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Z-string global gauge anomaly and Lorentz non-invariance

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Abstract

Certain (3+1)-dimensional chiral non-Abelian gauge theories have been shown to exhibit a new type of global gauge anomaly, which in the Hamiltonian formulation is due to the fermion zero-modes of a Z-string-like configuration of the gauge potential and the corresponding spectral flow. Here, we clarify the relation between this Z-string global gauge anomaly and other anomalies in both 3+1 and 2+1 dimensions. We then point out a possible trade-off between the (3+1)-dimensional Z-string global gauge anomaly and the violation of CPT and Lorentz invariance.

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1 Introduction

Chiral non-Abelian gauge theories [1, 2] can be probed in a variety of ways, some of which may lead to chiral anomalies [3, 4, 5, 6]. The physical origin of these anomalies, however, remains mysterious [7]. It may, therefore, be of interest that one further probe and corresponding anomaly have been presented recently in Ref. [8]. Whereas that paper was somewhat technical, the present paper aims to give a more general discussion of this new type of chiral gauge anomaly.

The basic theory to be considered in this article is the (3+1)-dimensional chiral $SU(2)$ Yang-Mills theory [1, 2], with a Lie algebra valued gauge potential $W_\mu(x)$ and a single isodoublet of left-handed Weyl fermions $\Psi_L(x)$. The 3-dimensional space manifold M is taken to be the product of a sphere and a circle, $M = S_2 \times S_1$, with coordinates $x^1, x^2 \in S_2$ and $x^3 \in S_1$. The length of the x^3 circle is denoted by L_3. In order to simplify the discussion later on, the time coordinate is also taken to range over a circle, $x^0 \in S_1$, but the case of $x^0 \in \mathbb{R}$ presents no substantial differences. Natural units with $c = \hbar = 1$ are used throughout. Greek indices μ, ν, etc. run over the coordinate labels 0, 1, 2, 3, and Latin indices k, l, etc. over 0, 1, 2. Repeated indices are summed over and the metric has Minkowskian signature $(+---)$.

As mentioned above, this particular gauge theory with non-Abelian gauge group $G = SU(2)$ and massless chiral fermions in a single irreducible representation $r = \mathbf{2}_L$ (isospin $I = \frac{1}{2}$) has been shown to display a new type of chiral gauge anomaly [8]. The idea is to test chiral Yang-Mills theory (gauge group G and unitary representation U_r) with gauge transformations [2],

$$\Psi_L(x) \to U_r\Big(g(x)\Big)\,\Psi_L(x)\;, \quad W_\mu(x) \to g(x)\,\Big(W_\mu(x) + \partial_\mu\Big)\,g^{-1}(x)\;\;, \tag{1}$$

based on gauge functions $g(x) \in G$ which are *independent* of the spatial x^3 coordinate.[2] The second-quantized fermionic vacuum state of the $G = SU(2)$ and $r = \mathbf{2}_L$ chiral Yang-Mills theory in the Hamiltonian formulation (temporal gauge $W_0 = 0$) [7] then has a Möbius bundle structure over a specific non-contractible loop of x^3-independent static gauge transformations. The relevant non-contractible loop of static gauge transformations (1), for $G = SU(2)$ and $r = \mathbf{2}_L$, has gauge functions [8]

$$g_{\rm KR} = g_{\rm KR}\Big(x^1, x^2\,;\,\omega\Big) \in G\;, \tag{2}$$

where $\omega \in [0, 2\pi]$ parametrizes the loop. (The map $g_{\rm KR} : S_2 \times S_1 \to G = SU(2)$ is topologically equivalent to the map $S_3 \to S_3$ with winding number $n = 1$.) The resulting

[2] This paper considers primarily (3+1)-dimensional gauge theories, but the same idea of using restricted gauge transformations may apply to higher dimensional theories with appropriate space-time topologies.

