3i-2j)!}\; \sigma_1^{r-2j-3i}\; \sigma_2^j \quad . \quad (2.9b)$$

There is an equivalent way of solving *Newton*'s equations for power sums, *viz.* in terms of *Riordan*'s [7] *Bell* polynomials (see also ref.[8], p.2762) which occur in the context of *Faà di Bruno*'s formula for higher derivatives of composite functions. This solution ([7],Exercise 27b), p.47) is (with our notation $Y_r(f_1,...,f_r;g_1,...,g_r)$ instead of *Riordan*'s $Y_r(fg_1,..,fg_r)$)

$$t_r^{(N)}(\sigma_1,...,\sigma_N) \;=\; -\frac{1}{N\;(r-1)!}\; Y_r(f_1,...,f_r;g_1,...,g_r) \quad , \qquad (2.10)$$

with $f_j=(-1)^{j-1}(j-1)!$, $g_j=(-1)^j\; j!\;\sigma_j$ for $j=1,...,N$ and $\sigma_j=0$ for $N<j\leq r$.

For polynomials of definite parity, *i.e.* if $P_N(-x)=(-1)^N\;P_N(x)$ holds, *Girard*'s formula can be simplified further because all σ_{2i+1} in (2.1) vanish.

Corollary 1: Moments for $\mathbf{P_N(-x) = (-1)^N\; P_N(x)}$ polynomials

i) $m_0(N)\equiv 1$, $N\in\mathbf{N}_0$ *ii)* $m_{2l+1}(N)\equiv 0$, $l\in\mathbf{N}_0$, $N\in\mathbf{N}$,

iii) $m_{2l}(N) \;=$

$$\frac{2}{N}\;(-\sigma_2)^l \Big(\prod_{k=1}^{\lfloor\frac{N}{2}\rfloor-1} \big(\sum_{i_k=0}^{\lfloor B_k\rfloor} (-1)^{(\lfloor\frac{N}{2}\rfloor-k)i_k}\frac{1}{(i_k)!}\big)\Big) \;<\lfloor\frac{N}{2}\rfloor, l, \{i_j\}_1^{\lfloor\frac{N}{2}\rfloor-1}>\; \prod_{p=2}^{\lfloor\frac{N}{2}\rfloor} \big(\frac{\sigma_{2p}}{\sigma_2^p}\big)^{i_{\lfloor\frac{N}{2}\rfloor-p+1}} \quad ,$$

$$= \frac{2}{N}\lfloor\frac{N}{2}\rfloor(-1)^l\; t_l^{(\lfloor\frac{N}{2}\rfloor)}(\sigma_2,\sigma_4,...,\sigma_{2\lfloor\frac{N}{2}\rfloor})$$

$$= \frac{2}{N}\lfloor\frac{N}{2}\rfloor(-1)^l\; \big(\sigma_{2\lfloor\frac{N}{2}\rfloor}\big)^{l/\lfloor\frac{N}{2}\rfloor}\; T_l^{(\lfloor\frac{N}{2}\rfloor-1)}(e_1,...,e_{\lfloor\frac{N}{2}\rfloor-1}) \quad , \qquad (2.11)$$

for $l\in\mathbf{N}_0$, $N=4,5,...,$

where B_k, resp. $<\lfloor\frac{N}{2}\rfloor, l, \{i_j\}_1^{\lfloor\frac{N}{2}\rfloor-1}>$, resp. $t_l^{(\lfloor\frac{N}{2}\rfloor)}$, is obtained from b_k of (2.3), resp. $<N,r,\{i_j\}_1^{N-1}>$ of (2.4), resp. $t_r^{(N)}$ of (2.2), by replacing $N\to\lfloor\frac{N}{2}\rfloor$ and $r\to l$. The variables of $T_l^{(\lfloor\frac{N}{2}\rfloor-1)}$, which is also obtained by the same replacement from $T_r^{(N-1)}$ given in (2.6), are

$$e_k\equiv e_k(N) \;=\; \sigma_{2k}/\big(\sigma_{2\lfloor\frac{N}{2}\rfloor}\big)^{k/\lfloor\frac{N}{2}\rfloor} \quad , \quad k=1,...,\lfloor\frac{N}{2}\rfloor-1 \quad . \qquad (2.12)$$

All σ_{2k} quantities depend on N.

In addition: $m_{2l}(1) = 0$, $m_{2l}(2) = (-\sigma_2(2))^l$, $m_{2l}(3) = \frac{2}{3}(-\sigma_2(3))^l$.

Proof:

i) holds by definition of $m_r(N)$. *ii)* and *iii)* are proven for N even and odd separately. The parity requirement of $P_N(x)$ is equivalent to $\sigma_{2i+1} \equiv 0$ in (2.1). Therefore, in (2.2.) only vanishing exponents of these odd numbered coefficients contribute. For both cases this implies even r, say $r = 2l$, which proves *ii)*. For *even* N $i_2 = i_4 = ... = i_{N-2} = 0$, and from the σ_1 exponent $i_{N-1} = l - \sum_{k=1}^{\frac{N}{2}-1}(k+1)\, i_{N-(2k+1)}$. For *odd* N, $i_1 = i_3 = ... = i_{N-2} = 0$ and $i_{N-1} = l - \sum_{k=1}^{\lfloor\frac{N}{2}\rfloor-1}(k+1) i_{N-(2k+1)}$. For *even* N one renames the summation indices after elimination of i_{N-1} according to $i_1 \to i_1$ $i_3 \to i_2$, $..., i_{N-3} \to i_{N/2-1}$. For *odd* N the replacement is $i_2 \to i_1$, $i_4 \to i_2$ $, ..., i_{N-3} \to i_{\lfloor\frac{N}{2}\rfloor-1}$. For example, if $N = 2M$ the sum with index i_k (new) runs up to $\lfloor B_k \rfloor$ with $B_k = \big(2l - (2Mi_1 + 2(M-1)i_2 + ... + 2(M-(k-2))i_{k-1})\big)/(2M - 2(k-1))$ which is $b_k(\lfloor\frac{N}{2}\rfloor, l, \{i_j\}_1^{k-1})$. For $N = 2M+1$ a similar result holds. The new combinatorial factor arises from the old rewritten one together with the replaced $1/i_{N-1}!$. The denominator of the old combinatorial factor becomes 1. This is true in the even and odd N case. The identification of formula *iii)* with the polynomials $t_l^{(\lfloor\frac{N}{2}\rfloor)}$ and $T_l^{(\lfloor\frac{N}{2}\rfloor-1)}$ follows from (2.2) and (2.6). □

Example 1:

Consider *Chebyshev*'s polynomials of the first kind, $T_N(x)$, which are classical orthogonal polynomials of the *Jacobi* type [9,10]. The coefficients of the monic $\tilde{T}_N(x)$ have been given in (2.8). For this example the zeros are known, and the non-vanishing moments are $m_{2l}^{(T)}(N) = \frac{2}{N}\sum_{k=1}^{\lfloor\frac{N}{2}\rfloor}(cos\big(\frac{2k-1}{2N}\pi\big))^{2l}$. *Corollary 1*, together with (2.11) and (2.12), gives an alternative expression for this sum.

For $l \in \mathbf{N}$, $M \in \mathbf{N}$:

$$m_{2l}^{(T)}(2M) = T_l^{(M-1)}(e_1(2M), ..., e_{M-1}(2M))/2^{(2-\frac{1}{M})l} \ , \tag{2.13a}$$

$$\text{with } e_k(2M) \equiv \frac{2M}{k}\, 2^{-\frac{k}{M}} \binom{2M-k-1}{k-1} \quad , \quad k = 1, ..., M-1, \tag{2.13b}$$

$$m_{2l}^{(T)}(2M+1) = \frac{2M}{2M+1}\, \Big(\frac{(2M+1)^{\frac{1}{M}}}{4}\Big)^l\, T_l^{(M-1)}(e_1(2M+1), ..., e_{M-1}(2M+1)), \tag{2.14a}$$

$$\text{with } e_k(2M+1) \equiv \frac{2M+1}{k}\, \frac{1}{(2M+1)^{\frac{k}{M}}} \binom{2M-k}{k-1}, \ k = 1, ..., M-1. \tag{2.14b}$$

and $m_0^{(T)}(N) = 1$, $m_{2l}^{(T)}(1) = 0$.

