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The Relations of Inner and Outer Differential Calculi on Quantum Groups

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Abstract The differential caluli (Γ, d) on quantum groups are classified due to the property of the generating element X of its differential d. There are, on the one hand, differential caluli which contain this element X in the basis of one- forms that span Γ , called Inner Differential Calculi. On the other hand, one has the differential caluli which do not contain the generating element X of its differential d; thus they are called Outer Differential Calculi. We show that this two classes of differential caluli, for a given quantum group $\mathcal{A}$, are related by homomorphisms, which map the elements of one class onto elements of the other class.

1.Introduction

The differential calulus on quantum groups, as a special branch of Noncommutative Geometry in the sense of A.Connes [Co1][Co2], should play a prominent role in the construction of physical models based on quantum groups. This stems from the general concept of physics that the dynamics (or sometimes the kinematics), i.e. the equations of motion, are expressed as (systems of) differential equations. Thus one expects in the noncommutative setting the same, or at least a natural counterpart of such structures.

The abstract structure of quantum groups is that of non-commutative non-cocommuta-

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tive Hopf algebras. The objects of our consideration are more precisely the noncommutative generalizations of the functionalgebras $Fun(G)$ of some Lie group G, which we will call for short quantum groups. General references on quantum groups are for example the books [KS][Ks].

The first example of a differential calculus on quantum groups was given by Woronowicz [Wo1]. Shortly afterwards he gave in [Wo2] the general theory of differential calculi on quantum groups. Later there were, and still are, many contributions from several authors to this field. A necessarily personal selection of such contributions is Jurčo [Ju], Carow-Watamura/Schlieker/Watamura/Weich [CSWW], Rosso [Ro], Aschieri/Castellani [AsCa], Schmüdgen/Schüler [SS1][SS2]. What can be extracted from all this papers is, as far as bicovariant differential calculi are concerned, that there are two classes of differential calculi. We will restrict ourself to this situation, because bicovariant differential calculi are the most natural generalizations of the commutative differential calculus.

The above mentioned classification of the bicovariant differential calculi (Γ, d) is due to the properties of the generating element X of the exterior differential d. On the one hand we have the possibility, that the generating element X does not belong to Γ, thus we will call such differential calculi Outer Differential Calculi[2]. On the other hand there are Inner Differential Calculi[3], which are characterized by the fact that the generating element X of its differential d is an element of the set of one-forms ω^i, which span Γ (i.e. $X \in \Gamma$).

In order to show how these two classes are related, we will proceed as follows. Firstly we will give a review of the construction of Outer Differential Calculi, which is nothing else but the original Woronowicz construction [Wo2]. Afterwards we will give a short description of the Inner Differential Calculi, as it was initiated independently by Jurčo [Ju] and Carow-Watamura/Schlieker/Watamura/Weich [CSWW]. This differential calculi[4] are classified by Schmüdgen/Schüler [SS1][SS2] for the q-analogues of the standard series of Lie groups. This review is mainly intended to make the paper selfcontained, and to fix the notation which will be used in the following. All this will be done in the section 2.

In section 3 we show, starting from the splitting of bicovariant differential calculi into Outer Differential Calculi and Inner Differential Calculi, that using one-dimensional bicovariant differential calculi, it is possible to map from one class into the other and vice versa. This is done by construction of the maps Δ_{In}^{Out} and Δ_{Out}^{In} which map the set of isomorphy-classes $\{(\Gamma, \partial)\}$ of Outer Differential Calculi into the set of isomorphy-classes $\{(\hat{\Gamma}, d)\}$ of Inner Differential Calculi and vice versa. Furthermore we give explicitly maps, which associate to any differential calculus in one class one in the other class, and show some of their properties.

In section 4 we give, as an application the decomposition of the q-de Rham complex of the Inner Differential Calculi in the differential bicomplex $(\Gamma^{r,s}, \partial, \delta)$, which consists of the q-de Rham complex of the Outer Differential Calculi and the q-de Rham complex of the one-dimensional bicovariant differential calculi whose relation was shown in section 3.

[2]In the sense of an outer automorphism.

[3]In the sense of an inner automorphism.

[4]Observe that in this context there was used for the first time, as far as I know, the term inner differential calculi, thus implying implicitly our splitting.

2.The Bicovariant Differential Calculi on Quantum Groups

The bicovariance of differential calculi on quantum groups is the substitute of the commutativity of the left and right action of differential forms in the case of Lie groups. So they are in a certain sense the most natural differential calculi on quantum groups. In subsection 2.1 we will give the general properties of bicovariant differential calculi, while describing the Outer Differential Calculi, which is nothing else but a short version of the original Woronowicz approach [Wo2]. In subsection 2.2 we will give a short review of the properties which distinguish the Inner Differential Calculi.

Having a look on the respective constructions of Inner and Outer Differential Calculi, we can see that there is also the possibility to give the name cotangent approach to the Outer Differential Calculi, and tangent approach to the Inner Differential Calculi.

2.1 The Outer Differential Calculi on Quantum Groups

In this section we shortly review the foundational article [Wo2], where the reader can find the details (and is recommended to do so), especially concerning proofs, which wo omit here. Only where we feel it absolutely necessity for the understanding will we sketch the proofs. Our intention is to be close to the original article, but to stress those points which justify the notion of Outer Differential Calculi. In order to facilitate a comparison with the original article, we will quote in brackets its equation numbers of Theorems, etc.

We start from the abstract form of the bundle theoretic point of view on first order differential forms. (This is the reason why we call this calculi cotangent approach differential calculi)

Definition 1[Wo2, Def.1.1]*Let $\mathcal{A}$ be an algebra with unity and Γ be a bimodule over $\mathcal{A}$ with*

$$\partial : \mathcal{A} \longrightarrow \Gamma \, . \tag{1}$$

a linear map. We say that (Γ, ∂) is a first order differential calculus over $\mathcal{A}$ if: -1. For any $a, b \in \mathcal{A}$ the Leibniz-rule holds :

$$\partial(ab) = \partial(a)b + a\partial b \, . \tag{2}$$

-2. Any element $\varrho \in \Gamma$ is of the form:

$$\varrho = \sum_{k=1}^{K} a_k \partial(b_k) \tag{3}$$

where $a_k, b_k \in \mathcal{A}$, ($k = 1, \ldots, K$ and K) is a positive integer.

Two first order differential calculi (Γ, ∂) and (Γ', ∂') are called isomorphic if there exists a bimodule-isomorphism $\mathrm{iso} : \Gamma \to \Gamma'$ such that $\mathrm{iso} \circ \partial a = \partial' a$ for any $a \in \mathcal{A}$. In the following we will identify such isomorphic differential calculi.

For every algebra $\mathcal{A}$ with the multiplication- map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ there exists a linear subset:

$$\mathcal{A}^2 = \{ q \in \mathcal{A} \otimes \mathcal{A} | \; mq = 0 \, \} \, . \tag{4}$$

of $\mathcal{A} \otimes \mathcal{A}$. With the relations

$$c\Big(\sum_k a_k \otimes b_k \Big) \;\; = \;\; \sum_k c a_k \otimes b_k \tag{5}$$

$$\Big(\sum_k a_k \otimes b_k \Big) c \;\; = \;\; \sum_k a_k \otimes b_k c \tag{6}$$

for any $c \in \mathcal{A}$, $\mathcal{A}^2$ has a $\mathcal{A}$-bimodule-structure. Setting, for any $b \in \mathcal{A}$ and using the unit-element I of $\mathcal{A}$, $Db = I \otimes b - b \otimes I$ one can show that $D : \mathcal{A} \to \mathcal{A}^2$ is a linear map and $(\mathcal{A}^2, D)$ is a first order differential calculus over $\mathcal{A}$ (If $\mathcal{A}$ has a unity, this is an universal construction.). Its importance stems from this universality.

Proposition 2[Wo2, Prop.1.1]: *Let $\mathcal{N}$ be a subbimodul of $\mathcal{A}^2$, with $\Gamma = \mathcal{A}^2/\mathcal{N}$ and π is the canonical epimorphism[5] $\mathcal{A}^2 \to \Gamma$ and $\partial = \pi \circ D$. Then (Γ, ∂) is a first order differential calculus over $\mathcal{A}$. Any first order differential calculus over $\mathcal{A}$ can be obtained in this way.*

Up to now these are totally general considerations. For quantum groups we have to introduce the following notations: ϕ is the comultiplication, κ is the antipode and ϵ is the unit. The bicovariance of some bimodule Γ of a quantum group $\mathcal{A}$ is given by the existence of the two linear maps ${}_\Gamma\phi : \Gamma \to \Gamma \otimes \mathcal{A}$ and $\phi_\Gamma : \Gamma \to \mathcal{A} \otimes \Gamma$. They have to fulfill the conditions

$$\begin{array}{lll} \text{left covariance:} & \phi_\Gamma(a\varrho b) & = \phi(a)\phi_\Gamma(\varrho)\phi(b) \\ & (id \otimes \phi_\Gamma) \circ \phi_\Gamma & = (\phi \otimes id) \circ \phi_\Gamma \\ & (\epsilon \otimes id) \circ \phi_\Gamma(\varrho) = \varrho & \\ \text{right covariance:} & {}_\Gamma\phi(a\varrho b) & = \phi(a)\, {}_\Gamma\phi(\varrho)\phi(b) \\ & ({}_\Gamma\phi \otimes id) \circ {}_\Gamma\phi & = (id \otimes \phi) \circ {}_\Gamma\phi \\ & (id \otimes \epsilon) \circ {}_\Gamma\phi(\varrho) = \varrho \, , & \end{array} \tag{7}$$

for all $\forall\, a, b \in \mathcal{A}$ and $\forall\, \varrho \in \Gamma$. If these two maps are compatible, that is if they obey the relation

$$(id \otimes {}_\Gamma\phi) \circ \phi_\Gamma = (\phi_\Gamma \otimes id) \circ {}_\Gamma\phi \tag{8}$$

the triple $(\Gamma, {}_\Gamma\phi, \phi_\Gamma)$ gives a bicovariant bimodule.