Möbius bundle twist, i.e. the relative phase factor -1 multiplying the fermionic state for $\omega = 2\pi$ compared to $\omega = 0$, makes that Gauss' law cannot be implemented *globally* over the space of gauge potentials, and the theory is said to have a *global* (non-perturbative) gauge anomaly.

The same problem with Gauss' law occurs for an entirely different non-contractible loop of static gauge transformations (1) with gauge functions

$$g_{\mathrm{W}} = g_{\mathrm{W}}\left(x^1, x^2, x^3\,;\,\omega\right) \in G\,, \tag{3}$$

which for $G = SU(2)$ and $r = \mathbf{2}_L$ gives rise to the global $SU(2)$ Witten anomaly in the Hamiltonian formulation [6]. (The map g_{W} for arbitrary but fixed x^3 has winding number $n = 0$.) In both cases, the Möbius bundle twist (Berry phase factor $\exp(i\,\pi) = -1$) of the fermionic vacuum state over the gauge orbit is due to the level-crossing of the Dirac Hamiltonian in a background bosonic field configuration 'encircled' by the loop [9, 10]. For the chiral $SU(2)$ Yang-Mills-Higgs theory relevant to the weak interactions [2, 11, 12], some of these special bosonic field configurations correspond, in fact, to non-trivial classical solutions, namely the Z-string [13] for the gauge orbit (2) and the sphaleron $\mathrm{S}^\star$ [14] for the gauge orbit (3). (The Z-string solution here is simply the non-Abelian embedding of a static Nielsen-Olesen magnetic flux tube [15] aligned along the x^3-axis.) Henceforth, we call the anomaly associated with the gauge transformations (2) the 'Z-string global gauge anomaly,' even for the case of Yang-Mills theories without massive vector fields.

For completeness, there is a third way to probe chiral Yang-Mills theories, which may give rise to the local (perturbative) Bardeen anomaly [5]. For the theory with enlarged gauge group $G = SU(3)$ and a single triplet $\mathbf{3}_L$ of left-handed Weyl fermions, this *local* anomaly results in the Hamiltonian framework from static gauge transformations (1) with gauge functions

$$g_{\mathrm{B}} = g_{\mathrm{B}}\left(x^1, x^2, x^3\,;\,\mathrm{d}^2\Omega\right) \in G \tag{4}$$

corresponding to an *infinitesimal* loop of solid angle $\mathrm{d}^2\Omega$ on a non-contractible sphere of static gauge transformations. (A non-contractible sphere of static gauge transformations can be constructed for gauge group $G = SU(3)$, but not for $G = SU(2)$.) Again, there is a background bosonic field configuration 'inside' the sphere with fermionic levels crossing, which gives rise to the Berry phase factor $\exp\left(i\,\mathrm{d}^2\Omega\,/2\right)$ responsible for the anomaly [10]. To our knowledge, the corresponding classical solution has not been identified, and the construction method of Ref. [14] may turn out to be useful.

The rest of this paper consists of two sections which can be read more or less independently. In Section 2, we compare the Z-string global gauge anomaly with the other gauge anomalies known in (3+1)-dimensional chiral Yang-Mills theory, namely the Witten and Bardeen anomalies already mentioned. In Section 3, we first relate the Z-string global gauge anomaly to previous results in genuinely 2+1 dimensions. Then, we discuss for the (3+1)-dimensional case a possible trade-off between the Z-string global gauge anomaly and the violation of CPT and Lorentz invariance.

2 Z-string global gauge anomaly

General (3+1)-dimensional chiral gauge theories with non-Abelian gauge group G and left-handed Weyl fermions in a (possibly reducible) pseudoreal or complex representation r also give rise to the Z-string global gauge anomaly, provided an anomalous $SU(2)$ theory can be embedded. This leads to the following conditions [8]:

1. a non-trivial third homotopy group of the Lie group G manifold, $\pi_3\,[G] \neq 0$;

2. an appropriate subgroup $SU(2) \subseteq G$, for which the chiral fermions are in an odd number of anomalous irreducible $SU(2)$ representations (these anomalous $SU(2)$ representations have dimensions $\mathbf{n} \equiv 2\,I+1 = 4\,k+2\,,\ k = 0,\,1,\,2,\,\ldots\,$).