Some instances for $l \in \mathbf{N}$ are:

$$m_{2l}^{(T)}(2) = (\frac{1}{2})^l \ , \ m_{2l}^{(T)}(4) = T_l(\sqrt{2})/(2\sqrt{2})^l \ ,$$

$$m_{2l}^{(T)}(6) = T_l^{(2)}(3 \cdot 2^{\frac{2}{3}}, 9 \cdot 2^{-\frac{2}{3}})/2^{\frac{5}{3}l} \ , \tag{2.15a}$$

$$m_{2l}^{(T)}(3) \;=\; 3^{l-1}/2^{2l-1} \quad , \quad m_{2l}^{(T)}(5) \;=\; (\sqrt{5})^{l-2}\, T_l(\sqrt{5}/2)/2^{2(l-1)} \quad ,$$
$$m_{2l}^{(T)}(7) \;=\; \frac{6}{7}\,\Big(\frac{7^{\frac{1}{3}}}{4}\Big)^l\, T_l^{(2)}(7^{\frac{2}{3}}, 2\cdot 7^{\frac{1}{3}}) \quad . \tag{2.15b}$$

$T_l^{(2)}$ is given by (2.9b). From these examples one finds for N up to 7:

$$m_2^{(T)}(N) \;=\; \begin{cases} 0 & \text{N=1} \\ \frac{1}{2} & \text{N=2,...} \end{cases} \quad , \quad m_4^{(T)}(N) \;=\; \begin{cases} 0 & \text{N=1} \\ 1/4 & \text{N=2} \\ 3/8 & \text{N=3,...} \end{cases} \quad , \quad m_6^{(T)}(N) \;=\; \begin{cases} 0 & \text{N=1} \\ 1/8 & \text{N=2} \\ 9/32 & \text{N=3} \\ 5/16 & \text{N=4,...} \end{cases} \quad .$$

This contradicts the statement of ref. [1], eq. (46). y_{2n}/N there is our $m_{2n}^{(T)}(N)$. Only the values for $N \geq n+1$ are correctly given by the Z_1, Z_2 and Z_3 of [1]. It will be seen later that all Z_n values given in [1] correspond to the ones of the asymptotic ($N \to \infty$) moments. These Z_n values are, in fact, correct for all $N \geq n+1$. The moments for $N \leq n$ cannot be found from the 'recursion relations' of ref. [1]. Here (2.13) and (2.14) determine the values for all moments.

Example 2:

The coefficients of the monic *Chebyshev* polynomial of the second type, $\tilde{U}_N(x)$, are

$$\sigma_{2k+1}^{\tilde{U}}(N) \equiv 0 \quad , \quad \sigma_{2k}^{\tilde{U}}(N) \;=\; (-1)^k \binom{N-k}{k}/2^{2k} \quad . \tag{2.16}$$

The non-vanishing moments are $m_{2l}^{(U)}(N) \;=\; \frac{2}{N}\,\sum_{k=1}^{\lfloor N/2 \rfloor}\big(cos(\frac{k}{N+1}\pi)\big)^{2l}$. From *Corollary 1* one finds, for $l \in \mathbf{N}_0$, $N \in \mathbf{N}$, $M = 2, 3, ...$, the alternative expression *

$$m_{2l+1}^{(U)}(N) \equiv 0 \tag{2.17a}$$
$$m_{2l}^{(U)}(2M) \;=\; T_l^{(M-1)}(e_1(2M), ..., e_{M-1}(2M))/2^{2l} \quad ,$$
$$\text{with} \;\; e_k(2M) \equiv \binom{2M-k}{k} \quad , k = 1, ..., M-1, \tag{2.17b}$$

and for $l \in \mathbf{N}$, $M = 2, 3, ...$,

$$m_{2l}^{(U)}(2M+1) \;=\; \frac{M(M+1)^{\frac{l}{M}}}{(2M+1)2^{2l-1}}\, T_l^{(M-1)}(e_1(2M+1), ..., e_{M-1}(2M+1)),$$
$$\text{with} \;\; e_k(2M+1) \equiv \frac{1}{(M+1)^{\frac{k}{M}}}\binom{2M-k+1}{k}, \;\; k = 1, ..., M-1. \tag{2.17c}$$

In addition, $m_{2l}^{(U)}(2) \;=\; 1/4^l \quad , \quad m_{2l}^{(U)}(3) \;=\; \frac{2}{3}\big(\frac{1}{2}\big)^l$.

---

* The superscripts (U) on the e_k quantities have been omitted. No confusion with the quantities (2.13b), (2.14b), relevant for the $T-$polynomials should arise.

For instance, $m_{2l}^{(U)}(4) \;=\; T_l(\frac{3}{2})/2^{2l}$, $m_{2l}^{(U)}(5) \;=\; \frac{1}{5}\,(\sqrt{3})^l\; T_l(\frac{2}{\sqrt{3}})/2^{2(l-1)}$, with *Chebyshev*'s ordinary T_l polynomial.

3. Moment generating functions

In *section 2* it has been shown that the moments of the zeros of polynomials are given by multivariate *Chebyshev* $T-$polynomials evaluated at points which depend on the coefficients of the considered polynomial. The generating function for these generalised *Chebyshev* polynomials are known [3,11,4]. Therefore one can derive the generating function for the moments or power sums of zeros. These functions will be very useful; for instance, in determining the asymptotic formula for the moments. This will be the topic of the next section.

It is well-known (*cf.* [3], Ch.I; [11], p.55) that the generating function of the sum of the $r-$th power of N variables $\{x_i\}_1^N$, *viz* $p_r(N)$, is formally given by the logarithmic derivative of the generating function of the elementary symmetric functions $\{\sigma_k\}$ of these variables. To be precise, if

$$E(z) \equiv E_N(z) := \prod_{i=1}^{N} (1 - x_i z) \;=\; \sum_{r=0}^{N} (-1)^r \sigma_r z^r \quad , \; \sigma_0 = 1 \;, \tag{3.1}$$

then

$$E'(z)/E(z) \;=\; (ln\; E(z))' \;=\; -\frac{1}{z}\sum_{r=1}^{\infty} p_r(N)\; z^r \;\; . \tag{3.2}$$

With *Girard's* fomula (2.2) for the moments $m_r(N) := p_r(N)/N$ this produces

$$\sum_{r=1}^{\infty} t_r^{(N)}(\sigma_1, ..., \sigma_N)\; z^r \;=\; \sum_{r=1}^{\infty} m_r(N)\; z^r \;=\; \frac{1}{N}\,\frac{\sum_{r=1}^{N}(-1)^{r-1}\; r\; \sigma_r\; z^r}{\sum_{r=0}^{N}(-1)^r\; \sigma_r\; z^r} \;\; . \tag{3.3}$$

Theorem 2: Moment generating function $\mathbf{G(N,z)}$

$$G(N,z) \;:=\; \sum_{r=0}^{\infty} m_r(N)\; z^r \;=\; \sum_{r=0}^{\infty} t_r^{(N)}(\sigma_1, ..., \sigma_N)\; z^r \;=\; \frac{1}{N}\,\frac{x\frac{d}{dx}P_N(x)}{P_N(x)}\Bigg|_{x=1/z} \quad , \tag{3.4}$$

with $P_N(x)$ given in (2.1).

Proof: With $m_0(N) \equiv 1$, $t_0^{(N)} \equiv 1$, and $E(z) \;=\; z^N \tilde{P}_N(\frac{1}{z})$, from (3.1) and (2.1), one finds from (3.3)

$$G(N,z) \;=\; 1 \;-\; zE'(z)/(NE(z)) \;=\; \frac{1}{N}\,\frac{\frac{1}{z}\frac{d}{d(1/z)}\tilde{P}_N(\frac{1}{z})}{\tilde{P}_N(\frac{1}{z})} \;\; .$$

Here the monic $\tilde{P}_N$ can be replaced by P_N. □

Note 1: (3.4) can explicitly be written with (3.3) as ($\sigma_0 = 1$)

$$G(N,z) \;=\; \frac{\sum_{r=0}^{N-1}(-1)^r(1-\frac{r}{N})\;\sigma_r(N)\;z^r}{\sum_{r=0}^{N}(-1)^r\sigma_r(N)\;z^r} \quad . \tag{3.5}$$

Note 2: (3.4) can also be written with the help of the dilation operator acting on a function of 'dimension' N, *viz* $\delta_N(x)\;f(x)\;:=\;(x\frac{d}{dx}\;-\;N)f(x)$, as

$$G(N,z) \;=\; 1 \;+\; [(\delta_N(x)\;P_N(x))/(N\;P_N(x))]_{x=1/z} \quad . \tag{3.6}$$

Therefore $G(N,z) \equiv 1$ iff $P_N(x) = x^N$ due to *Euler*'s theorem on homogeneous functions.