The application of the bicovariance condition on some differential calculus (Γ, ∂) over $\mathcal{A}$ gives us a representation of the maps ${}_\Gamma\phi, \phi_\Gamma$:

$$\phi_\Gamma(a\partial b) = \quad \phi(a)\phi_\Gamma(\partial b) \quad = \phi(a)(id \otimes \partial)\phi(b) \tag{9}$$

$${}_\Gamma\phi(a\partial b) = \quad \phi(a)\, {}_\Gamma\phi(\partial b) \quad = \phi(a)(\partial \otimes id)\phi(b) \; . \tag{10}$$

An element $\omega \in \Gamma$ is called left- (respectivly right-) invariant if and only if it obeys

$$\phi_\Gamma(\omega) \quad = \quad (I \otimes \omega) \tag{11}$$

$${}_\Gamma\phi(\omega) \quad = \quad (\omega \otimes I) \; . \tag{12}$$

If both conditions are fulfilled simultaneously $\omega \in \Gamma$ is called biinvariant.

Now one can show that there are sets of left- (respectively right-) invariant elements ${}_{\mathrm{inv}}\Gamma$ (respectively Γ_{inv}) of left- (respectivly right-) covariant bimodules Γ over $\mathcal{A}$. They form a basis $\omega^{\mathcal{I}}$ (respectively $\eta^{\mathcal{I}}$), with $\mathcal{I}$ an Indexset, of the bimodule Γ under consideration. With this basis it can be shown that every left- (respectively right-) covariant bimodule Γ over $\mathcal{A}$ has a unique decomposition. (We give the relations for the rightcovariant bimodule in brackets.)

$$\varrho = \sum_{i \in \mathcal{I}} a_i \omega_i \qquad \left(\varrho = \sum_{i \in \mathcal{I}} a_i \eta_i \right) \tag{13}$$

$$\varrho = \sum_{i \in \mathcal{I}} \omega_i b_i \qquad \left(\varrho = \sum_{i \in \mathcal{I}} \eta_i b_i \right) \, , \tag{14}$$

[5] An epimorphism is a surjective homomorphism.

which are related by a class of characteristic functionals $f_{ij}\,,\,\tilde{f}_{ij}\,\in\mathcal{A}'$ (where $\mathcal{A}'$ is the space of linear functionals over $\mathcal{A}$), respectively, according to:

$$\omega_i b=\sum_{j\in\mathcal{I}}(f_{ij}*b)\omega_j \qquad \left(\eta_i b=\sum_{j\in\mathcal{I}}(b*\tilde{f}_{ij})\eta_j\right) \tag{15}$$

$$a\omega_i=\sum_{j\in\mathcal{I}}\omega_i((f_{ij}\circ\kappa^{-1})*a) \qquad \left(a\eta_i=\sum_{j\in\mathcal{I}}\eta_j(a*(\tilde{f}_{ij}\circ\kappa^{-1})\right)\,. \tag{16}$$

where we have used the convolution product $(\xi * a)\stackrel{def}{=}(id\otimes\xi)\circ\phi(a)$ and $(b*\xi)\stackrel{def}{=}(\xi\otimes id)\circ\phi(b)$ for all $a,b\in\mathcal{A}$ and $\xi\in\mathcal{A}'$. This characteristic functionals have to fulfill the following compatibility relations,

$$f_{ij}(ab)=\sum_{k\in\mathcal{I}}f_{ik}(a)f_{kj}(b) \qquad \left(\tilde{f}_{ij}(ab)=\sum_{k\in\mathcal{I}}\tilde{f}_{ik}(a)\tilde{f}_{kj}(b)\right) \tag{17}$$

$$f_{ij}(I)=\delta_{ij} \qquad \left(\tilde{f}_{ij}(I)=\delta_{ij}\right) \tag{18}$$

where $a\,,a_i\,,b\,,b_i\,\in\mathcal{A}$.

Putting the decomposition and the bicovariance properties together, we obtain a relation for the characteristic functionals, which fixes them uniquely.

Theorem 3[Wo2, Thm.2.4]: *Let $(\Gamma\,,\phi_\Gamma\,,{}_\Gamma\phi)$ be a bicovariant bimodule over $\mathcal{A}$, and $\{\omega_i\,|\,i\in\mathcal{I}\}$ be the basis in the vector space of all left-invariant elements of Γ. Then we have*

1. For any $i\in\mathcal{I}$

$${}_\Gamma\phi(\omega_i)=\sum_{j\in\mathcal{I}}\omega_i\otimes R_{ji} \tag{19}$$

where $R_{ji}\in\mathcal{A}\,(i,j\in\mathcal{I})$ satisfy the following relations

$$\begin{aligned}\phi(R_{ji}) &= \sum_{k\in\mathcal{I}}R_{jk}\otimes R_{ki} && (20)\\ \epsilon(R_{ji}) &= \delta_{ji}\,. && (21)\end{aligned}$$

2. There exists a basis $\{\eta_i|i\in\mathcal{I}\}$ (indexed by the same Indexset $\mathcal{I}$) in the vector space of all right-invariant elements of Γ, such that

$$\omega_i=\sum_{j\in I}\eta_j R_{ji} \tag{22}$$

is fulfilled for all $i\in\mathcal{I}$.

3. With this choise of basis in $\Gamma_{\rm inv}$ the functionals f_{ij} and $\tilde{f}_{ij}$ with $(i,j\in\mathcal{I})$, introduced in eqns. (15)-(18), coincide.

4. For any $i,j,h\in\mathcal{I}$ and any $a\in\mathcal{A}$ we have the relation:

$$\sum_{i\in I}R_{ij}(a*f_{ih})=\sum_{i\in\mathcal{I}}(f_{ji}*a)R_{hi}\,. \tag{23}$$

The construction of the exterior algebra over the bimodule $(\Gamma\,,\phi_\Gamma\;{}_\Gamma\phi)$ is made as follows. We start with a graded algebra of $\mathcal{A}$-modules T^n (with $\mathcal{A}=T^0$)

$$T=\sum_{n=0}^{\infty}\!\oplus\,T^n \qquad \text{with} \qquad T^n=\bigotimes_{i=0}^{n}{}_{\mathcal{A}}\,T\,, \tag{24}$$

where the tensor-product of the $\mathcal{A}$-modules T (respectively Γ) is given by the relation $\omega \otimes_{\mathcal{A}} a\eta = \omega a \otimes_{\mathcal{A}} \eta$ for $\forall\, \omega, \eta \in T$ (respectively Γ) and $\forall\, a \in \mathcal{A}$. Associating the two grade preserving linear maps $\phi_T : T \to \mathcal{A} \otimes T$ and ${}_T\phi : T \to T \otimes \mathcal{A}$, that is:

$$\phi_T(T^n) \subset \mathcal{A} \otimes T^n \tag{25}$$
$${}_T\phi(T^n) \subset T^n \otimes \mathcal{A} \tag{26}$$

this makes $(T, \phi_T, {}_T\phi)$ a bicovariant, graded algebra. The restriction of ϕ_T (respectively ${}_T\phi$) on T^n is called ϕ_T^n (respectively ${}_T\phi^n$). If we have $T^0 = \mathcal{A}$ and $\phi_T^0 = {}_T\phi^0 = \phi$, then $(T^n, \phi_T^n, {}_T\phi^n)$ is a bicovariant bimodule over $\mathcal{A}$.

Let $(\Gamma, \phi_\Gamma, {}_\Gamma\phi)$ be a given bicovariant bimodule over $\mathcal{A}$, then we call$(T, \phi_T, {}_T\phi)$ a graded algebra built over $(\Gamma, \phi_\Gamma, {}_\Gamma\phi)$ if
1. $T^0 = \mathcal{A}$ and $\phi_T^0 = {}_T\phi^0 = \phi$.
2. The bicovariant bimodule $(T^1, \phi_T^1, {}_T\phi^1)$ coincides with $(\Gamma, \phi_\Gamma, {}_\Gamma\phi)$.
3. T is generated by an element of grade 1, i.e. any element $\tau \in T^n (n = 2, 3, \ldots)$ is of the form $\tau = \sum_i \tau^i$, where τ^i, for all i, is a product of n elements of Γ.

Applying this method it is possible to associate to any bimodule $(\Gamma, \phi_\Gamma, {}_\Gamma\phi)$ two bicovariant graded bimodules $(\Gamma^\otimes, \phi_\Gamma^\otimes, {}_\Gamma\phi^\otimes)$, the tensor-algebra, and $(\Gamma^\wedge, \phi_\Gamma^\wedge, {}_\Gamma\phi^\wedge)$, the exterior algebra. The exterior algebra $(\Gamma^\wedge, \phi_\Gamma^\wedge, {}_\Gamma\phi^\wedge)$ is obtained from the tensor algebra $(\Gamma^\otimes, \phi_\Gamma^\otimes, {}_\Gamma\phi^\otimes)$ due to the factorization of the symmetric part (for details see chapter 3 of [Wo2]).