In the Hamiltonian formulation, the first condition allows for a non-contractible loop of gauge transformations (2) and the second for a net twist of the fermionic state over the loop. In the Euclidean path integral formulation, the topologically non-trivial four-dimensional gauge transformation is then also given by (2), with ω replaced by the Euclidean coordinate $x^4 \in S_1$, and the fermionic functional integral (fermion 'determinant') picks up an over-all factor -1 from this gauge transformation. See Section 3 for further details on the Z-string global gauge anomaly in the path integral formulation.

The Witten global gauge anomaly has similar conditions [6]:

1. a non-trivial fourth homotopy group of G, $\pi_4\,[G] \neq 0$;

2. chiral fermions in an odd number of anomalous irreducible representations.

In the Euclidean path integral formulation, for example, the first condition allows for a topologically non-trivial four-dimensional gauge transformation (3), with ω replaced by x^4, and the second for an over-all factor -1 from the fermion 'determinant.'

The probe with gauge transformations (2) appears to be more effective in finding anomalous theories than the one with gauge transformations (3), since all compact connected simple Lie groups G have non-trivial π_3, whereas π_4 is non-trivial only for the symplectic groups $Sp(N)$ (recall $Sp(1) = SU(2)$). But, as will be seen shortly, the conditions on the fermion representations make the two types of global gauge anomalies equally effective in this respect.

As mentioned in the Introduction, there exists a further probe of the gauge invariance of chiral Yang-Mills theories, namely the gauge transformations (4) corresponding to the local Bardeen anomaly [5]. This perturbative anomaly, in general, is known to be absent for gauge theories with irreducible fermion representations r of the Lie algebra, t^a_r, $a = 1, \ldots, \dim(G)$, for which the symmetrized traces $D^{abc}[r] \equiv (1/3!)\,\mathrm{Str}\,(t^a_r, t^b_r, t^c_r)$ vanish identically or cancel between the different fermion species present [16]. The question, now, is which standard chiral Yang-Mills theories are free from perturbative gauge anomalies, but not from the Z-string or Witten global gauge anomalies. The answer turns out to be the same for both types of global gauge anomalies and follows from the group theoretic results of Okubo et al. [17], which leave only the sympletic groups $Sp(N)$ with fermion representations containing an odd number of anomalous irreducible represention of an $SU(2)$ subgroup. This can be verified by inspection of Table 58 in Ref. [18], for example.

The main interest of the Z-string global gauge anomaly is therefore not which standard chiral Yang-Mills theory is ruled out or not, but the fact that a different sector of the theory is probed compared to what is done for the other chiral gauge anomalies. In other words, the loop of gauge transformations (2) 'encircling' the Z-string configuration can be used as a new diagnostic tool to investigate (3+1)-dimensional chiral non-Abelian gauge theory. This can be illustrated by the following example.

Consider, again, the standard $SU(3)$ Yang-Mills theory [2] with a single triplet $\mathbf{3}_L$ of left-handed Weyl fermions [1] in the Euclidean path integral formulation. This theory has no genuine Witten anomaly, since $\pi_4\,[SU(3)] = 0$. The non-trivial $SU(2)$ gauge transformation (3), with ω replaced by the Euclidean coordinate x^4, can be embedded in $SU(3)$ to become contractible. But the cumulative effect of the perturbative $SU(3)$ Bardeen anomaly still gives a factor -1 in the path integral for this $SU(2) \subset SU(3)$ gauge transformation [19]. Hence, the $SU(2)$ gauge transformation (3) embedded in $SU(3)$ gives a factor -1 in the fermion 'determinant,' even though it is not guaranteed by a direct topological argument to do so. On the other hand, the gauge transformation (2), with ω replaced by x^4, is topologically non-trivial also in $SU(3)$ and gives directly a global anomaly factor -1, apparently without appeal to the perturbative anomaly. This last observation will be important later on.