Corollary 2: $\mathbf{G(N,z)}$ for $\mathbf{T_N(x)}$ moments

For $N = 2, 3, ...$:

$$G^{(T)}(N,z) \;=\; \sum_{l=0}^{\infty} m_{2l}^{(T)}(N)\;z^{2l} \;=\; \frac{\sum_{k=0}^{\lfloor (N-1)/2\rfloor}(-1)^k\binom{N-1-k}{k}\left(\frac{z}{2}\right)^{2k}}{1\;+\;N\sum_{k=1}^{\lfloor N/2\rfloor}(-1)^k\binom{N-k-1}{k-1}\frac{1}{k}\left(\frac{z}{2}\right)^{2k}} \tag{3.7a}$$

$$=\; \frac{\frac{1}{z}\;U_{N-1}(\frac{1}{z})}{T_N(\frac{1}{z})} \;=\; \frac{U_N(\frac{1}{z})\;+\;U_{N-2}(\frac{1}{z})}{U_N(\frac{1}{z})\;-\;U_{N-2}(\frac{1}{z})} \tag{3.7b}$$

$$=\; \frac{1}{\sqrt{1-z^2}}\;\frac{z^{2N}\;-\;(1-\sqrt{1-z^2})^{2N}}{z^{2N}\;+\;(1-\sqrt{1-z^2})^{2N}} \tag{3.7c}$$

$$=\; \frac{1}{\sqrt{1-z^2}}\;tanh\Big(N\;ln\frac{z}{1-\sqrt{1-z^2}}\Big) \quad . \tag{3.7d}$$

In addition, $G^{(T)}(1,z) \equiv 1$.

Proof: The first eq. of a) holds because due to *Corollary 1* $m_{2l+1} \equiv 0$. The second eq. of a) follows from *Note 1* with the explicit $\tilde{T}_N$ coefficients (2.8). The identity $\binom{N-k}{k}\;(N-2k)/(N-k) \;=\; \binom{N-k-1}{k}$ has been used. The first eq. of b) flows from *Theorem 2*, which yields $G^{(T)}(N,z) \;=\; \frac{1}{N}\frac{xT_N'(x)}{T_N(x)}|_{x=1/z}$, together with $T_N' \;=\; N\;U_{N-1}$ with *Chebyshev*'s polynomial U_{N-1} of the second kind. The second eq. of b) is due to $2T_N = U_N - U_{N-2}$ and the recursion relation $2xU_{N-1}(x) \;=\; U_N(x) + U_{N-2}(x)$. Eq. c) stems from the explicit forms $2T_N(x) \;=\; (x-\sqrt{x^2-1})^N \;+\; (x-\sqrt{x^2-1})^{-N}$ and $-2\sqrt{x^2-1}\;U_{N-1}(x) \;=\; (x-\sqrt{x^2-1})^N \;-\; (x-\sqrt{x^2-1})^{-N}$. A reference on *Chebyshev*'s polynomials is [12]. Finally, eq. d) is a blown up version of the preceding expression. □

In *Table 1* one finds $G^{(T)}(N,z)$ for $N = 1, ..., 8$ together with the first five expansion terms.

Corollary 3: $\mathbf{G(N,z)}$ for $\mathbf{U_N(x)}$ moments

For $N = 2, 3, ...$:

$$G^{(U)}(N,z) \;=\; \sum_{l=0}^{\infty} m_{2l}^{(U)}(N)\;z^{2l} \;=\; \frac{\sum_{k=0}^{\lfloor (N-1)/2\rfloor}(-1)^k\frac{k+1}{N}\binom{N-k}{k+1}\left(\frac{z}{2}\right)^{2k}}{\sum_{k=0}^{\lfloor N/2\rfloor}(-1)^k\binom{N-k}{k}\left(\frac{z}{2}\right)^{2k}} \tag{3.8a}$$

$$= \frac{\frac{1}{z}\, U_N'(\frac{1}{z})}{N\, U_N(\frac{1}{z})} \;=\; \frac{1}{1-z^2}\, \frac{-U_N(\frac{1}{z}) \;+\; (N+1)\; z\; T_{N+1}(\frac{1}{z})}{N\; U_N(\frac{1}{z})} \qquad (3.8b)$$

$$= \frac{(1+(N+1)\sqrt{1-z^2})(1-\sqrt{1-z^2})^{2(N+1)} - (1-(N+1)\sqrt{1-z^2})\; z^{2(N+1)}}{(1-z^2)\; N\; (z^{2(N+1)} - (1-\sqrt{1-z^2})^{2(N+1)})} \qquad (3.8c)$$

$$= \frac{1}{\sqrt{1-z^2}}\; \Big\{(1+\frac{1}{N})\; tanh\Big((N+1)\; ln\frac{z}{1-\sqrt{1-z^2}}\Big) \;-\; \frac{1}{N}\; \frac{1}{\sqrt{1-z^2}}\Big\} \;. \qquad (3.8d)$$

In addition, $G^{(U)}(1,z) \equiv 1$.

The proof is analogous to the one of *Corollary 2*.

In *Table 2* one finds $G^{(U)}(N,z)$ for $N = 1, ..., 8$ together with the first five expansion terms.

Theorem 3: Riccati equation for $\mathbf{G^{(T)}(N,z)}$

$G^{(T)}(N,z)$ of (3.7) satisfies for every $N \in \mathbf{N}$

$$\frac{(1-z^2)z}{N}\frac{\partial}{\partial z}G^{(T)}(N,z) \;=\; \frac{z^2}{N}\; G^{(T)}(N,z) \;+\; (1-z^2)\big(G^{(T)}(N,z)\big)^2 \;-\; 1 \;. \qquad (3.9)$$

Theorem 4: Riccati equation for $\mathbf{G^{(U)}(N,z)}$

$G^{(U)}(N,z)$ of (3.8) satisfies for all $N \in \mathbf{N}$

$$\frac{(1-z^2)z}{N}\frac{\partial}{\partial z}G^{(U)}(N,z) \;=\; \frac{2+z^2}{N}\; G^{(U)}(N,z) + (1-z^2)\big(G^{(U)}(N,z)\big)^2 - \frac{N+2}{N} \;. \qquad (3.10)$$

Proofs: Both *Theorems* can be checked, for example, with the version (3.7*d*), resp. (3.8*d*). (We have checked all *Riccati* eqs. in this paper for $N = 1, ..., 10$ with *Maple* [13].) □

Note 3: Equivalent second order linear equations

With $G(N,z) \;=\; -\frac{z}{N}\frac{\partial}{\partial z}\big(ln\, \mathcal{U}(N,z)\big)$, or $\mathcal{U}(N,z) = exp(-N\int \frac{dz}{z} G(N,z))$, one finds from (3.9) and (3.10) *

$$\Big(z^2(1-z^2)\frac{\partial^2}{\partial z^2} \;+\; z(1-2z^2)\frac{\partial}{\partial z} \;-\; N^2\Big)\mathcal{U}^{(T)}(N,z) \;=\; 0 \;, \qquad (3.11)$$

$$\Big(z^2(1-z^2)\frac{\partial^2}{\partial z^2} \;-\; z(1+2z^2)\frac{\partial}{\partial z} \;-\; N(N+2)\Big)\mathcal{U}^{(U)}(N,z) \;=\; 0 \;. \qquad (3.12)$$

Note 4: Case's equations

The *Riccati* eqs.(3.9) and (3.10) for the moment generating functions follow also from the recursion type eqs. for the moments derived in ref.[1]. For example, in the *Chebyshev* T_N case *Case*'s eqs. are (remember that y_{2n} is our $N m_{2n}(N)$), for $N \in \mathbf{N}$, $r \in \mathbf{N}$,

$$(2N-(1+r))\; m^{(T)}_{r+1}(N) \;=\; -r\; m^{(T)}_{r-1}(N) + N\sum_{s=0}^{r-1}\big(m^{(T)}_{r-1-s}(N)\; m^{(T)}_{s}(N) \;-\; m^{(T)}_{r-s}(N)\; m^{(T)}_{s+1}(N)\big) \;, \qquad (3.13)$$

---

* The solution of (3.11) corresponding to (3.7*d*) is $\mathcal{U}(N,z) = cosh\Big(N\; ln\frac{z}{1-\sqrt{1-z^2}}\Big)$.
The one of (3.12) related to (3.8*d*) is more involved.

with inputs $m_0^{(T)}(N) \equiv 1$, $m_1^{(T)}(N) \equiv 0$.
If the generating function $G^{(T)}(N,z) := \sum_{r=0}^{\infty} m_r^{(T)}(N)\ z^r$ is introduced these eqs. yield the *Riccati* eq. (3.9).