This exterior algebras are characterized as follows:
Theorem 4[Wo2, Thm.3.3]: *Let $(\tilde{\Gamma}, \tilde{\phi}_\Gamma, {}_\Gamma\tilde{\phi})$ be a bicovariant bimodule over $\mathcal{A}$ and $(\tilde{\Gamma}^\wedge, \tilde{\phi}_\Gamma^\wedge, {}_\Gamma\tilde{\phi}^\wedge)$ be the exterior algebra built over $(\tilde{\Gamma}, \tilde{\phi}_\Gamma, {}_\Gamma\tilde{\phi})$, with the left- and right-invariant sub-bimodule Γ of $\tilde{\Gamma}$. The maps ϕ_Γ (respectively ${}_\Gamma\phi$) are restrictions of the maps $\tilde{\phi}_\Gamma$ (respectivly ${}_\Gamma\tilde{\phi}$) on Γ and $(\tilde{\Gamma}^\wedge, \tilde{\phi}_\Gamma^\wedge, {}_\Gamma\tilde{\phi}^\wedge)$ on the exterior algebra built over $(\Gamma, \phi_\Gamma, {}_\Gamma\phi)$. Then there exists a gradepreserving multiplicative embedding*

$$\Gamma^\wedge \subset \tilde{\Gamma}^\wedge\,, \tag{27}$$

coinciding with id on elements of grade 0 (i.e. on $\mathcal{A}$) and with the inclusion $\Gamma \subset \tilde{\Gamma}$ on elements of grad 1.Moreover this inclusion intertwines the left and right action of $\mathcal{A}$ on $\Gamma^\wedge$ and $\tilde{\Gamma}^\wedge$, respectively.

The use of this construction on first order differential calculi gives then higher order differential calculi , which are described in the following theorem.
Theorem 5[Wo2, Thm.4.1]: *Let $G = (A, u)$ be compact quantum group, and (Γ, ∂) be a first order differential calculus on G; let ϕ_Γ and ${}_\Gamma\phi$ be the left- respective right- actions of G on Γ given in eqns.(7), and let $(\Gamma^\wedge, \phi_\Gamma^\wedge, {}_\Gamma\phi^\wedge)$ be the exterior algebra built over $(\Gamma, \phi_\Gamma, {}_\Gamma\phi)$. Then there exists one and only one lineare map*

$$\partial : \Gamma^\wedge \longrightarrow \Gamma^\wedge\,, \tag{28}$$

such that
1. ∂ increases the grade by one.
2. On elements of grade 0 , ∂ coincides with the original derivation from eqn.(1).
3. ∂ is a graded derivative:

$$\partial(\Theta \wedge \Theta') = \partial(\Theta) \wedge \Theta' + (-)^k \Theta \wedge \partial\Theta' \tag{29}$$

for any $\Theta \in \Gamma^{\wedge k}$ and $\Theta' \in \Gamma^\wedge$ with $k \in \mathbf{N}_0$.

4. ∂ is a coboundary

$$\partial(\partial\Theta) = 0 \tag{30}$$

for any $\Theta \in \Gamma^\wedge$.
5. ∂ is bicovariant:

$$\begin{aligned} \phi_\Gamma^\wedge(\partial\Theta) &= (id \otimes \partial)\phi_\Gamma^\wedge(\Theta) && (31) \\ {}_\Gamma\phi^\wedge(\partial\Theta) &= (\partial \otimes id)\,{}_\Gamma\phi^\wedge(\Theta) && (32) \end{aligned}$$

for any $\Theta \in \Gamma^\wedge$.
Proof (sketch): The applicatin of the extended bimodule construction given in chapter 4 of [Wo1], proofs the theorem.

We associate to the bimodule Γ a free left $\mathcal{A}$-module Γ_0,which is generated by one element X:

$$\tilde{\Gamma} = \Gamma_0 \bigoplus \Gamma = \mathcal{A}X \bigoplus \Gamma \tag{33}$$

thus any element $\tilde{\xi} \in \tilde{\Gamma}$ has the form

$$\tilde{\xi} = cX + \xi\ , \tag{34}$$

uniquely determined for any $c \in \mathcal{A}$ and $\xi \in \Gamma$.

The right-multiplication of elements of $\mathcal{A}$ on $\tilde{\Gamma}$ is defined by the relation

$$\tilde{\xi}a \stackrel{def}{=} caX + c(\partial a + \xi a) \qquad\qquad \forall\, a \in \mathcal{A}\ . \tag{35}$$

From this one can check that $\tilde{\Gamma}$ is a bimodule over $\mathcal{A}$ and furthermore, with an appropriate choice (namely $(c = I\ , \xi = 0)$), one obtains the defining relation of the exterior derivative ∂:

$$\partial a = Xa - aX \tag{36}$$

The fact that $X \notin \Gamma$ is the justification to call such calculi **Outer Differential Calculi** (in the sense of an outer automorphism).

The choice

$$\begin{aligned} \tilde{\phi}_\Gamma(\tilde{\xi}) &= \phi(c)(X \otimes I) + \phi_\Gamma(\xi)\ , && (37) \\ {}_\Gamma\tilde{\phi}(\tilde{\xi})) &= \phi(c)(I \otimes X) + {}_\Gamma\phi(\xi) && (38) \end{aligned}$$

of the extended left- (respectively right-) action of G makes $(\tilde{\Gamma}\, , \tilde{\phi}_\Gamma\, , {}_\Gamma\tilde{\phi})$ a bicovariant bimodule over $\mathcal{A}$, which contains $(\Gamma\, , \phi_\Gamma\, , {}_\Gamma\phi)$ as a invariant sub-bimodule. The extension of $(\tilde{\Gamma}\, , \tilde{\phi}_\Gamma\, , {}_\Gamma\tilde{\phi})$ to the exterior bicovariant bimodule $(\tilde{\Gamma}^\wedge\, , \tilde{\phi}_\Gamma^\wedge\, , {}_\Gamma\tilde{\phi}^\wedge)$ makes $\Gamma^\wedge$ according to theorem 4 an exterior bicovariant differential bimodule.

From eqns.(37) and (38) one obtains

$$X \wedge X = 0\ . \tag{39}$$

This allows us to put, as a generalization of eqn. (36) to $\forall\Theta \in \tilde{\Gamma}$,

$$\partial\Theta = [X\ , \ \Theta]_{grad} \tag{40}$$

where the graded commutator $[\ ,\ ]_{grad}$ is given by

$$[X\ , \ \Theta]_{grad} \stackrel{\rm def}{=} \begin{cases} X \wedge \Theta - \Theta \wedge X & \text{if grad } \Theta \text{ is even} \\ X \wedge \Theta + \Theta \wedge X & \text{if grad } \Theta \text{ is odd}\ . \end{cases} \tag{41}$$

Thus we have the following mapping property

$$[\ ,\ ]_{grad} : \tilde{\Gamma}_0 \times \tilde{\Gamma}^{\wedge} \longrightarrow \Gamma^{\wedge}\ . \tag{42}$$

The coboundary condition

$$\partial^2 = 0 \tag{43}$$

is then implied by this construction. ■

Now one constructs the vector space of vectorfields $T \subset \mathcal{A}'$ on $\mathcal{A}$. It is characterized by the right ideal $\Re$ of $\mathcal{A}$, which is defined as the subset of the kernel of the counit $\ker(\epsilon)$ that gives the subbimodule $\mathcal{N}$ from Prop.2 as follows

$$T = \left\{ \chi \in \mathcal{A}' \,|\, \chi(I) = 0 \;\bigwedge\; \chi(a) = 0 \;\text{ for }\; \forall\, a \in \Re \right\}\ . \tag{44}$$

Then we have

Theorem 6[Wo2, Thm.5.1]: *There exists a unique bilinear form:*

$$\langle\ ,\ \rangle : \Gamma \times T \to \mathbf{C} \qquad\qquad (\varrho, \chi) \in \Gamma \times T :\ \langle \varrho, \chi \rangle \in \mathbf{C}\ , \tag{45}$$

such that for any $a \in \mathcal{A}\,, \varrho \in \Gamma$ and $\chi \in T$ we have

$$\begin{aligned} \langle a\varrho, \chi \rangle &= \epsilon(a)\, \langle \varrho, \chi \rangle & (46)\\ \langle \partial a, \chi \rangle &= \chi(a)\ . & (47)\end{aligned}$$

Moreover, denoting by ${}_{inv}\Gamma$ the set of all leftinvariant elements of Γ, we have

1. For any $\omega \in {}_{inv}\Gamma$,

$$\left(\begin{array}{c} \langle \omega, \chi \rangle = 0 \\ \textit{for}\ \forall\, \chi \in T \end{array} \right) \Longrightarrow (\omega = 0)\ . \tag{48}$$

2. For any $\chi \in T$,

$$\left(\begin{array}{c} \langle \omega, \chi \rangle = 0 \\ \textit{for}\ \forall\, \omega \in {}_{inv}\Gamma \end{array} \right) \Longrightarrow (\chi = 0)\ . \tag{49}$$

This makes it natural that T and ${}_{inv}\Gamma$ are a dual pair of vector spaces with respect to $\langle\ ,\ \rangle$. Due to this bilinear form it is possible to obtain a dual basis χ_i of vector fields, which one obtains from the basis ω^i of ${}_{inv}\Gamma$.

$$\left\langle \omega^i\ ,\ \chi_j \right\rangle = \delta^i_j \qquad\qquad (i, j \in \mathcal{I}, \text{with } \mathcal{I} \text{ is an index- set})\ . \tag{50}$$

From eqn.(47) one obtains a family $a_i (i \in \mathcal{I})$ of elements of $\ker(\epsilon)$ which fulfill the relation

$$\chi_i(a_j) = \delta_{ij}\ . \tag{51}$$

They give us a connection of the basis ω^i and the exterior differential ∂.