3 Counterterm and space-time symmetries

In the previous section, we have compared, for general (3+1)-dimensional chiral Yang-Mills theories, the Z-string global gauge anomaly and the other known chiral gauge anomalies. In this section, we return to the basic chiral $SU(2)$ Yang-Mills theory with a single isodoublet $\mathbf{2}_L$ of left-handed Weyl fermions over the space manifold $M = S_2 \times S_1$, and look for a modification of the theory to get rid of the Z-string global gauge anomaly. One modification would be to impose (anti-)periodic boundary conditions on the (fermionic) bosonic fields at $x^3 = 0$ and $x^3 = L_3$, which would rule out x^3-independent fermionic fields and the associated Z-string global gauge anomaly. Here, we keep the global space-time structure of the theory unchanged ($x^3 \in S_1$ and periodic boundary conditions for all fields) and try to remove the Z-string global gauge anomaly in some other way. But, first, we recall the situation in 2+1 dimensions.

There are no chiral anomalies in (2+1)-dimensional $SU(2)$ Yang-Mills theory with massless fermions, simply because there are no chiral fermions (the Lorentz group $SO(2,1)$ has only one type of spinor representation). Still, the $SU(2)$ Yang-Mills theory with a single isodoublet of massless fermions does have a global gauge anomaly as pointed out, in the path integral formulation, by Redlich [20] and, independently, by Alvarez-Gaumé and Witten [21]. The fermionic functional integral (fermion determinant) picks up a factor -1 from topologically non-trivial three-dimensional gauge transformations with odd winding number n. The $SU(2)$ gauge anomaly can, however, be cancelled by a *local* counterterm in the action. This term in the Lagrangian density is none other than the Chern-Simons three-form known from instanton studies, see for example Ref. [7]. Taking the 2-dimensional space manifold $\tilde{M} = S_2$, the extra term in the action is

$$\Delta I_{2+1} \;=\; \pi\, \Omega_{\mathrm{CS}}\,[W_k] \equiv \pi \int_{S_1} \mathrm{d}x^0 \int_{S_2} \mathrm{d}x^1 \mathrm{d}x^2\; \omega_{\mathrm{CS}}\,(W_k)\;, \tag{5}$$

$$\omega_{\mathrm{CS}}\,(W_k) \;\equiv\; \frac{1}{16\,\pi^2}\,\epsilon^{lmn}\,\mathrm{tr}\left(W_{lm}\, W_n - \frac{2}{3}\, W_l\, W_m\, W_n \right)\;, \tag{6}$$

with the completely antisymmetric Levi-Civita symbol ϵ^{lmn} (Latin indices run over 0, 1, 2, and $\epsilon^{012} = 1$) and the Yang-Mills field strength and gauge potential

$$W_{lm} \equiv \partial_l\, W_m - \partial_m W_l + [\, W_l\,, W_m\,]\;, \quad W_n \equiv \tilde{g}\, W_n^a\, T^a\;, \tag{7}$$

in terms of the dimensionful coupling constant $\tilde{g}$ and the anti-Hermitian Lie group generators T^a normalized by $\mathrm{tr}\,(T^a T^b) = -\,\frac{1}{2}\,\delta^{ab}$ (for gauge group $G = SU(2)$, $T^a \equiv \tau^a/(2\,i)$ with the Pauli matrices τ^a, $a = 1, 2, 3$). For a gauge transformation $W_k \to g\,(W_k + \partial_k)\, g^{-1}$ with gauge function $g(x^0, x^1, x^2) \in G = SU(2)$ of winding number n, the Chern-Simons term (5) gives an extra factor in the path integral