For the general *Jacobi*-type polynomials, $P_N^{(\alpha,\beta)}(x)$, *Case*'s eqs. are for $N \in \mathbf{N}$, $r \in \mathbf{N}$, $\alpha > -1$, $\beta > -1$ *

$$
\begin{aligned}
(\alpha+\beta+2N-r)\ m_{r+1}^{(J)}(N) \ &= \ (\beta-\alpha)\ m_r^{(J)}(N) - r\ m_{r-1}^{(J)}(N) \\
&\quad + N\sum_{s=0}^{r-1}\big(m_{r-1-s}^{(J)}(N)\ m_s^{(J)}(N) \ - \ m_{r-s}^{(J)}(N)\ m_{s+1}^{(J)}(N)\big) \ , \quad (3.14)
\end{aligned}
$$

with $m_0^{(J)}(N) \equiv 1$, $m_1^{(J)}(N) \equiv (\beta-\alpha)/(2N+\alpha+\beta)$, leading to the following *Riccati* eq. for the moment generating function ** $G^{(J)}(N,z) := \sum_{r=0}^{\infty} m_r^{(J)}(N)\ z^r$

$$
\begin{aligned}
\frac{(1-z^2)z}{N}\frac{\partial}{\partial z}G^{(J)}(N,z) \ &= \ (z^2+(\alpha-\beta)z+\alpha+\beta+1)\frac{1}{N}\ G^{(J)}(N,z) \\
&\quad + \ (1-z^2)\ \big(G^{(J)}(N,z)\big)^2 \ - \ (1+\frac{1+\alpha+\beta}{N}) \ . \quad (3.15)
\end{aligned}
$$

(3.9), resp. (3.10), is the $\alpha=\beta=-1/2$, resp. $\alpha=\beta=+1/2$, instance. The equivalent second order linear differential eq. is †

$$
\Big(z^2(1-z^2)\frac{\partial^2}{\partial z^2} - z(2z^2+(\alpha-\beta)z+\alpha+\beta)\frac{\partial}{\partial z} - N(N+1+\alpha+\beta)\Big)\,\mathcal{U}^{(J)}(N,z) \ = \ 0\,. \quad (3.16)
$$

Again, (3.11) and (3.12) are instances. *Case*'s eqs. for moments of *Jacobi* polynomials, (3.14), are not sufficient to determine all moments. This is due to the factor appearing on the *l.h.s.* which can vanish. For given r only the moments with $N > (r-\alpha-\beta)/2$ can be determined recursively from (3.14). These eqs. are therefore incomplete. Nevertheless, from these eqs. the *Riccati* eq. (3.15) for the moment generating function can be derived for all N and $\alpha,\beta > -1$. From the experience gained in the *Chebyshev* T_N and U_N case it is to be expected that the (correct) generating function $G^{(J)}(N,z)$ obtained from *Theorem 2* (or (3.5)) satisfies the *Riccati* eq. (3.15).

Corollary 4: $\mathbf{G^{(J)}(N,z)}$ for $\mathbf{P_N^{(\alpha,\beta)}(x)}$ moments

$$
\begin{aligned}
G^{(J)}(N,z) \ &= \ \frac{1}{2N}(1+N+\alpha+\beta)\ \frac{\frac{1}{z}P_{N-1}^{(\alpha+1,\beta+1)}(\frac{1}{z})}{P_N^{(\alpha,\beta)}(\frac{1}{z})} \\
&= \ \frac{N+\alpha+\beta+1}{2(\alpha+1)}\ \frac{1}{z}\ \frac{{}_2F_1(-N+1,N+\alpha+\beta+2,\alpha+2;\frac{z-1}{2z})}{{}_2F_1(-N,N+\alpha+\beta+1,\alpha+1;\frac{z-1}{2z})} \ , \quad (3.17)
\end{aligned}
$$

* We use the superscript (J) (*Jacobi*) instead of indicating the α,β dependence.

** Because of $P_N^{(\alpha,\beta)}(-x) = (-1)^N\ P_N^{(\beta,\alpha)}(x)$ *Corollary 1* applies only for $\alpha=\beta$, and all moments are present for $\alpha \neq \beta$.

† (3.16) is, for $\alpha=\beta$ (*Gegenbauer*), of type 2.1.69 of [16],p.96.

with the hypergeometric function ${}_2F_1(a,b,c;x)$.
Proof: The first eq. follows from *Theorem 2* with
$\frac{d}{dx}P_N^{(\alpha,\beta)}(x) = \frac{1}{2}(1+N+\alpha+\beta)\ P_{N-1}^{(\alpha+1,\beta+1)}(x)$ (*cf.* [10],p.149).
The second one from the identification
$P_N^{(\alpha,\beta)}(x) \ = \ \Gamma(N+\alpha+1)\ {}_2F_1(-N, N+\alpha+\beta+1, \alpha+1; \frac{1-x}{2})/(\Gamma(N+1)\Gamma(\alpha+1))$
with the Gamma function $\Gamma(z)$ (*cf.*[14],Vol.I, p.89, or [15], p.283). □

Theorem 5: $G^{(J)}(N,z)$ of (3.17) satisfies the *Riccati* eq. (3.15).

Proof: The first eq. of (3.17) is used in (3.15) written in terms of $x = 1/z$. Besides the identity for $\frac{d}{dx}P_N^{(\alpha,\beta)}$ which occured in the last proof, also another one for $\frac{d}{dx}P_N^{(\alpha+1,\beta+1)}$, found in [10], eq.(2.21), is used. Application of the contiguous identities,*e.g.* [14], Vol.II, p.173, (35) and (36), then leads to the recursion formula for the *Jacobi* polynomials, found *e.g.* in [10] eq.(2.26). □

For the sake of completeness we now list the *Riccati* eqs. which the moment generating functions for the *Hermite* and *Laguerre* polynomials obey. *Case*'s eqs. determine all moments of these polynomials, *i.e.* they are complete.

Corollary 5: $\mathbf{G(N,z)}$ for $\mathbf{H_N(x)}$ moments

The moments of the zeros of the *Hermite* polynomial $H_N(x)$ are generated by

$$G^{(H)}(N,z) \ := \ \sum_{r=0}^{\infty} m_r^{(H)}(N)\ z^r \ = \ \sum_{l=0}^{\infty} m_{2l}^{(H)}(N)\ z^{2l} \ = \ \frac{2}{z}\ \frac{H_{N-1}(\frac{1}{z})}{H_N(\frac{1}{z})} \tag{3.18a}$$

$$= \frac{1}{z}\ \frac{F(-\frac{N-1}{2},\frac{1}{2};\frac{1}{z^2})/\Gamma(\frac{2-N}{2}) \ - \ \frac{2}{z}\ F(\frac{2-N}{2},\frac{3}{2};\frac{1}{z^2})/\Gamma(\frac{1-N}{2})}{F(-\frac{N}{2},\frac{1}{2};\frac{1}{z^2})/\Gamma(\frac{1-N}{2}) \ - \ \frac{2}{z}\ F(\frac{1-N}{2},\frac{3}{2};\frac{1}{z^2})/\Gamma(-\frac{N}{2})}\ , \tag{3.18b}$$

with the confluent hypergeometric function $F(a,c;x) \equiv \ {}_1F_1(a,c;x)$.
Proof: The second eq. of $a)$ follows from *Corollary 1.* The third one from *Theorem 2* with $H'_N(x) = 2N\ H_{N-1}(x)$. $b)$ is due to the identification
$H_N(x)/\sqrt{\pi} \ = \ 2^N\ F(-\frac{N}{2},\frac{1}{2};x^2)/\Gamma(\frac{1-N}{2}) \ - \ 2^{N+1}\ x\ F(\frac{1-N}{2},\frac{3}{2};x^2)/\Gamma(-\frac{N}{2})$.
(*Cf.* [14], p.267, or [15], p.275.) □