Theorem 7[Wo2, Thm.5.2]: *1. For any $a \in \mathcal{A}$*

$$\partial a = \sum_{i \in \mathcal{I}} (\chi_i * a) \omega^i\ . \tag{52}$$

2. Let $\{f_{ij} \,|\, i, j \in I\}$ be a family of linear functionals on $\mathcal{A}$, ($f_{ij} \in \mathcal{A}'$), introduced in eqns.(15)-(18). Then we have, for any $i \in I$ and $a, b \in \mathcal{A}$

$$\chi_i(ab) = \sum_{j \in \mathcal{I}} \chi_j(a) f_{ji}(b) + \epsilon(a)\chi_i(b)\ . \tag{53}$$

In particular we have for any $i, j \in \mathcal{I}$ and $b \in \mathcal{A}$

$$\chi_i(a_j b) = f_{ji}(b) \ .$$

3. Let $\{R_{ij} \,|\, i,j \in \mathcal{I}\}$ the family of elements of $\mathcal{A}$ introduced in Theorem 3. Then

$$R_{ij} = (\chi_i \otimes id)(\mathrm{ad}(a_j)) \,. \tag{54}$$

Here we have used the definition $\mathrm{ad}(a) = \sum_k b_k \otimes \kappa(a_k) c_k$, with $a_k, b_k, c_k \in \mathcal{A}$ such that $(id \otimes \phi) \circ \phi = \sum_k a_k \otimes b_k \otimes c_k$.

Due to the structure of the exterior algebra $\Gamma^\wedge$ it is possible to construct a higher order differential calculus. The q-Lie-bracket is then given by

$$[\chi' \,,\, \chi''] \stackrel{\mathrm{def}}{=} \chi' * \chi'' - \sum_s \chi''_s * \chi'_s \ , \tag{55}$$

with χ'_s, χ''_s $(s = 1, \ldots, S)$ and $S \in \mathbf{N}$ are elements of T, such that[6]

$$\sigma^t(\chi' \otimes \chi'') = \sum_s \chi''_s \otimes \chi'_s \,. \tag{56}$$

Then we have
Theorem 8[Wo2, Thm.5.3]:*1. For $\forall \chi', \chi'' \in T$*

$$[\chi' \,,\, \chi''] \in T \,. \tag{57}$$

2. If $\chi'_s, \chi''_s (s = 1, \ldots, S)$ are elements of T such that $\sigma^t(\sum_s \chi'_s \otimes \chi''_s) = \sum_s \chi'_s \otimes \chi''_s$ then

$$\sum_s [\chi'_s \,,\, \chi''_s] = 0 \,. \tag{58}$$

3. For $\forall \chi, \chi', \chi'' \in T$

$$[\chi \,, [\chi' \,,\, \chi'']] = [[\chi \,,\, \chi'] \,,\, \chi''] - \sum_s [[\chi \,,\, \chi''_s] \,,\, \chi'_s] \tag{59}$$

where $\chi'_s, \chi''_s (s = 1, \ldots, S)$ are elements of T introduced in eqn.(56).

We now have that eqn.(58) is the q-deformed antisymmetry of the commutator, while eqn.(59) gives the q-analog of the Jacobi-identity.

In terms of the basis elements ω^i of ${}_{inv}\Gamma$ this reads

$$\sigma_\Gamma(\omega_i \otimes \omega_j) = \sum_{k.l \in \mathcal{I}} \lambda_{ij}{}^{kl} (\omega_k \otimes \omega_l) \tag{60}$$

with $\lambda_{ij}{}^{kl} \in \mathbf{C}$ $(i, j, k, l \in I)$, which implies on the dual space T the corresponding relations

$$\sigma^t(\chi_k \otimes \chi_l) = \sum_{i,j \in \mathcal{I}} \lambda_{ij}{}^{kl} (\chi_i \otimes \chi_j) \ . \tag{61}$$

Thus we obtain for the q-Lie-bracket in terms of the dual basis χ_i

$$[\chi_k \,,\, \chi_l] = \chi_k * \chi_l - \sum_{i,j \in \mathcal{I}} \lambda^{ij}{}_{kl} (\chi_i * \chi_j) \ , \tag{62}$$

[6] σ^t is the twistmap in the space $T \otimes T$ dual to the twist-map σ_Γ on the space $\Gamma \otimes \Gamma$.

and the corresponding form of the q-Jacobi-identity

$$[\chi_i\,,[\chi_j\,,\,\chi_k]] = [[\chi_i\,,\chi_j]\,\,,\,\chi_k] - \sum_{l,m\in\mathcal{I}} \lambda^{lm}{}_{jk}\,[[\chi_i\,,\chi_l]\,\,,\,\chi_m]\,. \tag{63}$$

The same construction works as well for right invariant differential calculi.

2.2 The Inner Differential Calculi on Quantum Groups

The Inner Differential Calculi on quantum groups are based on the FRT-construction [FRT] of quantum groups, which uses the duality of the algebra of functions (quantum groups) $\mathrm{A_R}$ and the Universal Envelopping Algebra (QUEA) $\mathrm{U_R}$. The regular functionals $(L^{\pm})^a_b$ form a basis of the QUEA $\mathrm{U_R}$. Relying on this structure [Ju],[CSWW] constructed bicovariant differential calculi on the quantum groups, generalizing the A_n-, B_n-, C_n-, D_n-series of Lie groups. We will in our presentation follow the paper [AsCa], where the reader can find the details.

The regular functionals $(L^{\pm})^a_b$ have the characteristic contractions with the generators $t^c{}_d$ of the function-algebra A_R, which gives the fundamental R-matrix

$$(L^{\pm})^a_b(t^c{}_d) = R^{\pm ac}{}_{bd}\,. \tag{64}$$

This fixes the QUEA $\mathrm{U_R}$. (The specific form of the R-matrices can be found in [AsCa]).

The coalgebra-structure of $\mathrm{U_R}$, in this representation, is given by the following relations (in the following we will use the Einstein-summation-convention),

$$\begin{aligned}
\phi_d((L^{\pm})^a_b) &= (L^{\pm})^a_g \otimes (L^{\pm})^g_b &(65)\\
\epsilon_d((L^{\pm})^a_b) &= \delta^a_b &(66)\\
\kappa_d((L^{\pm})^a_b) &= (L^{\pm})^a_b \circ \kappa_c &(67)
\end{aligned}$$

where ϕ_d is the comultiplcation in $\mathrm{U_R}$, ϵ_d is the counit in $\mathrm{U_R}$ and κ_d is the antipode in $\mathrm{U_R}$ (the subskripts d (respectively c) refer to the dual $\mathrm{U_R}$-Hopf-algebra (respectively the original function algebra

A_R-Hopf-algebra)) structures. This Hopf-algebra properties fixes the relations among the generators of the differental calculus.

Due to the consistency condition of the exterior differential d and the bimodule-structure, one obtains relations among the vectorfields χ_i and the characteristic functionals f^i_j of the bimodule (compare this with eqns.(15)-(18) and (44),(50)). These bicovariance conditions are

$$\begin{aligned}
[\chi_i\,,\,\chi_j] &\equiv \chi_i\chi_j - \Lambda^{kl}{}_{ij}(\chi_k\chi_l) = \mathrm{C}_{ij}{}^k\chi_k &(68)\\
\Lambda^{nm}{}_{ij}f^i{}_pf^j{}_q &= f^n_if^m_j\Lambda^{ij}{}_{pq} &(69)\\
\mathrm{C}_{mn}{}^if^m{}_jf^n{}_k + f^i{}_j\chi_k &= \Lambda^{pq}{}_{jk}\chi_pf^i{}_q + \mathrm{C}_{jk}{}^lf^i{}_l &(70)\\
\chi_kf^n{}_l &= \Lambda^{ij}{}_{kl}f^n{}_i\chi_j\,. &(71)
\end{aligned}$$

Here are the $\mathrm{C}_{ij}{}^k$ the q-structure constants (which are implicit in eqn.(57)) and the $\Lambda^{kl}{}_{ij}$ are the q-commutation functions of the vectorfields (like in eqn.(62)), (For convenience we have changed the notation of the $\lambda^{kl}{}_{ij}$ from eqn.(62) to $\Lambda^{kl}{}_{ij}$). The Hopf-algebra costructure of $\mathrm{U_R}$ expressed in the vectorfields χ_i and the characteristic functionals f^i_j gets the following form

$$
\begin{aligned}
\phi_d(\chi_i) &= \chi_j \otimes f^j{}_i + \epsilon_c \otimes \chi_i \qquad\qquad \text{(observe: } \epsilon_c = I_d) && (72)\\
\epsilon_d(\chi_i) &= 0 && (73)\\
\kappa_d(\chi_i) &= -\chi_j \kappa_d(f^j{}_i) && (74)\\
\phi_d(f^i{}_j) &= f^i{}_k \otimes f^k{}_j && (75)\\
\epsilon_d(f^i{}_j) &= \delta^i{}_j && (76)\\
\kappa_d(f^k{}_j) f^j{}_i &= \delta_i{}^k \epsilon_c = f^k{}_j \kappa_d(f^j{}_i)\,. && (77)
\end{aligned}
$$

The functionals $\chi_i, f^i{}_j, \mathrm{C}_{ij}{}^k, \Lambda^{ij}{}_{kl}$, which fulfill the bicovariance conditions, can be rephrased in terms of the regular functionals $(L^\pm)^a_b$ and the R-matrices $R^{\pm ac}{}_{bd}$. This can be done with the help of the eqns.(65)-(67) and (72)-(77). There one has used the following convention $\{ {}_a{}^b \hat{=}\, {}^i,\ {}^b{}_a \hat{=}\, {}_i \}$ for the replacement of the "vector indices" by "spinor indices", i.e. double indices which reflect the matrix structur of the $L^\pm$- and $R^\pm$-functionals. Now, using the normalization constant λ (the simplest choice of which is $q - q^{-1}$), one obtains

$$
\begin{aligned}
\chi_i &\equiv \chi^{c_1}{}_{c_2} = \frac{1}{\lambda}\left\{\kappa_d((L^+)^{c_1}{}_b)(L^-)^b{}_{c_2} - \delta^{c_1}{}_{c_2}\epsilon_c\right\} && (78)\\
\Lambda^{ij}{}_{kl} &\equiv \Lambda_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} = d^{f_2} d^{-1}{}_{c_2} R^{f_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{e_1 a_1} (R^{-1})^{a_2 e_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 f_2} && (79)\\
\mathrm{C}_{ij}{}^k &\equiv \mathrm{C}^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}{}^{c_2} = -\frac{1}{\lambda}\left\{\Lambda_a{}^a{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} - \delta^{b_1}{}_{b_2}\delta^{c_1}{}_{d_1}\delta_{c_2}{}^{d_2}\right\} && (80)\\
f^i{}_j &\equiv f_{a_1}{}^{a_2 b_1}{}_{b_2} = \kappa_d((L^+)^{b_1}{}_{a_1})(L^-)^{a_2}{}_{b_2}\,, && (81)
\end{aligned}
$$

with the notation

$$
d^a \stackrel{\text{def}}{=} \begin{cases} q^{2a-1} & \text{for the } A_{n-1} - \text{series} \\ D^a{}_a & \text{for the } B_n, C_n, D_n - \text{series, with } D^a{}_a = C^{ae}C_{ae} \end{cases}
$$

the details of which can be found in [AsCa]. So we have the tools at hand to construct a bicovariant differential calculus from the $L^\pm$-representation of $\mathrm{U_R}$, which is the reason to call it a tangent approach.