$$\exp\left(\, i\,\pi\, \Omega_{\mathrm{CS}} \left[-\partial_k g\, g^{-1} \right] \right) = \exp\,(i\,\pi\, n) = (-1)^n\;, \tag{8}$$

which cancels against the factor $(-1)^n$ from the fermion determinant. But the Chern-Simons density (6) is odd under the (2+1)-dimensional space-time inversion ('parity') transformation $W_n^a(x) \to -W_n^a(-x)$, whereas the standard Yang-Mills action density [2] is even. In other words, the counterterm (5) removes the $SU(2)$ gauge anomaly and generates the 'parity' anomaly instead. See Ref. [20] for further discussion.

The Z-string global gauge anomaly results of Ref. [8] were obtained in the Hamiltonian formulation of (3+1)-dimensional $SU(2)$ Yang-Mills(-Higgs) theory with a single isodoublet of left-handed Weyl fermions. Dropping the x^3 coordinate altogether (or, more physically, taking $L_3 \to 0$ while keeping the corresponding momenta finite), these Hamiltonian results are directly relevant to the (2+1)-dimensional theory discussed above. For fixed isospin, the single left-handed Weyl spinor $\psi_L(x^0, x^1, x^2, x^3)$ has two components [1], which under the reduction of x^3 correspond precisely to the two components of the usual Dirac spinor $\psi(x^0, x^1, x^2)$ of the (2+1)-dimensional theory. The inconsistency of the theory appeared as a gauge anomaly in our calculation, since 'parity' invariance was maintained throughout.[3] Moreover, the Hamiltonian results [8] on the Z-string global gauge anomaly for *other* fermion representations or gauge groups match those of the three-dimensional gauge anomaly established in the Euclidean path integral formulation [22]. The connection between the (3+1)-dimensional Z-string global gauge anomaly in the basic chiral $SU(2)$ Yang-Mills theory and the (2+1)-dimensional $SU(2)$ gauge anomaly also answers some questions raised in Ref. [8], notably on the derivation of the Z-string global gauge anomaly by Euclidean path integral methods and the index theorem responsible. These answers are essentially provided by the three-dimensional results of Refs. [20, 22].[4]

There is, however, a crucial difference between these (2+1)- and (3+1)-dimensional $SU(2)$ gauge anomalies. In 2+1 dimensions, the local counterterm in the action is given by (5). In 3+1 dimensions, on the other hand, the local counterterm is given by

$$\Delta I_{3+1} = \int_{S_1} \mathrm{d}x^0 \int_{S_2} \mathrm{d}x^1 \mathrm{d}x^2 \int_{S_1} \mathrm{d}x^3 \; \frac{\pi}{L_3} \; \omega_{\rm CS} \left(W_0, W_1, W_2 \right) \ , \tag{9}$$

with the three gauge potentials in the Chern-Simons density (6) depending on *all* spacetime coordinates x^μ, $\mu = 0,\, 1,\, 2,\, 3$, and the coupling constant $\tilde{g}$ in (7) replaced by the dimensionless coupling constant g (not to be confused with the Lie group element $g \in G$). A (3+1)-dimensional gauge transformation (1) of $W_\mu(x)$ with *smooth* gauge function $g(x^0, x^1, x^2, x^3) \in G = SU(2)$ shifts the action term (9) by $\pi\, n$, where n is the integer winding number $\Omega_{\rm CS}[-\partial_k g(x)\, g^{-1}(x)]$ evaluated over a surface of constant but arbitrary x^3.

[3] See Appendix A and Section 4 of Ref. [8] for the role of the reflection symmetry R_1 in establishing the spectral flow and the Berry phase, respectively.