Note 5: Case's eqs. and Riccati eq. for $\mathbf{G^{(H)}(N,z)}$

Case's recurrence eqs. for the scaled moments $q_r^{(H)}(N) \ := \ m_r^{(H)}(N)/N^{r/2}$, and for $N \in \mathbf{N}$, $r \in \mathbf{N}_0$, are

$$q_{r+2}^{(H)}(N) \ = \ -\frac{r+1}{2N}\ q_r^{(H)}(N) \ + \ \frac{1}{2}\sum_{s=0}^{r} q_{r-s}^{(H)}(N)\ q_s^{(H)}(N) \ , \tag{3.19}$$

with inputs $q_0^{(H)}(N) \equiv 1$, $q_1^{(H)}(N) \equiv 0$.
The generating function $Q^{(H)}(N,z) \ := \ \sum_{r=0}^{\infty}\ q_r^{(H)}(N)\ z^r$ satisfies the *Riccati* eq.

$$\frac{z^3}{2N}\ \frac{\partial}{\partial z}\ Q^{(H)}(N,z) \ = \ \frac{z^2}{2}\big(Q^{(H)}(N,z)\big)^2 \ - \ (1+\frac{z^2}{2N})Q^{(H)}(N,z) \ + \ 1 \ , \tag{3.20}$$

The connection to the moment generating function $G^{(H)}(N,z)$, defined in (3.18a), is

$$G^{(H)}(N,z) \;=\; Q^{(H)}(N,\sqrt{N}z) \quad . \tag{3.21}$$

The equivalent second order linear eq. for $\mathcal{U}^{(H)}(N,z)$ defined by $Q^{(H)}(N,z) = -\frac{z}{N}\,\frac{\partial}{\partial z}\big(ln\,\mathcal{U}^{(H)}(N,z)\big)$ is *

$$\Big(z^4\,\frac{\partial^2}{\partial z^2} \;+\; 2z(z^2+N)\frac{\partial}{\partial z} \;+\; 2N^2\Big)\,\mathcal{U}^{(H)}(N,z) \;=\; 0 \quad . \tag{3.22}$$

Theorem 6: Riccati equation for $\mathbf{G^{(H)}(N,z)}$

$G^{(H)}(N,z)$, given by (3.18), satisfies

$$\frac{z^3}{2}\,\frac{\partial}{\partial z}\,G^{(H)}(N,z) \;=\; \frac{z^2}{2}N\big(G^{(H)}(N,z)\big)^2 \;-\; (1+\frac{z^2}{2})G^{(H)}(N,z) \;+\; 1 \quad . \tag{3.23}$$

Proof: This eq. is (3.20), rewritten with (3.21). Because in this instance *Case*'s eqs. are complete $G^{(H)}(N,z)$ of (3.18), which followed from *Theorem 2*, is identical with $G^{(H)}(N,z)$ derived from *Case*'s eqs. $\square$

A direct proof that (3.18a) (hence (3.18b)) solves (3.20) uses the identity $H'_N \;=\; 2N\,H_N$ and the recursion relation $H_N(x) \;=\; 2xH_{N-1}(x) - 2(N-1)H_{N-2}$.

Maple has also been employed to check this *Theorem* for $N = 1, ..., 10$.

Corollary 6: $\mathbf{G(N,z)}$ for $\mathbf{L_N^{(\alpha)}(x)}$ moments

The moments of the (generalised) *Laguerre* polynomial $L_N^{(\alpha)}(x)$ are generated by

$$G^{(\alpha)}(N,z) \;:=\; \sum_{r=0}^{\infty} m_r^{(\alpha)}(N)\;z^r \tag{3.24a}$$

$$= \;-\;\frac{1}{N}\;\frac{\frac{1}{z}\,L_{N-1}^{(\alpha+1)}(\frac{1}{z})}{L_N^{(\alpha)}(\frac{1}{z})} \;=\; 1-(1+\frac{\alpha}{N})\;\frac{L_{N-1}^{(\alpha)}(\frac{1}{z})}{L_N^{(\alpha)}(\frac{1}{z})} \tag{3.24b}$$

$$= 1 - \frac{F(-N+1,\alpha+1;\frac{1}{z})}{F(-N,\alpha+1;\frac{1}{z})} \quad , \tag{3.24c}$$

with the confluent hypergeometric function $F(a,c;x) \equiv \;{}_1F_1(a,c;x)$.

Proof: The first eq. of b) follows from *Theorem 2* and $\frac{d}{dx}L_N^{(\alpha)}(x) = -L_{N-1}^{(\alpha+1)}(x)$. In the second one $x\frac{d}{dx}L_N^{(\alpha)}(x) = N\;L_N^{(\alpha)}(x) - (N+\alpha)\;L_{N-1}^{(\alpha)}(x)$ ([10],p.149) was used. Eq. c) is due to the identification

$L_N^{(\alpha)}(x) = \Gamma(\alpha+1+N)\;F(-N,\alpha+1;x)/(\Gamma(N+1)\;\Gamma(\alpha+1))$ ([16],p.80). $\square$

---

* (3.22) is of type 2.1.69 of [16],p.96.

Note 6: Case's eqs. and Riccati eq. for $\mathbf{G}^{(\alpha)}(\mathbf{N},\mathbf{z})$

The moments of the *Laguerre* polynomials $L_N^{(\alpha)}(x)$ satisfy *Case*'s eqs. which for the scaled moments $q_r^{(\alpha)}(N) \; := \; m_r^{(\alpha)}(N)/N^r$ and $N \in \mathbf{N}, r \in \mathbf{N}_0$ are

$$q_{r+1}^{(\alpha)}(N) \; = \; \frac{\alpha - r}{N} \, q_r^{(\alpha)}(N) \; + \; \sum_{s=0}^{r} q_{r-s}^{(\alpha)}(N) \, q_s^{(\alpha)}(N) \quad , \tag{3.25}$$

with input $q_0^{(\alpha)}(N) \equiv 1$.

The generating function $Q^{(\alpha)}(N,z) \; := \; \sum_{r=0}^{\infty} q_r^{(\alpha)}(N) \; z^r$ satisfies therefore the *Riccati* eq.

$$\frac{z^2}{N} \, \frac{\partial}{\partial z} \, Q^{(\alpha)}(N,z) \; = \; z\big(Q^{(\alpha)}(N,z)\big)^2 \; + \; (\frac{\alpha}{N} z - 1) \; Q^{(\alpha)}(N,z) \; + \; 1 \quad . \tag{3.26}$$

The equivalent second order linear differential eq. is, with $Q^{(\alpha)}(N,z) = -\frac{z}{N} \, \frac{\partial}{\partial z}\big(ln \, \mathcal{U}^{(\alpha)}(N,z)\big)$,*

$$\Big(z^3 \, \frac{\partial^2}{\partial z^2} \; + \; z(N + (1-\alpha)z) \, \frac{\partial}{\partial z} \; + \; N^2\Big) \, \mathcal{U}^{(\alpha)}(N,z) \; = \; 0 \quad . \tag{3.27}$$

The connection with the moment generating function $G^{(\alpha)}(N,z)$, defined in $(3.24a)$, is

$$Q^{(\alpha)}(N,z) \; = \; G^{(\alpha)}(N,z/N) \quad . \tag{3.28}$$

Theorem 7: Riccati eq. for $\mathbf{G}^{(\alpha)}(\mathbf{N},\mathbf{z})$

$G^{(\alpha)}(N,z)$, given by (3.24), satisfies

$$z^2 \, \frac{\partial}{\partial z} \, G^{(\alpha)}(N,z) \; = \; Nz\big(G^{(\alpha)}(N,z)\big)^2 \; + \; (\alpha z - 1) \; G^{(\alpha)}(N,z) \; + \; 1 \quad . \tag{3.29}$$

Proof: Analogous to the proof of *Theorem 5.* Or directly, using the first eq. of $(3.24b)$, the identity for the derivative of $L_N^{(\alpha)}(x)$ used at the beginning of the proof of *Corollary 6*, the recursion formula of the *Laguerre* polynomials and the identity $L_N^{(\alpha)}(x) \; - \; L_N^{(\alpha+1)}(x) \; + \; L_{N-1}^{(\alpha)}(x) \equiv 0$. $\square$

Next, we comment on the $1/N$ expansion of *Case*'s eqs. which is most conveniently performed for the moment generating functions obeying the given *Riccati* eqs.

---

* (3.27) is of type 2.1.57 of [16],p.94.