The exterior differential calculus[7], dual to this construction, is obtained by the application of the bilinear form $\langle\ ,\ \rangle$ given in theorem 6. We define the A_R-bimodule $\hat\Gamma$ by the basis $\omega_a{}^b$ of differential-one-forms $(a, b = 1, \ldots, N)$, where N is the dimension of the fundamental representation of A_R. This basis is assumed to be left invariant[8], that is ${}_{inv}\hat\Gamma$ has dimension N^2 and we have (like in eqn.(11))

$$
\phi_{\hat\Gamma}(\omega_a{}^b) = I \otimes \omega_a{}^b \qquad , \qquad \omega_a{}^b \in {}_{inv}\hat\Gamma\,, \tag{82}
$$

which fixes $\phi_{\hat\Gamma}$ on the whole of $\hat\Gamma$. With the characteristic functionals (of the (A_R - $\hat\Gamma$)-commutation) f^i_j, as given in eqn.(81), we have

$$
\omega_{a_1}{}^{a_2} a = (f_{a_1}{}^{a_2 b_1}{}_{b_2} * a)\omega_{b_1}{}^{b_2}\,. \tag{83}
$$

[7]In the sense of exterior algebra

[8]Analogously one can give right invariant calculi

To obtain, in this formalism, the exterior algebra $\hat{\Gamma}$, we need the exterior multiplication of the elements ${\omega_a}^b$. This is, in the case of the B_n-, C_n-, D_n- series, for example, given by

$$\omega_i \wedge \omega_j = -{Z^{kl}}_{ij} \omega_k \wedge \omega_l \, , \tag{84}$$

where we have used the identity

$${Z^{ij}}_{kl} \equiv \frac{1}{q^2 - q^{-2}} \left[{\Lambda^{ij}}_{kl} - {(\Lambda^{-1})^{ij}}_{kl} \right] \, . \tag{85}$$

Here we have, for notational simplicity, returned to the "vector indices". This reflects the generalized commutation properties of the quantum group.

The exterior differential $d : \hat{\Gamma}^{\wedge k} \to \hat{\Gamma}^{\wedge k+1}$ is constructed similar to theorem 5 and the eqns.(34),(35) as well as (40),(41), due to the biinvariant element $\hat{X} = \sum_a {\omega_a}^a$. But observe, in contrary to the considerations given there, we have

$$\hat{X} = \sum_a {\omega_a}^a \in \hat{\Gamma} \, , \qquad\qquad \left\langle \hat{X} \right\rangle = \hat{\Gamma}_0 \, . \tag{86}$$

Thus we have **not** an extended bimodule construction. Nevertheless, we have $\forall\, \Theta \in \hat{\Gamma}^{\wedge k}$, analogously to eqns.(40),(41)

$$d\Theta \stackrel{\text{def}}{=} \frac{1}{\lambda} \left[\hat{X} \, , \, \Theta \right]_{grad} = \frac{1}{\lambda} \left(\hat{X} \wedge \Theta - (-)^k \Theta \wedge \hat{X} \right) \, , \tag{87}$$

with the graded commutator

$$[\ ,\ ]_{grad} : \hat{\Gamma}_0 \times \hat{\Gamma}^{\wedge} \longrightarrow \hat{\Gamma}^{\wedge} \, ,$$

and the normalization constant λ as in eqn.(78) (respectively (81)). Especially in the case $\hat{\Gamma}^{\wedge 0} = \mathrm{A}_R$ we obtain

$$da = \frac{1}{\lambda} \left[\hat{X} \, , \, a \right] \qquad\qquad \forall a \in \mathrm{A}_R \, , \tag{88}$$

which obeys the Leibniz rule

$$d(ab) = (da)b + adb \, . \tag{89}$$

Due to the above construction we have a differential calculus $(\hat{\Gamma}, d)$ over A_R, which is different from the differential calculi obtained in theorem 5, in so far as the generating element $\hat{X}$ of the exterior differential d is contained in $\hat{\Gamma}$. This is the reason to call these differential calculi **Inner Differential Calculi** (in the sense of an inner automorphism).

3. The Reconstruction Theorem

The reconstruction theorem gives the relation between the Outer- and Inner Differential Calculi on quantum groups, described in the previous section. In order to find such a relation we start by a comparison of these two classes of differential calculi on the level of their respective structure of differential forms. That is, we consider their bicovariant bimodules and the associated exterior differentials, respectively.

As we have seen in section 2.1, following the original Woronowicz- construction of the differential calculi (Γ, ∂), the Outer Differential Calculi on quantum groups are characterized by the bicovariant bimodule Γ and the exterior differential ∂, which is explicit in theorem 5. There one finds that, in order to define the exterior differential ∂ one has to apply the extended bimodule construction, i.e. one has to complete the bimodule Γ with a bimodule Γ_0, which is generated by the element X, to obtain the extended bimodule $\tilde{\Gamma}$. As a consequence of eqns.(33)-(36) the differential ∂ is implied by the element X, which is, due to this fact, called the generating element (of ∂) (see also the eqns.(40),(41) for the higher differential calculi).

The first order differential calculi $(\hat{\Gamma}, d)$ of the class of Inner Differential Calculi, on the other hand, as is shown in section 2.2, are given due to the bicovariant bimodule $\hat{\Gamma}$ and the exterior differential d. In contrast to the construction of the Outer Differential Calculi, the exterior differential d is characterized by the canonical element $\hat{X} = \sum_a {\omega_a}^a$, which is an element of the bicovariant bimodule $\hat{\Gamma}$. (Compare with the eqns.(86),(88) and for the higher differential calculi eqn.(87)).

The application of these two constructions of differential calculi on the same quantum group $\mathcal{A}$ leads us, as was shortly sketched above, to two different results, namely to $(\hat{\Gamma}, d)$ in the case of Inner Differential Calculi and to (Γ, ∂) in the case of Outer Differential Calculi. Now comparing the vector space dimension of the respective bimodules of differential-one-forms of the both classes, under the assumption that the subspaces orthogonal to the generating element of the respective differential are isomorphic, we find them different. This is obviously due to to fact that the generating element $\hat{X}$ of the differential d is contained in $\hat{\Gamma}$, contrary to Γ which does not contain the generating element X of its differential ∂. On the other hand, the bimodule Γ is imbedded in the extended bimodule $\tilde{\Gamma}$

$$\tilde{\Gamma} = \Gamma \oplus \Gamma_0 \supset \Gamma \ ,$$

that contains the generating element X of ∂ (compare chapter 4 in [Wo1]). This observation makes it natural to try for an identification of the bimodules $\tilde{\Gamma}$ and $\hat{\Gamma}$. The deeper reason for such an identification is due to the fact that it is always possible to find a Γ which is isomorphic to $\hat{\Gamma}/\hat{\Gamma}_0$. This is true because of the universal character of Prop.2. Furthermore the vector space dimension of $\tilde{\Gamma} = \Gamma \oplus \Gamma_0$ and $\hat{\Gamma}$ coincide. Starting on the contrary from any Γ we have analogously a subbimodule $\hat{\Gamma}/\hat{\Gamma}_0$ isomorphic to Γ. All this is complemented by the fact that the generators of $\hat{\Gamma}_0$ and Γ_0 obey the same relations, and furthermore, as will be shown in Lemma 10, there is a possibility to make a completion of Γ_0 to a one-dimensional bicovariant differential calculus (Γ_0, δ).

Having considered the bimodule part of the definition of the two classes of differential calculi, we have now to turn to the differentials. The exterior differentials ∂, respectively d, show also different properties. While on the one hand the Image $\mathrm{Im}(\partial)$ of the differential ∂, looked upon as a map, does not contain terms proportional to its generator X (see eqn.(42))

$$\Gamma_0 \not\subset \mathrm{Im}(\partial) = \Gamma \ , \tag{90}$$

we have on the other hand that the image $\mathrm{Im}(d)$ of the differential d, generated by $\hat{X}$, contains its generator (see eqn.(86)), thus we have

$$\begin{array}{rcl} \mathrm{Im}(\partial) & \subseteq & \mathrm{Im}(d) \\ & \wedge & \\ \mathrm{Im}(\partial) & \neq & \mathrm{Im}(d) \ . \end{array} \tag{91}$$

In order to investigate the relations of the two classes of differential calculi it is useful to formalize the decomposition of the bimodules $\hat{\Gamma}$ and $\tilde{\Gamma}$ in the parts that generate the respective differentials[9], that is Γ_0 respectively $\hat{\Gamma}_0$, and the subspaces which complement them.