[4] In the Euclidean path integral approach of Ref. [6], the four-dimensional Z-string global gauge anomaly originates in the $k_3 = 2\, l\, \pi/L_3 = 0$ momentum sector of the Dirac eigenvalues, whereas the anomalies of the other sectors $l = \pm 1, \pm 2, \ldots$, cancel in pairs (for an appropriate choice of regularization). Alternatively, the $SU(2)$ Z-string global gauge anomaly follows directly from the partially regularized Weyl 'determinant' given by Eq. (14) of Ref. [23], again with only the $k_3 = 0$ momentum sector contributing to the anomaly. As mentioned at the beginning of this section, the theory with anti-periodic boundary conditions on the fermionic fields at $x^3 = 0,\, L_3$ has no such anomaly, since all k_3 momenta come in pairs, $k_3 = \pm\, (2\, m + 1)\, \pi/L_3$ for $m = 0,\, 1,\, 2,\, \ldots$.

For an x^3-independent gauge function $g(x^0, x^1, x^2)$ in particular, the resulting factor $(-1)^n$ in the path integral cancels against the same factor $(-1)^n$ from the fermion 'determinant,' which proves (9) to be a suitable counterterm for the Z-string global gauge anomaly.

Four remarks are in order. First, the counterterm (9) violates CPT and $SO(3,1)$ Lorentz invariance. In 2+1 dimensions, the integrand of the counterterm (5) is CPT even and manifestly Lorentz invariant [24]. In 3+1 dimensions, the integrand of the counterterm (9) is CPT odd, whereas the standard Yang-Mills action density [2] is CPT even. (The (3+1)-dimensional CPT transformation resembles in this respect the (2+1)-dimensional 'parity' transformation discussed above.) Furthermore, having added the counterterm (9) to the standard Yang-Mills action [2], the gauge field propagation is clearly different in the x^1, x^2 directions and the x^3 direction ($\omega_{\rm CS}$ has, for example, no partial derivatives with respect to x^3). Remarkably, this local anisotropy, i.e. Lorentz non-invariance, is controlled by a mass parameter g^2/L_3 which depends on the global structure of the space-time manifold.[5]

Second, the gauge anomaly viewed as a Berry phase factor [9, 10] over the gauge orbit (2) 'originates' in the Z-string configuration, which has a preferred direction (x^3) dictated by the topology of the space manifold ($S_2 \times S_1$). (Of course, there does not have to be a real Z-string running through the universe; what matters here is the mere possibility of having such a configuration.) This makes it plausible that the corresponding local counterterm in the action also carries a preferred direction and thereby violates Lorentz invariance.

Third, it remains to to be determined under which conditions the local counterterm (9) is unique. The local counterterm given by (9) respects, for example, translation invariance in the x^3 direction and has the smallest possible magnitude, with a factor π/L_3 instead of $(2\,m+1)\,\pi/L_3$ for some positive integer m. Also, the over-all sign of the counterterm (9) may be irrelevant as far as the global gauge anomaly is concerned, but matters for the propagation of the gauge fields [24, 28]. (Inspection of (9) suggests an additional counterterm with $(\pi/L_0)\,\omega_{\rm CS}\,(W_1, W_2, W_3)$ in the integrand, which would, however, be absent if x^0 had an infinite range ($x^0 \in \mathbb{R}$), or if the fermionic fields had anti-periodic boundary conditions in the time direction. Different space manifolds M may also lead to different counterterms. The manifold $M = S_1 \times S_1 \times S_1$, for example, has three possible counterterms similar to (9) and $M = S_3$ apparently none.)

[5]See Refs. [25] and [26] for the experimental tests of CPT and local Lorentz invariance, respectively. Further tests of special relativity are discussed in Ref. [27]. The authors of Ref. [28] have also considered the effect of a purely electromagnetic Lagrangian density term which is essentially the same as the $SU(2)$ counterterm (9), with $W^1_\mu = W^2_\mu = 0$ identically. To first order, this particular Lagrangian density term causes the linear polarization of an electromagnetic plane wave *in vacuo* to rotate in a manner dependent on the direction of propagation but not on the wavelength. Linear polarization measurements of distant radio sources can perhaps be used to constrain the mass parameters of this Lagrangian density term, see Refs. [28, 29] and references therein.