Note 7: Concerning the 1/N expansion for $\mathbf{T_N}$- and $\mathbf{U_N}$- Chebyshev polynomials

The $1/N$ expansion applied to the *Riccati* eq.(3.9) leads to the zeroth order solution

$$G_0(z) \ := \ 1/\sqrt{1-z^2} \quad , \tag{3.30}$$

which generates the moments

$$m_{0,2l} \ = \ \binom{2l}{l}/2^{2l} \quad . \tag{3.31}$$

All higher expansion terms $G_k^{(T)}(z)$, $k = 1,2,...$, vanish. This is seen after defining $G^{(T)}(N,z) \ := \ G_0(z)\ P^{(T)}(N,z)$ and deriving from (3.9) the *Riccati* eq. for $P^{(T)}$

$$\frac{1}{N} z\sqrt{1-z^2}\ \frac{\partial}{\partial z}\ P^{(T)}(N,z) \ - \ \left(P^{(T)}(N,z)\right)^2 \ + \ 1 \ = \ 0 \quad , \tag{3.32}$$

with $P^{(T)}(N,z) \equiv 1$ as result of the $1/N$ expansion.
However, it is clear from the result (3.7d) that this $1/N$ expansion is not reliable. In fact, the $1/N$ expansion of the $tanh(Nf(z))$ factor of (3.7d) does not represent this function. $P^{(T)}(N,z) \equiv 1$, resp. (3.30), can be used, however, as *special* solution of the *Riccati* eq. (3.32), resp. (3.9), in order to find its general solution. In this way one can recover the correct solution (3.7d) after imposing certain boundary conditions. The asymptotic moments $m_{0,2l}$ given by (3.31), which are generated by (3.30), are common to all *Jacobi* polynomials, as can be seen from the *Riccati* eq. (3.15). In this limit the incompleteness of *Case*'s eqs. does not matter.* These asymptotic moments have been found in ref.[1].

For the moment generating functions of *Chebyshev*'s U_N polynomials the situation is similar. Here the sum of the zeroth plus first order expansion terms of (3.10) turns out to be a special solution of this *Riccati* eqs., and all higher expansion terms vanish. This special solution is **

$$G_{sp}^{(U)}(N,z) \ := \ G_0(z) \ + \ \frac{1}{N}\ \frac{\sqrt{1-z^2}-1}{1-z^2} \ = \ \frac{(N+1)\sqrt{1-z^2}-1}{N(1-z^2)} \quad . \tag{3.33}$$

Again, the $1/N$ expansion is not reliable. The correct solution (3.8d) can, in principle, be recovered from the general solution of (3.10). This computation, with the special solution (3.33) as input, is expected to be rather tedious.

---

* Note that in this way a new asymptotic formula involving hypergeometric functions $_2F_1$ is found, *viz* the one obtained from the *r.h.s.* of (3.17).

** One might suspect that in the *Gegenbauer* case, $\alpha = \beta =: \lambda - 1/2$, a simple pattern emerges. This, however, is not true; for $\lambda = 2$ the sum of the first three $1/N$ expansion terms of (3.15) does not solve this eq. for arbitrary N.

Note 8: Asymptotic moments for $\mathbf{Hs_N(x) := H_N(x\sqrt{N})}$
The leading term in the $1/N$ expansion of the $Q^{(H)}(N,z)$ *Riccati* eq. (3.20) is, with $Q^{(H)}(N,0) \equiv 1$, *

$$Q_0^{(H)}(z) \;=\; \lim_{N\to\infty} Q^{(H)}(N,z) \;=\; c(\frac{1}{2}z^2) \;=\; \frac{1}{z^2}(1-\sqrt{1-2z^2}) \quad , \tag{3.34}$$

with the generating function $c(y) \;:=\; \frac{1}{2y}(1-\sqrt{1-4y})$ of the *Catalan* numbers $\{C_n\}_0^\infty = \{1,1,2,5,14,42,...\}$. $C_n \;=\; \binom{2n}{n}/(n+1)$. Because of (3.21) the asymptotic moments of the scaled $Hs_N(x) \;:=\; H_N(x\sqrt{N})$ polynomials (which have the scaled zeros of the *Hermite* polynomials: $x_i^{(N)}/\sqrt{N}$) are therefore generated by $c(\frac{1}{2}z^2)$. In other words $\lim_{N\to\infty} m_{2l}^{(H)}(N)/N^l \;=\; C_l/2^l$. This agrees with the result found in [17], p.1191 .

Note 9: Asymptotic moments for $\mathbf{Ls_N^{(\alpha)}(x) \;:=\; L_N^{(\alpha)}(xN)}$
The leading term in the $1/N$ expansion of the $Q^{(\alpha)}(N,z)$ *Riccati* eq.(3.26) is, with $Q^{(\alpha)}(N,0) \equiv 1$, for all α

$$Q_0^{(\alpha)}(z) := \lim_{N\to\infty} Q^{(\alpha)}(N,z) \;=\; c(z) \quad , \tag{3.35}$$

with $c(z)$ given above. Because of (3.28) the asymptotic moments of the scaled $Ls_N^{(\alpha)}(x) \;:=\; L_N^{(\alpha)}(xN)$ polynomials are therefore generated by $c(z)$, *i.e.* the scaled asymptotic moments are *Catalan* numbers: $\lim_{N\to\infty} m_r^{(\alpha)}(N)/N^r \;=\; C_r$ for all α. This is in agreement with ref.[17], p.1191, after a nice binomial sum representation for the numbers C_r has been used.

4. Distribution of zeros and Perron-Stieltjes inversion

The moment generating functions $G(N,z)$, elaborated on in the last *section*, are related in a simple way to a certain *Stieltjes* integral $\chi(N,z)$ with the measure given by the distribution function $\rho_N(x)$ of zeros of the polynomial $P_N(x)$. Using the *Perron-Stieltjes* inversion formula (e.g. [10], p.90, [18], [19], [22]) one may (under certain assumptions) recover or find $\rho_N(x)$ from the *Stieltjes* integral which turns out to be $\chi(N,z) \;=\; \frac{1}{z}\,G(N,\frac{1}{z})$. Especially, in the limit $N\to\infty$ one can find in this way $\rho(x) := \lim_{N\to\infty}\; \rho(N,x)$ directly from the moment generating function provided the $N\to\infty$ limit can be performed. For this it is necessary to know the functional dependence on N. (See *section 3* for the examples of classical orthogonal polynomials.)

One may think of the rth moments of zeros of a polynomial $P_N(x)$ as rth moments around the origin $x=0$ of a purely discontinuous probability distribution

$$m_r(N) \;:=\; \frac{1}{N}\sum_{i=1}^{N}\big(x_i^{(N)}\big)^r \;=\; \int_{-\infty}^{+\infty} x^r\, d\rho_N(x) \quad , \tag{4.1}$$

$$d\rho_N(x) \;:=\; \frac{1}{N}\sum_{k=1}^{N}\delta(x-x_k^{(N)})\; dx \quad . \tag{4.2}$$

* The $N\to\infty$ limit of $G^{(H)}(N,z/\sqrt{N})$ obtained *via* (3.18*b*) is expected to be $Q_0^{(H)}(z)$ of (3.34).

Dirac's δ distribution has been used. For this section we assume that the zeros of P_N are real, not necessarily simple, in order to have a measure on the real line. $\int_{-\infty}^{+\infty} d\rho_N(x) \; = \; 1$. The $r = 1$ moment $m(N)$ is the mean (around $x = 0$). The corresponding variance is $\sigma_N^2 \; = \; \int_{-\infty}^{+\infty} |x - m(N)|^2 d\rho_N(x)$.

Lemma 1:

The ordinary (as opposed to exponential) moment generating function $G(N, z) \; := \; \sum_{r=0}^{\infty} m_r(N) \; z^r$ satisfies (in the sense of formal power series) for $z \neq 1/x_k^{(N)}$, $k = 1, 2, .., N$,

$$G(N, z) \; = \; \int_{-\infty}^{\infty} \frac{1}{1 - z \; x} \; d\rho_N(x) \quad = \; \frac{1}{N} \sum_{k=1}^{N} \frac{1}{1 - z \; x_k^{(N)}} \quad . \tag{4.3}$$

Proof: Insert the moments (4.1) into the definition of $G(N, z)$, interchange summation and integration, and use the geometric series (in the formal sense). The last eq. results from (4.2). The denominators never vanish due to the assumption on z.