Definition 9: *Let J be the projection operator on the bimodules $\hat{\Gamma}$, respectively $\tilde{\Gamma}$*

$$J : \begin{cases} \hat{\Gamma} \longrightarrow \hat{\Gamma}_0 \\ \tilde{\Gamma} \longrightarrow \Gamma_0 \ , \end{cases} \tag{92}$$

with the projection property $J^2 = J$, and $J^{\perp}$ the complementary operator:

$$J^{\perp} = \begin{cases} (id_{\hat{\Gamma}} - J) : \hat{\Gamma} \longrightarrow \hat{\Gamma}/\hat{\Gamma}_0 \\ (id_{\tilde{\Gamma}} - J) : \tilde{\Gamma} \longrightarrow \tilde{\Gamma}/\Gamma_0 \ . \end{cases} \tag{93}$$

The projection property of $J^{\perp}$ follows easily from that of J.

The projector J will now be used to establish the Lemma which describes the one-dimensional differential calculi associated with Γ_0, respectively $\hat{\Gamma}_0$.

Lemma 10: *Let $\mathcal{A}$ be a quantum group and Γ_0 , respectively $\hat{\Gamma}_0$, bimodules as defined in eqn.(33), respectively eqn.(86). Then we have*
-(1) The one-dimensional bimodules Γ_0 and $\hat{\Gamma}_0$ are isomorphic.
-(2) The bimodule Γ_0 can be completed to a first order differential calculus over $\mathcal{A}$, with the differential δ. This differential calculus (Γ_0, δ) is isomorphic to the restriction $(\hat{\Gamma}_0, d|_{\hat{\Gamma}_0})$ of the Inner Differential Calculus $(\hat{\Gamma}, d)$ on its one-dimensional sub-bimodule $\hat{\Gamma}_0$.

Proof: (1) The one-dimensional bimodule Γ_0 is defined as a free left-$\mathcal{A}$-module generated by the element X. This element obeys the characteristic condition $X \wedge X = 0$ from eqn.(39). The right-multiplication of any $a \in \mathcal{A}$ is given by the restriction of the eqns.(35),(36) to the subspace Γ_0

$$Xa - aX = \partial|_{\Gamma_0} \equiv 0 \ . \tag{94}$$

This is an implication of the fact that the differential ∂ has no image in Γ_0 ($\mathrm{Im}(\partial) \not\supset \Gamma_0$). The eqn.(94) can thus be replaced by the suggestive form

$$Xa - (\epsilon_c * a)X = 0 \tag{95}$$

using the fact that $(\epsilon_c * a) = a$) for any $a \in \mathcal{A}$.

The bimodule $\hat{\Gamma}_0$ is the restriction of the bimodule $\hat{\Gamma}$, given by eqn.(82), to the one-dimensional sub-bimodule $\hat{\Gamma}_0$, which is generated by $\hat{X} = \sum_a {\omega_a}^a$, the canonical element of the bimodule $\hat{\Gamma}$. This sub-bimodule $\hat{\Gamma}_0$ has, up to the right-multiplication rule of elements $a \in \mathcal{A}$

$$\hat{X}a - ({f_a}^{ab}{}_b * a)\hat{X} = 0 \ , \tag{96}$$

by construction, the same properties as the bimodule Γ_0, i.e. we have

$$\hat{X} \wedge \hat{X} = 0 = X \wedge X \ . \tag{97}$$

The comparison of the eqns.(94) and (95) shows that one obtains an isomorphism of the left-pairs (a, X) and $(a, \hat{X})$ by simply identifying the generating elements X and $\hat{X}$. Concerning the associated right-pairs (X, b) and $(\hat{X}, a)$ the isomorphism is given by the relation

$$b = \sum_{a,b} ({f_a}^{ab}{}_b * a) \ , \tag{98}$$

[9] In the case of $\tilde{\Gamma}$ the generated differential is that of Γ, i.e. ∂.

that is, the isomorphism is the convolution product of any $a \in \mathcal{A}$ with the characteristic functionals given in eqn.(83), and the identification of the generating elements X and $\hat{X}$. In this case the isomorphism is relative to the induced left-pairs (b, X) and $((f_a{}^{ab}{}_b * a), \hat{X})$.

(2) The right-multiplication rule described in eqn.(35) is due to the above given isomorphism of Γ_0 and $\hat{\Gamma}_0$ to be interpreted as the simplest choice of a right-multiplication rule. Thus we can define by an appropriate choice of a right-multiplication rule (we add simply a term $\delta a \sim X$) a differential δ on the bimodule Γ_0

$$0 \stackrel{\mathrm{def}}{=} Xa - (f^0{}_0 * a)X \ , \tag{99}$$

where $f^0{}_0$ generalizes the counit ϵ_c in eqn.(95). Thus we have analogously to eqn.(40)

$$\begin{aligned} \delta a &= Xa - aX \\ \delta \Theta &= [X, \Theta]_{grad}|_{\Gamma_0} \ , \qquad\qquad \forall\, \Theta \in \Gamma_0^\wedge \ , \end{aligned} \tag{100}$$

with the graded Lie-bracket

$$[\ ,\ ]_{grad} : \Gamma_0 \times \Gamma_0^\wedge \longrightarrow \Gamma_0^\wedge \ ,$$

and if one constructs, as in theorem 7, a vector field χ_X which is dual to X, then one has

$$\delta a = (\chi_X * a)X \ .$$

This makes (Γ_0, δ) a bicovariant differential calculus on $\mathcal{A}$.

The restriction of the Inner Differential Calculus $(\hat{\Gamma}, d)$ on its one-dimensional subbimodule $\hat{\Gamma}_0$, due to the projection operator J given in definition 9, gives

$$J : \hat{\Gamma} \longrightarrow \hat{\Gamma}_0 \qquad\qquad J \circ d = d|_{\hat{\Gamma}_0} \ , \tag{101}$$

which is an one-dimensional differential calculus $(\hat{\Gamma}_0, J \circ d)$ over $\mathcal{A}$. Due to the fact that the characteristic functionals $f^0{}_0$ and $f_a{}^{ab}{}_b$ involved in the eqns.(96),(99) are elements of the Hopf-algebra $\mathcal{A}'$ dual to the quantum group $\mathcal{A}$, there must exist inverse functionals (in the sense of the antipode) of them.Thus we can define functionals $\tilde{f}^0{}_0$ such that

$$\tilde{f}^0{}_0 : f_a{}^{ab}{}_b \longrightarrow f^0{}_0 \ .$$

Explicitly, this means that for all pairs $f^0{}_0, f_a{}^{ab}{}_b \in \mathcal{A}'$ there exists a

$$\tilde{f}^0{}_0 \stackrel{\mathrm{def}}{=} f^0{}_0\, \kappa_d(f_a{}^{ab}{}_b) \ , \tag{102}$$

which obeys the relation

$$m_d(\tilde{f}^0{}_0 \otimes f_a{}^{ab}{}_b) = \tilde{f}^0{}_0 f_a{}^{ab}{}_b = f^0{}_0 \kappa_d(f_a{}^{ab}{}_b) f_a{}^{ab}{}_b = f^0{}_0 \ .$$

The converse mapping from $f^0{}_0 \rightarrow f_a{}^{ab}{}_b$ is defined analogously. Observing that the the characteristic functionals $f^0{}_0$ and $f_a{}^{ab}{}_b$ respectively fix, by the eqn.(99) respectively eqn.(96), the respective differentials δ and $J \circ d$ of the bimodules Γ_0 and $\hat{\Gamma}_0$, we see due to eqn.(102), the possibility to rewrite the differentials δ and $J \circ d$ into one another.

The combination of the results of the first part of our proof, that is the isomorphism of the bimodules Γ_0 and $\hat{\Gamma}_0$, and the possibility to rewrite $J \circ d$ into δ and vice versa makes the one-dimensional differential calculi $(\hat{\Gamma}_0, J \circ d)$ and (Γ_0, δ) over $\mathcal{A}$ isomorphic. ■

Remark: A realization of the characteristic functionals $f^0{}_0$, given in eqn.(99) in the proof of our Lemma, is given for example by the functional f_z, which is defined in Thm.5.6 of [Wo3]. This functional has all the properties, which one has to have for the characteristic functionals f_{ij}, as setteled in the eqns.(15)-(18) and in theorem 3. That is, the funcional has to obey

$$f_z(I) = 1$$

and

$$f_z(ab) = f_z(a) f_z(b) \ ,$$

for any $a, b \in \mathcal{A}$. This funcionals approach in the limit $z \to 0$ the counit ϵ_c, that is one recovers eqn.(95) in this limit. A further example of such functionals (in the case of $SL_q(N)$ quantum groups) is described in [SS1], in the remark 4 following theorem 2.2.

In order to map the Outer Differential Calculi (Γ, ∂) onto an Inner Differential Calculi $(\hat{\Gamma}, d)$ one has to extend the bimodule Γ by the bimodule which is characterized by the generating element of the exterior differential ∂, i.e. by Γ_0 . Thus we obtain the extended bimodule $\tilde{\Gamma}$, given in eqn.(33)

$$\tilde{\Gamma} = \Gamma \oplus \Gamma_0 \ .$$

Furthermore we have to complete the differential ∂ in such a manner, that the image of this extended differential ∂', $\mathrm{Im}(\partial')$, with $\mathrm{Im}(\partial) \subset \mathrm{Im}(\partial')$ and $\mathrm{Im}(\partial')/\mathrm{Im}(\partial) \sim X$ contains a term proportional to X. This is obtained due to a modification of the right-multiplication rule eqn.(35), in the following way

$$\begin{aligned} \tilde{\xi} a &\stackrel{\text{def}}{=} c({f^0}_0 * a) X + c(\partial a + \xi a) \\ &= c\left[({f^0}_0 - \epsilon_c) * a\right] X + c(\epsilon_c * a) X + c(\partial a + \xi a) \ , \end{aligned} \tag{103}$$

where $\tilde{\xi}$ and ξ are defined as in eqn.(35). The functional ${f^0}_0 \in \mathcal{A}_d$, is as in eqn.(99) the characteristic functional of the $(\mathcal{A} - \Gamma_0)$-commutation, ruling the commutation of the elements $a \in \mathcal{A}$ with the generator X of the bimodule Γ_0.