Fourth and finally, the counterterm (9) for space manifold $M = S_2 \times S_1$ and gauge group $G = SU(2)$ may perhaps cure the Z-string global gauge anomaly, but the gauge invariance is still threatened by the remaining global $SU(2)$ Witten anomaly. The potential violation of Lorentz invariance needs to be taken seriously only if the global $SU(2)$ Witten anomaly can be eliminated first (possibly by the introduction of new fields or by some other means).

To enlarge upon this last remark, consider now the standard Yang-Mills theory [2] with gauge group $G = SU(3)$ and a single triplet $\mathbf{3}_L$ of left-handed Weyl fermions [1] in the path integral formulation. The space manifold M is again $S_2 \times S_1$. The perturbative Bardeen anomaly [5] and the resulting Witten anomaly [6] of this theory have already been discussed at the end of the previous section. Here, we introduce a further octet $\mathbf{8}_{PS}$ of *elementary* pseudoscalar fields Π^a, $a = 1, \ldots, 8$, transforming according to a non-linear realization of $SU(3)_L \times SU(3)_R$, with $U \to g_L\, U\, g_R^{-1}$ for $U \equiv \exp\left(T^a\, \Pi^a/\Lambda\right) \in SU(3)$ and energy scale Λ. These pseudoscalar fields have a special Wess-Zumino-like interaction [30, 31] which cancels the perturbative $SU(3)_L$ gauge anomaly of the chiral fermions. (See in particular Eqs. (17), (24), (25) of Ref. [31], with only $SU(3)_L$ gauged. For the anomaly cancellation, see also Ref. [32].) As far as the pseudoscalars are concerned, the theory is now considered fixed. However, the effective action for the bosonic fields is still non-invariant under the $SU(3)_L$ gauge transformation (2), with ω replaced by x^0. This last anomaly, the Z-string global gauge anomaly, needs to be cancelled by an additional contribution to the action of the gauge bosons. A suitable counterterm is given by (9) in terms of the $SU(3)_L$ gauge potentials.

The resulting theory of $SU(3)_L$ gauge bosons, left-handed fermions and pseudoscalars has no longer the $SU(3)_L$ gauge anomalies mentioned, but violates CPT and Lorentz invariance through the counterterm (9) in the action. In fact, each of the $SU(3)_L$ gauge bosons W^a_μ, $a = 1, \ldots, 8$, has two transverse degrees of freedom with (different) anisotropic energy-momentum dispersion relations [28]. The counterterm (9) thus removes the Z-string global gauge anomaly and generates the 'isotropy anomaly' instead. Of course, the theory considered is most likely non-renormalizable and may require substantial modifications at the energy scale Λ of the pseudoscalar interactions. In an elementary particle physics context, Λ might be related to the energy scale at which gravitational effects become important. This would then suggest a possible gravitational origin for the Lorentz non-invariance of the 'low-energy' theory as given by the isotropy anomaly term (9) in the effective action.[6]

In conclusion, the existence of the (3+1)-dimensional Z-string global gauge anomaly depends on both the non-Abelian gauge group G and the product space M, the latter of

[6]Other Lorentz non-invariant terms running with energy have been considered before, see Ref. [33] and references therein.

which allows for one spatial coordinate to become temporarily 'irrelevant' (here, $x^3 \in S_1$ for $M = S_2 \times S_1$). Essentially the same non-Abelian gauge anomaly occurs in 2+1 dimensions. The crucial difference, however, is that in 2+1 dimensions the local counterterm available violates only certain reflection symmetries, whereas in 3+1 dimensions the corresponding local counterterm violates both CPT and Lorentz invariance.

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