We mention that in general an exponential generating function $E(N, z)$ is related to the ordinary one, $G(N, z)$, by a *Laplace* transform $\mathcal{L} \, : \; \frac{1}{z} \; G(N, \frac{1}{z}) \; = \; \mathcal{L}[E(N, z)]$ with certain restrictions on the real part of z (*cf. e.g.* [20] Vol I, p.143 (1)).

It is useful to define what we shall call the *Stieltjes* transform of the discrete measure $\rho_N(x)$ *

$$\chi(N, z) \; := \; \int_{-\infty}^{+\infty} \frac{1}{z - x} \; d\rho_N(x) \quad , \quad z \notin supp(d\rho_N) \quad . \tag{4.4}$$

Theorem 8: Stieltjes transform of $\rho_{\mathbf{N}}(\mathbf{x})$

$$\chi(N, z) \; = \; \frac{1}{z} \; G(N, \frac{1}{z}) \quad . \tag{4.5}$$

Proof: This follows from the definition (4.4) and *Lemma 1.* □

Corollary 7: Independent proof of Theorem 2

$$N \; \chi(N, z) \; = \; \left(ln \; \tilde{P}_N(z)\right)' \; = \; \left(ln \; P_N(z)\right)' \; , \tag{4.6}$$

where $\tilde{P}_N(x) \; = \; \prod_{k=1}^{N} (x - x_k^{(N)})$.

Proof: Use (4.5) together with *Lemma 1.* □

$N \; \chi(N, z) \; = \; P_N'(z)/P_N(z)$ is the $N-$th approximation of a continued fraction.

Lemma 2: Continued fraction approximation for Stieltjes transform

With the Wronskian defined by $W(A(z), B(z)) \; := \; A(z) \; B'(z) \; - A'(z) \; B(z)$ one finds

$$N \; \chi(N, z) \; = \; \frac{\tilde{P}_N'(z)}{\tilde{P}_N(z)} \; = \; \frac{a_1(z) \mid}{\mid b_1(z)} \; + \frac{a_2(z) \mid}{\mid b_2(z)} \; + \; . \; . \; . \; + \; \frac{a_N(z) \mid}{\mid b_N(z)} \quad , \tag{4.7}$$

---

* A *Stieltjes* transform, which is a double *Laplace* transform, is usually defined as in (4.4) but integrated along the positive $x-$axis. One could (4.4) also call a *Hamburger* or *Nevanlinna* transform.

with

$$a_k(z) \equiv -W(\tilde{P}_{k-1}(z), \tilde{P}_k(z))/W(\tilde{P}_{k-2}(z), \tilde{P}_{k-1}(z)) \quad , \tag{4.8a}$$

$$b_k(z) \equiv W(\tilde{P}_{k-2}(z), \tilde{P}_k(z))/W(\tilde{P}_{k-2}(z), \tilde{P}_{k-1}(z)) \quad , \tag{4.8b}$$

where one has to put $\tilde{P}_{-1} = 0$, $\tilde{P}_0 = 1$, $\tilde{P}'_{-1} = 1$, $\tilde{P}'_0 = 0$.

Proof: The standard procedure to compute the continued fraction for a given $N-$th numerator/denominator approximation is applied (*cf. e.g.* [21], §3). The necessary condition $W(\tilde{P}_k(z), \tilde{P}_{k+1}(z)) \neq 0$, for $k = -1, 0, 1, 2, ...$ is fulfilled because the polynomials have different degree. □

We mention that for orthogonal polynomials

$$W(\tilde{P}_k(z), \tilde{P}_{k+1}(z)) \;=\; \sum_{j=0}^{k} \lambda_{j+1} \cdots \lambda_k \; \tilde{P}_j^2(z) \quad , \tag{4.9}$$

if the recursion formula is $\tilde{P}_{n+1} \;=\; (z - c_n)\; \tilde{P}_n \;-\; \lambda_n\; \tilde{P}_{n-1}$ with inputs $\tilde{P}_{-1} = 0$ and $\tilde{P}_0 = 1$. This follows from the confluent *Christoffel-Darboux* identity.

The *Perron-Stieltjes* inversion formula for a real measure of bounded variation can be applied to $\chi(N, z) \;=\; P'_N(z)/(N\; P_N(z))$. Thereby one will recover $\rho_N(x)$. The inversion formula is ([10],[18], [19],[22])

$$\rho_N(t_2) \;-\; \rho_N(t_1) \;=\; -\frac{1}{\pi} \lim_{\eta \to +0} \int_{t_1}^{t_2} dt\; Im\; \chi(N, t + i\eta) \quad , \tag{4.10}$$

with $\rho_N(t_k) \;:=\; \frac{1}{2}(\rho_N(t_k + 0) \;+\; \rho_N(t_k - 0))$ for $k = 1, 2$. $\bar{\chi}_N = \chi_N(\bar{z})$ for $z \in \mathbf{C}$, and χ_N is analytic off the real axis. For N finite χ_N has poles only at the zeros of P_N. The inversion formula yields for ρ_N jumps at these real zeros $x_i^{(N)}$ with height $m_i^{(N)}/N$ where m_i counts the multiplicity of the zero x_i. (*cf.*[22] for the proof that poles in χ lead to jumps in ρ. The converse statement is obvious from (4.4).) In this way one recovers the purely discontinuous measure (4.2).

The inversion formula becomes more interesting in the $N \to \infty$ limit, provided the analytic structure of $\chi(z) \;:=\; \lim_{N\to\infty} \chi(N, z)$ is known. For example, for *Chebyshev*'s T_N polynomials

$$\chi^{(T)}(z) \;=\; \lim_{N\to\infty} \frac{1}{z} G^{(T)}(N, \frac{1}{z}) \;=\; \frac{1}{\sqrt{z^2 - 1}} \tag{4.11}$$

due to (4.5) and the asymptotic form of (3.7d) which is (3.30). With the usual definition of the square-root in $\mathbf{C}$, with branch points at 0 and ∞, one finds from the inversion formula the absolutely continuous (a.c.) measure

$$\begin{aligned} \rho^{(T)}(x) \;:=\; \lim_{N\to\infty} \rho_N^{(T)}(x) \;&=\; \frac{d}{dx} \rho_{\text{a.c.}}^{(T)}(x) = -\frac{1}{\pi} \lim_{\eta\to+0} \; Im\; \chi^{(T)}(x + i\eta) \\ &= \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}} \quad , \quad \text{for} \quad |x| < 1 \quad \text{and} \quad 0 \quad \text{otherwise} \quad . \end{aligned} \tag{4.12}$$

This happens to be the orthogonality weight function of the $\{T_N\}$ polynomials (the '$T-$cup', also called inverted semi-circle, see [25], I,(8)). However, for *Chebyshev's* U_N-polynomials one finds the same asymptotic distribution of zeros (in fact for all *Jacobi* polynomials) because the asymptotic moment generating function, hence $\mathcal{X}$, is the same as for the $T-$polynomials. This universality property is known ([23], p.282, *cf.* [9] Ch.XII,p.309, (12.7.9); see also [24]).

Finally we state the asymptotic distribution of zeros for the scaled *Hermite*, resp. *Laguerre* polynomials, $Hs_N(x) \ := \ H_N(\sqrt{N}x)$, resp. $Ls_N^{(\alpha)}(x) \ := \ L_N^{(\alpha)}(Nx)$, as obtained from (3.34), resp. (3.35), with the help of the inversion formula.

$$\rho^{(Hs)}(x) \ = \ \frac{1}{\pi}\sqrt{2-x^2} \ , \quad \text{for} \quad |x| < \sqrt{2} \ \text{ and } \ 0 \ \text{ otherwise} \ . \qquad (4.13)$$

$$\rho^{(Ls^{(\alpha)})}(x) \ = \ \frac{1}{2\pi}\sqrt{\frac{4}{x}-1} \ , \quad \text{for} \quad 0 < x \leq 4 \ \text{ and } \ 0 \ \text{ otherwise} \ . \qquad (4.14)$$

The asymptotic distribution for the scaled *Laguerre* polynomials is α independent. These examples are of the type given in [25], I, (9), and II, (9),respectively. They are both normalised to 1.

Acknowledgements

The author would like to thank Dr. Stephen Bedding for a collaboration on multivariate *Chebyshev*-polynomials in the context of powers of matrices. He is grateful to Dr. Bedding and the School of Mathematics at La Trobe University, Bundoora, Victoria, for the hospitality extended to him during a stay in the Australian spring of 1995.