Observe that eqn.(103) becomes identical to eqn.(35) in the case ${f^0}_0 = \epsilon_c$. Here seems to be the right place to remark that the counit ϵ_c, at least on the formal level, has also the properties of the characteristic functionals f_{ij}, as described in the conditions eqns.(15)-(18). But the vector field associated to such a differential calculus is degenerate (i.e. the vector field is identically 0). Thus, in a certain sense, we can think about Outer Differential Calculi as Inner Differential Calculi, which are degenerate in the sector generating the differential.

Following now the argumentation, which leads from eqn.(35) to eqn.(36), we obtain, applying it to eqn.(103)

$$Xa - (\epsilon_c * a) X = \partial a + \underbrace{\left[({f^0}_0 - \epsilon_c) * a\right] X}_{\delta a} = \partial a + \delta a \ , \tag{104}$$

where we have set $\delta a = ({f^0}_0 - \epsilon_c) * aX = (\chi_X * a) X$. Here χ_X is the vectorfield dual to the generating element X of Γ_0, analogously to the constructions subjet of theorem 6. (Compare the definition of χ_X with eqn.(78) in the case $\chi_X \equiv {\chi^c}_c$.)

As in the eqns.(40)-(42) we obtain for any $\tilde{\Theta} \in \tilde{\Gamma}^\wedge$

$$(\partial + \delta) \tilde{\Theta} = \left[X \, , \, \tilde{\Theta}\right]_{grad} \ , \tag{105}$$

where $[X\,,\;]_{grad}$ now maps onto the whole of $\tilde{\Gamma}^{\wedge}$

$$[X\,,\;]_{grad}:\tilde{\Gamma}^{\wedge n}\longrightarrow\tilde{\Gamma}^{\wedge n+1}\,. \tag{106}$$

This is nothing but the encoding of our right-multiplication rule (103) in the graded bracket.

Due to this construction we get the extended Outer Differential Calculi[10] $(\tilde{\Gamma}\,,\,\partial+\delta)=(\Gamma\,,\,\partial)\oplus(\Gamma_0\,,\,\delta)$ over $\mathcal{A}$, this is the completion of the Outer Differential Calculi $(\Gamma\,,\,\partial)$ by some complementary one-dimensional differential calculus $(\Gamma_0\,,\,\delta)$ as given in Lemma 10.

This differential calculi $(\tilde{\Gamma}\,,\,\partial+\delta)$ are isomorphic to Inner Differential Calculi $(\hat{\Gamma},d)$, which follows from the isomorphy of $(\Gamma_0\,,\,\delta)$ and the one-dimensional sub- differential calculi $(\hat{\Gamma}_0,J\circ d)$ shown in Lemma 10, and Proposition 2. Because, due to Proposition 2, the part of the Inner Differential Calculi $(\hat{\Gamma},d)$ which is complementary to $(\hat{\Gamma}_0,J\circ d)$ (that is $(J^{\perp}\circ\hat{\Gamma},J^{\perp}\circ d)$), which is defined by the projection operator $J^{\perp}$ given in eqn.(93), is isomorphic to some Outer Differential Calculi $(\Gamma\,,\,\partial)$. Thus we have shown the isomorphism of the differential calculi $(\tilde{\Gamma}\,,\,\partial+\delta)$ and a certain Inner Differential Calculi $(\hat{\Gamma},d)$.

This construction allows us to give a family of natural, injective maps

$$\Delta_{In}^{Out}:\{(\Gamma,\partial)\}\longrightarrow\{(\hat{\Gamma},d)\}\,, \tag{107}$$

which maps the set of isomorphy-classes $\{(\Gamma,\partial)\}$ of Outer Differential Calculi into the set of isomorphism-classes $\{(\hat{\Gamma},d)\}$ of Inner Differential Calculi. These maps are parametrized by the characteristic functionals ${f^0}_0$, or equivalently by the choice of δ, as was shown in the eqns.(103), (104).

The map of an arbitrarily chosen, but then fixed, Outer Differential Calculus $(\Gamma\,,\,\partial)$ is given by

$$\Psi_{\Gamma,\Delta_{In}^{Out}(\Gamma)}:(\Gamma\,,\,\partial)\longrightarrow\Delta_{In}^{Out}(\Gamma\,,\,\partial)\,. \tag{108}$$

This map is given by

$$\Psi_{\Gamma,\Delta_{In}^{Out}(\Gamma)}:\Gamma\longrightarrow\Gamma\oplus\Gamma_0 \tag{109}$$

and

$$\Psi_{\Gamma,\Delta_{In}^{Out}(\Gamma)}:\partial\longrightarrow\partial\oplus\delta\,, \tag{110}$$

as in our previous construction.

The map from an Inner Differential Calculus $(\hat{\Gamma},d)$ onto an Outer Differential Calculus $(\Gamma\,,\,\partial)$ is obtained by the restriction of $(\hat{\Gamma},d)$ on the part which is complementary to $(\hat{\Gamma}_0,J\circ d)$. This is done with the help of the projection operators given in def. 9. Applying $J^{\perp}$ from eqn.(93) on $\hat{\Gamma}$ we obtain

$$J^{\perp}\hat{\Gamma}=\hat{\Gamma}/\hat{\Gamma}_0\,, \tag{111}$$

which is the sub-bimodule of $\hat{\Gamma}$ spanned by that basis-elements of $\hat{\Gamma}$, which are the ortho-complement of the generator $\hat{X}$. Thus we have the situation which we have met in the case of Outer Differential Calculi.

Now we split the differential d into two pieces, such that the first part will map onto the sub-bimodule $\hat{\Gamma}/\hat{\Gamma}_0$, and the second part maps onto the complementary sub-bimodule

[10] This differential calculi are Inner Differential Calculi in the sense of our classification.

$\hat{\Gamma}_0$. This splitting is given by the projectors J and $J^{\perp}$ defined in eqns.(92),(93), which complete each other to the identity-map on $\hat{\Gamma}$ $((id_{\hat{\Gamma}} = J + J^{\perp}))$

$$d = (J + J^{\perp}) \circ d \, . \tag{112}$$

With the properties given in Def. 9 we have the maps

$$J \circ d \qquad \hat{\Gamma}^{\wedge n} \longrightarrow \hat{\Gamma}^{\wedge n} \wedge \hat{\Gamma}_0 \qquad \subset \hat{\Gamma}^{\wedge n+1} \tag{113}$$

$$J^{\perp} \circ d \quad \hat{\Gamma}^{\wedge n} \longrightarrow \hat{\Gamma}^{\wedge n} \wedge (\hat{\Gamma}/\hat{\Gamma}_0) \quad \subset \hat{\Gamma}^{\wedge n+1} \, . \tag{114}$$

Then we can define the exterior differential

$$(J^{\perp} \circ d) \, \hat{\Theta} = J^{\perp} \circ \frac{1}{\lambda} \left[\hat{X} \, , \, \hat{\Theta} \right]_{grad} = \frac{1}{\lambda} \left[\hat{X} \, , \, \hat{\Theta} \right]_{grad} |_{\hat{\Gamma}^{\wedge}/\hat{\Gamma}_0} \, , \tag{115}$$

on $\hat{\Gamma}$, where $\hat{\Theta} \in \hat{\Gamma}^{\wedge}$ and $[\ ,\ ]_{grad}$ corresponds to the graded Lie-bracket given in eqn.(87), while $\hat{\Gamma}^{\wedge}/\hat{\Gamma}_0$ is the part of the exterior algebra over $\hat{\Gamma}$ which contains no $\hat{\Gamma}_0$-factors. This implies readily that eqn.(115) has the same structure like eqn.(40). Thus it makes the differential calculus $(J^{\perp} \circ \hat{\Gamma}, J^{\perp} \circ d)$ an Outer Differential Calculus. Remembering now the universality, stated in Prop. 2, there must exist an Outer Differential Calculus (Γ, ∂) which is isomorphic to $(J^{\perp} \circ \hat{\Gamma}, J^{\perp} \circ d)$.

The considerations from above give us a natural, surjective map

$$\Delta^{In}_{Out} : \{(\hat{\Gamma}, d)\} \longrightarrow \{(\Gamma, \partial)\} \, , \tag{116}$$

which maps the set of isomorphism-classes $\{(\hat{\Gamma}, d)\}$ of Inner Differential Calculi onto the set of isomorphism-classes $\{(\Gamma, \partial)\}$ of Outer Differential Calculi.

For an arbitrarily chosen, but then fixed, Inner Differential Calculus $(\hat{\Gamma}, d)$ the map

$$\Phi_{\hat{\Gamma}, \Delta^{In}_{Out}(\hat{\Gamma})} : (\hat{\Gamma}, d) \longrightarrow (\Gamma, \partial) \tag{117}$$

associates to it an Outer Differential Calculus (Γ, ∂) by

$$\Phi_{\hat{\Gamma}, \Delta^{In}_{Out}(\hat{\Gamma})} : \hat{\Gamma} \longrightarrow J^{\perp} \circ \hat{\Gamma} = \hat{\Gamma}/\hat{\Gamma}_0 \equiv \Gamma \tag{118}$$

and

$$\Phi_{\hat{\Gamma}, \Delta^{In}_{Out}(\hat{\Gamma})} : d \longrightarrow J^{\perp} \circ d \equiv \partial \, , \tag{119}$$

given due to the projector, like in our construction above.

Now we are prepared to state the reconstruction theorem.