He also thanks Prof. J. S. Dehesa for sending him reprints of the work of the Granada group on density of zeros of polynomials and functions.

References

[1] K. M. Case: "Sum rules for zeros of polynomials. I ", *J.Math. Phys.* 21(1980) 702-708 and part II, ibid. 709-714

[2] E. Buendia, J. S. Dehesa and F. J. Galvez: "The distribution of zeros of the polynomial eigenfunctions of ordinary differential operators of arbitrary order", pp. 222-235 in:
M. Alfaro, J. S. Dehesa, F. Marcellán, J. L. Rubio de Francia, and J. Vinuesa, edts., Lecture Notes in Mathematics, Vol. 1329, Springer, Heidelberg, 1988

[3] P. A. Macmahon, *"Combinatorial Analysis"*, Chelsea Publ. Comp., New York, 1960

[4] R. Lidl and Ch. Wells: "Chebyshev polynomials in several variables", J.f.d.reine u. angewandte Math. 255 (1972) 104-111

[5] R. Lidl: "Tschebyscheffpolynome in mehreren Variablen", J.f.d. reine u. angew. Math. 273 (1975) 178-198

[6a] H. C. Williams: "Some Properties of the General Lucas Polynomials", The Matrix and Tensor Quarterly, 21(1971)91-93

[6b] I. V. V. Raghavacharyulu and A. R. Tekumalla: "Solution of the Difference Equation of Generalized Lucas Polynomials", J. Math. Phys. 13 (1972) 321-324

[7] J. Riordan: *"An Introduction to Combinatorial Analysis"*, Wiley, New York, 1958

[8] F. Gálvez and J. S. Dehesa: "Some open problems of generalized Bessel polynomials", J. Phys. A 17 (1984) 2759-2766

[9] G. Szegö: *"Orthogonal Polynomials"*, Am. Math. Soc., New York, 1959

[10] T. S. Chihara: *"An Introduction to orthogonal polynomials"*, Gordon and Breach, New York, 1978

[11] V. Krishnamurthy: *"Combinatorics: Theory and Applications"*, Ellis Horwood, Chichester, U.K.,1986

[12] T. J. Rivlin: *"Chebyshev Polynomials"*, 2^{nd} ed., Wiley, New York, 1990

[13] *Maple* is a registered trademark of Waterloo Maple Software

[14] A. Erdélyi, ed., *"Higher Transcendental Functions"*, Vol.I, Vol.II, McGraw-Hill, New York, 1953

[15] A. F. Nikiforov and V. B. Uvarov: *"Special Functions of Mathematical Physics"*, Birkhäuser, Basel, 1988

[16] A. D. Poljanin and V.F. Sajzew: *"Sammlung gewöhnlicher Differentialgleichungen"*, Harri Deutsch, Frankfurt a.M., 1996

[17] P. Nevai and J. S. Dehesa: "On asymptotic average properties of zeros of orthogonal polynomials", SIAM J. Math. Anal. 10 (1979) 1184-1192

[18] A. Wintner: *"Spektraltheorie der unendlichen Matrizen"*, Hirzel, Leipzig, 1929

[19] N. I. Akhiezer: *"The Classical Moment Problem"*, Oliver & Boyd, Edinburgh, 1965

[20] A. Erdélyi, ed., *"Tables of Integral Transforms"*, Vol. I, McGraw-Hill, New York, 1954

[21] O. Perron: *"Die Lehre von den Kettenbrüchen"*, Bd. II, Teubner, Stuttgart, 1957

[22] R. Askey and M. Ismail: "Recurrence relations, continued fractions and orthogonal polynomials", Memoirs of the Am. Math. Soc., Nr. 300, Vol. 49, (1984)

[23] J. S.-Dehesa: "The asymptotical spectrum of Jacobi matrices", J. Comp. Appl. Maths. 3 (1977) 167-171; *ibid.* "The spectrum of Jacobi matrices in terms of its associated weight function", 4 (1978) 275-283

[24] J. S. Dehesa: "The asymptotic behaviour of zeros of orthogonal polynomials", Lett.al Nuovo Cim. 24 (1979) 151-157

[25] F. J. Gálvez and J. S. Dehesa: I: "Quantum systems with a common density of levels", Phys. Lett. 113*A* (1986) 454-458, *ibid.* part II, 122*A* (1987) 385-388

Table 1: Moment generating functions for T_N Chebyshev polynomials, $N = 1, ..., 8$

N	$G^{(T)}(N, z)$	Expansion up to $O(z^{10})$
1	1	1
2	$\frac{1}{1-\frac{1}{2}z^2}$	$1 + \frac{1}{2}z^2 + \frac{1}{4}z^4 + \frac{1}{8}z^6 + \frac{1}{16}z^8$
3	$\frac{1-\frac{1}{4}z^2}{1-\frac{3}{4}z^2}$	$1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{9}{32}z^6 + \frac{27}{128}z^8$
4	$\frac{1-\frac{1}{2}z^2}{1-z^2+\frac{1}{8}z^4}$	$1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \frac{17}{64}z^8$
5	$\frac{1-\frac{3}{4}z^2+\frac{1}{16}z^4}{1-\frac{5}{4}z^2+\frac{5}{16}z^4}$	$1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \frac{35}{128}z^8$
6	$\frac{1-z^2+\frac{3}{16}z^4}{1-\frac{3}{2}z^2+\frac{9}{16}z^4-\frac{1}{32}z^6}$	$1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \frac{35}{128}z^8$
7	$\frac{1-\frac{5}{4}z^2+\frac{3}{8}z^4-\frac{1}{64}z^6}{1-\frac{7}{4}z^2+\frac{7}{8}z^4-\frac{7}{64}z^6}$	$1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \frac{35}{128}z^8$
8	$\frac{1-\frac{3}{2}z^2+\frac{5}{8}z^4-\frac{1}{16}z^6}{1-2z^2+\frac{5}{4}z^4-\frac{1}{4}z^6+\frac{1}{128}z^8}$	$1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \frac{35}{128}z^8$

Table 2: Moment generating functions for $\mathbf{U_N}$ Chebyshev polynomials, $N = 1, ..., 8$

N	$G^{(U)}(N,z)$	Expansion up to $O(z^{10})$
1	1	1
2	$\frac{1}{1-\frac{1}{4}z^2}$	$1 + \frac{1}{4}z^2 + \frac{1}{16}z^4 + \frac{1}{64}z^6 + \frac{1}{256}z^8$
3	$\frac{1-\frac{1}{6}z^2}{1-\frac{1}{2}z^2}$	$1 + \frac{1}{3}z^2 + \frac{1}{6}z^4 + \frac{1}{12}z^6 + \frac{1}{24}z^8$
4	$\frac{1-\frac{3}{8}z^2}{1-\frac{3}{4}z^2+\frac{1}{16}z^4}$	$1 + \frac{3}{8}z^2 + \frac{7}{32}z^4 + \frac{9}{64}z^6 + \frac{47}{512}z^8$
5	$\frac{1-\frac{3}{5}z^2+\frac{3}{80}z^4}{1-z^2+\frac{3}{16}z^4}$	$1 + \frac{2}{5}z^2 + \frac{1}{4}z^4 + \frac{7}{40}z^6 + \frac{41}{320}z^8$
6	$\frac{1-\frac{5}{6}z^2+\frac{1}{8}z^4}{1-\frac{5}{4}z^2+\frac{3}{8}z^4-\frac{1}{64}z^6}$	$1 + \frac{5}{12}z^2 + \frac{13}{48}z^4 + \frac{19}{96}z^6 + \frac{39}{256}z^8$
7	$\frac{1-\frac{15}{14}z^2+\frac{15}{56}z^4-\frac{1}{112}z^6}{1-\frac{3}{2}z^2+\frac{5}{8}z^4-\frac{1}{16}z^6}$	$1 + \frac{3}{7}z^2 + \frac{2}{7}z^4 + \frac{3}{14}z^6 + \frac{19}{112}z^8$
8	$\frac{1-\frac{21}{16}z^2+\frac{15}{32}z^4-\frac{5}{128}z^6}{1-\frac{7}{4}z^2+\frac{15}{16}z^4-\frac{5}{32}z^6+\frac{1}{256}z^8}$	$1 + \frac{7}{16}z^2 + \frac{19}{64}z^4 + \frac{29}{128}z^6 + \frac{187}{1024}z^8$