Theorem 11: *Let $\mathcal{A}$ be a quantum group, which allows for the construction of Inner- as well as Outer Differential Calculi. Then we have*

-(1) There exists a natural, injective family of maps

$$\Delta^{Out}_{In} : \{(\Gamma, \partial)\} \longrightarrow \{(\hat{\Gamma}, d)\} \, ,$$

which maps the set of isomorphy-classes $\{(\Gamma, \partial)\}$ of Outer Differential Calculi into the set of isomorphy-classes $\{(\hat{\Gamma}, d)\}$ of Inner Differential Calculi.

For an arbitrarily chosen, but then fixed, Outer Differential Calculus $(\Gamma\, ,\, \partial)$, the map

$$\Psi_{\Gamma, \Delta^{Out}_{In}(\Gamma)} : (\Gamma \, , \, \partial) \longrightarrow (\Gamma \oplus \Gamma_0 \, , \, \partial \oplus \delta) \, .$$

is an $\mathcal{A}$-bimodule-monomorphism of Γ, which is parametrized by ${f^0}_0$. This parametrization fixes also the differential δ.
-(2) There exists a natural, surjective map

$$\Delta^{In}_{Out} : \{(\hat{\Gamma}, d)\} \longrightarrow \{(\Gamma, \partial)\} \ ,$$

which maps the set of isomorphy-classes $\{(\hat{\Gamma}, d)\}$ of Inner Differential Calculi onto the set of isomorphy-classes $\{(\Gamma, \partial)\}$ of Outer Differential Calculi.

For an arbitrarily chosen, but then fixed, Inner Differential Calculus $(\hat{\Gamma}, d)$, the map

$$\Phi_{\hat{\Gamma}, \Delta^{In}_{Out}(\hat{\Gamma})} : (\hat{\Gamma}, d) \longrightarrow (\Gamma, \partial)$$

is an $\mathcal{A}$-bimodule-epimorphism of $\hat{\Gamma}$. The differential ∂ is fixed by the given differential d.
-(3) On the set of isomorphy-classes $\{(\Gamma, \partial)\}$ of Outer Differential Calculi we have the relation

$$id_{\{(\Gamma,\partial)\}} = \Delta^{In}_{Out} \circ \Delta^{Out}_{In} \ .$$

Proof: (1) The family Δ^{Out}_{In} is given in eqn.(107), due to the construction of an extended bimodule $\tilde{\Gamma}$ and an associated extension of the differential given in eqns.(103),(104) by a modified right-multiplication rule.

The map $\Psi_{\Gamma, \Delta^{Out}_{In}}$ given for an arbitrarily fixed Outer Differential Calculus $(\Gamma\,,\,\partial)$ is parametrized by the characteristic functional ${f^0}_0$, ruling the $(\Gamma_0 - \mathcal{A})$-commutation. It is fixed in the eqns.(108)-(110). The fact that

$$ker(\Psi_{\Gamma, \Delta^{Out}_{In}(\Gamma)}) = \{\emptyset\} \tag{120}$$

is due to the property of mapping the Outer Differential Calculus $(\Gamma\,,\,\partial)$ identically onto itself.

Now let Γ_i ($i \in \mathcal{I}$, $\mathcal{I}$ an indexset) denote the representatives of the isomorphy-classes $\{(\Gamma, \partial)\}$, then we have by eqn.(109)

$$\Psi_{\Gamma, \Delta^{Out}_{In}(\Gamma)}(\Gamma_i) = \Gamma_i \oplus \Gamma_0 \equiv \hat{\Gamma}_i \ . \tag{121}$$

Thus, because the Γ_i are by assumption from different isomorphy-classes, the $\hat{\Gamma}_i$ are in different isomorphy-classes too, and have all a unique Γ_i as inverse image. Thus the maps $\Psi_{\Gamma, \Delta^{Out}_{In}(\Gamma)}$ are $\mathcal{A}$-bimodule-monomorphisms of the bimodules Γ. The image ofthe differential ∂ by $\Psi_{\Gamma, \Delta^{Out}_{In}(\Gamma)}$ is uniquely fixed by the functionals ${f^0}_0$. This finishes the proof of (1).
(2) The map Δ^{In}_{Out} is given in eqn.(116) and is fixed due to the projector $J^\perp$ given in eqn.(93), which projects $\hat{\Gamma}$, as well as d to the corresponding objects (Γ and ∂). The surjectivity is seen due to the fact that, with the map Δ^{Out}_{In} we can allways get an Inner Differential Calculus, which is mapped to a prescribed Outer Differential Calculus $(\Gamma\,,\,\partial)$ by Δ^{In}_{Out}.

The map $\Phi_{\hat{\Gamma}, \Delta^{In}_{Out}(\hat{\Gamma})}$ applied to any Inner Differential Calculus $(\hat{\Gamma}, d)$ given in the eqns. (117)-(119) has a nontrivial kernel

$$ker(\Phi_{\hat{\Gamma}, \Delta^{In}_{Out}(\hat{\Gamma})}) = (\hat{\Gamma}_0, \delta) \ . \tag{122}$$

Its $\mathcal{A}$-bimodule-epimorphism property follows from the fact that, having any Outer Differential Calculus $(\Gamma\,,\,\partial)$, one can with the application of the map $\Psi_{\Gamma, \Delta^{Out}_{In}(\Gamma)}$ associate an Inner Differential Calculus $(\tilde{\Gamma}\,,\,\partial+\delta)$. The application of the map $\Phi_{\hat{\Gamma},\Delta^{In}_{Out}(\hat{\Gamma})}$ on this Inner Differential Calculus gives us back the calculus we have started with. Thus it is possible to obtain any Outer Differential Calculi, using that $\Phi_{\hat{\Gamma},\Delta^{In}_{Out}(\hat{\Gamma})}$ is an $\mathcal{A}$-bimodule-epimorphism. Furthermore we have

$$\Phi_{\hat{\Gamma},\Delta^{In}_{Out}(\hat{\Gamma})} \circ \Psi_{\Gamma,\Delta^{Out}_{In}(\Gamma)}(\partial) = \partial\ , \tag{123}$$

and we have finished the proof of (2).
(3) The set of isomorphy-classes $\{(\Gamma,\partial)\}$ of Outer Differential Calculi is mapped by Δ^{Out}_{In} into the set of isomorphy-classes $\{(\hat{\Gamma},d)\}$ of Inner Differential Calculi as was shown in (1). The application of the map Δ^{In}_{Out} given in (2) on an Inner Differential Calculus obtained as above gives obviously, due to the projection property, back the original calculus we started with. Thus we have verified the relation

$$id_{\{(\Gamma,\partial)\}} = \Delta^{In}_{Out} \circ \Delta^{Out}_{In}\ , \tag{124}$$

which finishes the proof. ■

Remark: All our considerations which lead to Theorem 11 can be applied to left- as well as right-invariant differential calculi.

4. A Differential Bicomplex on Quantum Groups

The eqn.(87) taking into account the results obtained in Lemma 10 and Theorem 11 allows for a splitting of the Cartan condition

$$0 = d^2 = (\partial+\delta)^2 = \partial^2 + \partial\delta + \delta\partial + \delta^2\ . \tag{125}$$

Remembering the relation

$$\partial^2 = 0\ , \tag{126}$$

that is eqn.(43), and using a combination of the eqns.(97) and (100) to obtain the identity

$$\delta^2 = 0\ , \tag{127}$$

we have at hand three independent Cartan conditions. Thus one obtains, by the insertion of the eqns.(126) and (127) in eqn.(125) the relation

$$\partial\delta + \delta\partial = 0\ . \tag{128}$$

The splitting of the exterior differential $d = (\partial+\delta)$ and the corresponding decomposition of the Cartan condition eqn.(125) allows us to rewrite the q-de Rham complex

$$0 \longrightarrow \mathcal{A} \xrightarrow{d} \hat{\Gamma}^{\wedge 1} \xrightarrow{d} \hat{\Gamma}^{\wedge 2} \xrightarrow{d} \cdots \xrightarrow{d} \hat{\Gamma}^{\wedge(\dim\Gamma+1)} \xrightarrow{d} 0\ , \tag{129}$$

of the Inner Differential Calculus $(\hat{\Gamma},d)$ in a differential bicomplex $(\tilde{\Gamma},\partial,\delta)$

$$\begin{array}{ccccccccccc}
\tilde{\Gamma}^{0,0} & \xrightarrow{\partial} & \tilde{\Gamma}^{0,1} & \xrightarrow{\partial} & \tilde{\Gamma}^{0,2} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \tilde{\Gamma}^{0,\dim\Gamma} & \xrightarrow{\partial} & 0 \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} & & \\
\tilde{\Gamma}^{1,0} & \xrightarrow{\partial} & \tilde{\Gamma}^{1,1} & \xrightarrow{\partial} & \tilde{\Gamma}^{1,2} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \tilde{\Gamma}^{1,\dim\Gamma} & \xrightarrow{\partial} & 0 \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} & & \\
0 & & 0 & & 0 & & & & 0 & & .
\end{array} \tag{130}$$

where we have used the notation $\tilde{\Gamma}^{0,0} = \mathcal{A}, \tilde{\Gamma}^{0,1} = \Gamma_0$ and $\tilde{\Gamma}^{1,0} = \Gamma$ from eqn.(33). The spaces $\hat{\Gamma}^{\wedge q+1}$ of the q-de Rham complex are decomposed into two complementary spaces $\tilde{\Gamma}^{0,q+1}$ and $\tilde{\Gamma}^{1,q}$. This reflects the splitting of the spaces of $q+1$-forms $\hat{\Gamma}^{\wedge q+1} = \tilde{\Gamma}^{0,q+1} \oplus \tilde{\Gamma}^{1,q}$ into parts which are zero-forms with respect to Γ_0 (in the first case), respectively one-forms with respect to Γ_0 (in the second case).

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